Nagoya University, G30 program Fall 2018

Mathematics Tutorial Ia (calculus) Instructor : Serge Richard

---

Homework 8

Exercise 1 *Determine the maximal domain on which the following functions are defined and sketch their graph as precisely as possible:*

$$a)\ f(x) = \frac{x-3}{x^2+1}, \qquad b)\ g(x) = \frac{2x^2-1}{x^2-2}, \qquad c)\ h(x) = x + \frac{1}{x}\ .$$

Recall that the logarithm function has been introduced as the inverse of the strictly increasing and differentiable function $\mathbb{R} \ni x \mapsto \mathrm{e}^x \in \mathbb{R}_+$. More precisely, $\ln : \mathbb{R}_+ \to \mathbb{R}$ satisfies $\mathrm{e}^{\ln(y)} = y$ and $\ln(\mathrm{e}^x) = x$ for any $x \in \mathbb{R}$ and $y \in \mathbb{R}_+$. Recall also that the following properties have been proved:

(i) $\ln(y)' = \frac{1}{y}$ for any $y \in \mathbb{R}_+$,

(ii) $\ln(yz) = \ln(y) + \ln(z)$ for any $y, z \in \mathbb{R}_+$,

(iii) $\ln(y^x) = x \ln(y)$ for any $y \in \mathbb{R}_+$ and $x \in \mathbb{Q}$.

Based on this, it is natural to set for any $x \in \mathbb{R}$ and $y \in \mathbb{R}_+$

$$y^x \equiv \mathrm{e}^{\ln(y^x)} := \mathrm{e}^{x \ln(y)}$$

Exercise 2 *Let us set* $\varepsilon := \mathrm{e}^1 = 2.718....$ *Check that* $\ln(\varepsilon) = 1$ *and that* $\varepsilon^x = \mathrm{e}^x$*.*

Exercise 3 *Compute the following limits:*

$$a)\ \lim_{x\to 0_+} x \ln(x), \qquad b)\ \lim_{x\to 0_+} x^x, \qquad c)\ \lim_{x\to +\infty} \frac{\ln(x)}{x}, \qquad d)\ \lim_{x\to +\infty} x^{1/x}\ .$$

What can you say for $\lim_{x\to 0_+} x^r \ln(x)$ *for any* $r > 0$ *?*

Exercise 4 *Compute the following limits:*

$$a)\ \lim_{x\to 0} \frac{\ln(1+x)}{x}, \quad b)\ \lim_{x\to 0} (1+x)^{1/x}, \quad c)\ \lim_{x\to \infty} \big(1+\frac{1}{x}\big)^x, \quad d)\ \lim_{x\to \infty} \big(1+\frac{r}{x}\big)^x \text{ for any } r>0.$$

Exercise 5 *Compute the derivative of the following functions:*

$$f : \mathbb{R} \ni x \mapsto a^x \in \mathbb{R} \ \text{ for any } a > 0, \qquad g : \mathbb{R}_+^* \ni x \mapsto x^x \in \mathbb{R}.$$