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No.

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Thm: Let $F: M \to N$ be a smooth map between smooth manifold

For any $p \in M$, we set:

$F^*: C^\infty(F(p)) \to C^\infty(p)$: $F^*(f) = f \circ F$

$F_*: T_p(M) \to T_{F(p)}(N)$ $[F_*(X_p)](f) = X_p(F^* f)$

Then:

1, F^* is a homomorphism of algebra

2, F_* is a homomorphism of vector space.

Proof:

f, g are defined on some $V_f, V_g \in \mathcal{V}_{F(p)}$ respectively

1) One has:

$$\begin{aligned}
[F^*(f + \alpha g)](q) &= [(f + \alpha g) \circ F](q) \\
&= [f + \alpha g]((F(q)) \\
&= f(F(q)) + \alpha g(F(q)) \\
&= [f \circ F](q) + \alpha [g \circ F](q) \\
&= [F^*(f)](q) + \alpha [F^*(g)](q) \quad \forall q \in F^{-1}(V_f) \cap F^{-1}(V_g)
\end{aligned}$$

$$\begin{aligned}
[F^*(fg)](q) &= [(fg) \circ F](q) \\
&= [fg](F(q)) \\
&= f(F(q)) \cdot g(F(q)) \\
&= [f \circ F](q) [g \circ F](q) \\
&= [F^*(f) F^*(g)](q) \quad \forall q \in F^{-1}(V_f) \cap F^{-1}(V_g)
\end{aligned}$$

$\Rightarrow$ F^* is a homomorphism of algebra $\square$

2, $$\begin{aligned}
[F_*(X_p + \alpha Y_p)](f) &= [X_p + \alpha Y_p](F^*(f)) \\
&= X_p(F^*(f)) + \alpha Y_p(F^*(f)) \quad (\text{since } T_p(M) \text{ is a vector space}) \\
&= [F_*(X_p)](f) + \alpha [F_*(Y_p)](f) \quad \forall f \in C^\infty(F(p))
\end{aligned}$$

$\Rightarrow$ F_* is a homomorphism of vector space. $\square$