Nagoya University, G30 program Fall 2018

Mathematics Tutorial Ia (calculus) Instructor : Serge Richard

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Homework 4

Exercise 1 *Consider the curve in* $\mathbb{R}^2$ *defined by the relation*

$$\frac{(x-1)^2}{9} + \frac{y^2}{4} = 1, \qquad (x,y) \in \mathbb{R}^2.$$

Sketch this curve and determine the slope of the tangent at each point of it.

Exercise 2 *Compute the derivative of the functions* $f : \mathbb{R} \to \mathbb{R}$ *with* $f(x)$ *provided by the following expressions:*

$$a)\ 5x^4 + 4x^2 - 1, \qquad b)\ (x^5+1)(x^2-1), \qquad c)\ \frac{5x-1}{x-5}\ \text{ for } x \neq 5, \qquad d)\ \frac{x^{25}-2x}{x^2+3}.$$

Exercise 3 *Consider the function* $f : \mathbb{R} \ni x \mapsto x^3 - 6x^2 + 8x \in \mathbb{R}$, *and show that the graph of the function* $\ell : \mathbb{R} \ni x \mapsto -x \in \mathbb{R}$ *is tangent to the graph of the function* f. *Find the point of tangency.*

Exercise 4 *Consider the function* f *defined by* $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ *for any* $x \in \mathbb{R}$.

1. *For any fixed* $x \in \mathbb{R}$ *show that the sum is convergent,*
2. *Compute the derivative of* f,
3. *What can you say about this function ?*

Exercise 5 *1) For* $n \in \mathbb{N}^*$ *let* $P_{\frac{1}{n}} : \mathbb{R}_+ \to \mathbb{R}$ *be the function defined by* $P_{\frac{1}{n}}(x) = x^{\frac{1}{n}}$. *Show that the following equality holds:*

$$P'_{\frac{1}{n}}(x) = \frac{1}{n} x^{\frac{1}{n}-1}$$

For the proof you can use the equality

$$(a^n - b^n) = (a-b) \sum_{k=0}^{n-1} a^{n-k-1} b^k$$

for $a = (x+h)^{\frac{1}{n}}$ *and* $b = x^{\frac{1}{n}}$.

2) Deduce that if $P_{\frac{m}{n}} : \mathbb{R}_+ \to \mathbb{R}$ *is defined for* $m, n \in \mathbb{N}^*$ *by* $P_{\frac{m}{n}}(x) = x^{\frac{m}{n}}$, *then*

$$P'_{\frac{m}{n}}(x) = \frac{m}{n} x^{\frac{m}{n}-1} \ .$$

Exercise 6 (optional) *We say that a function* $f : \mathbb{R} \to \mathbb{R}$ *is* uniformly continuous *if for any* $x \in \mathbb{R}$ *and any* $\varepsilon > 0$ *there exists* $\delta > 0$ *(which depends on* f *and* ε *but NOT on* x*) such that* $|f(x+h) - f(x)| \leq \varepsilon$ *for any* $|h| \leq \delta$. *Show that the function defined by* $f(x) = x^2$ *is not uniformly continuous but that the function defined by* $f(x) = x$ *is uniformly continuous. What about the function* f *defined by* $f(x) = \cos(x)$ *?*