Homework 12

Exercise 1 *Consider $f : [a, b] \to \mathbb{R}$ continuous, and differentiable on (a, b), and suppose that f' is also continuous on $[a, b]$. Show that the length ℓ of the curve defined by $\big\{\big(x, f(x)\big) \mid x \in [a, b]\big\}$ is given by the expression*

$$\ell = \int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x\ .$$

Exercise 2 *Let $f : [a, b] \to \mathbb{R}_+$ be continuous and consider the volume of revolution generated by the rotation of $\big\{\big(x, f(x)\big) \mid x \in [a, b]\big\}$ around the x-axis. Show that the volume V of this solid is given by the expression*

$$V = \pi \int_a^b f(x)^2\,\mathrm{d}x$$

Exercise 3 *Let $f : [a, b] \to \mathbb{R}_+$ be continuous, and differentiable on (a, b), and suppose that f' is also continuous on $[a, b]$. Consider the surface of revolution generated by the rotation of the points $\big\{\big(x, f(x)\big) \mid x \in [a, b]\big\}$ around the x-axis. Show that the surface S is given by the expression*

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,\mathrm{d}x\ .$$

Exercise 4 (The painter's paradox) *Consider the function $f : [1, b] \ni x \mapsto \frac{1}{x} \in \mathbb{R}_+$ with $b > 1$. In the setting of the previous two exercises show that for any $b > 1$ one has $V = \pi\big(1 - \frac{1}{b}\big)$ while $S > 2\pi \ln(b)$. By considering the limit $b \to \infty$ why do we get an apparent paradox ?*