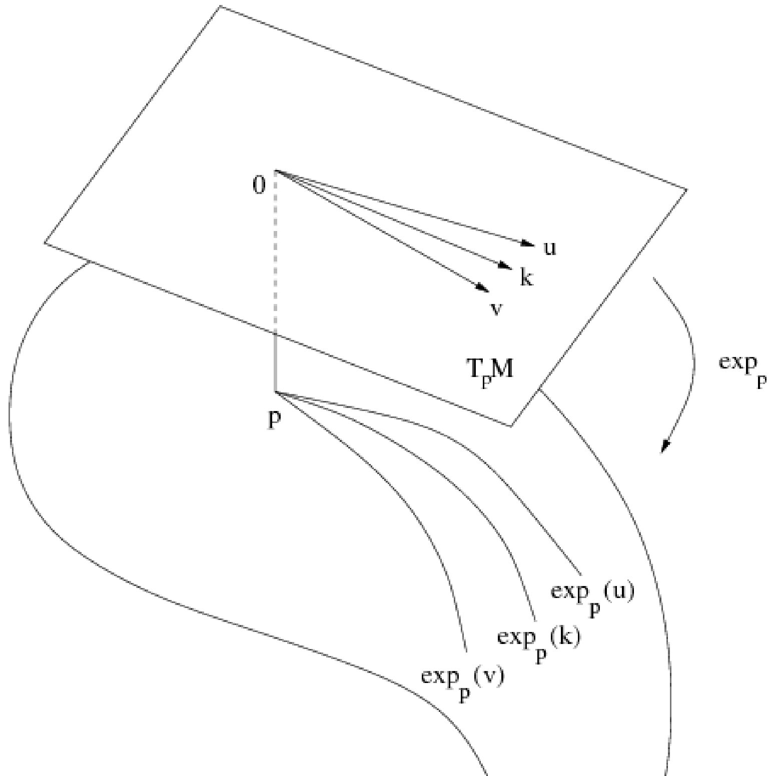
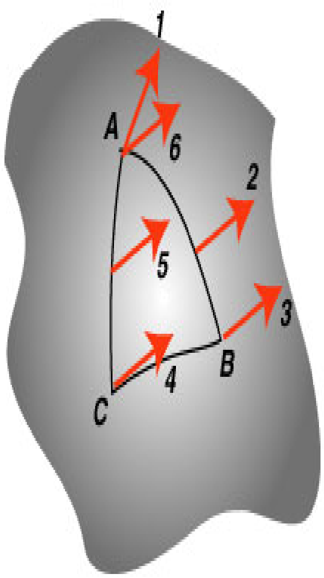
The exponential map



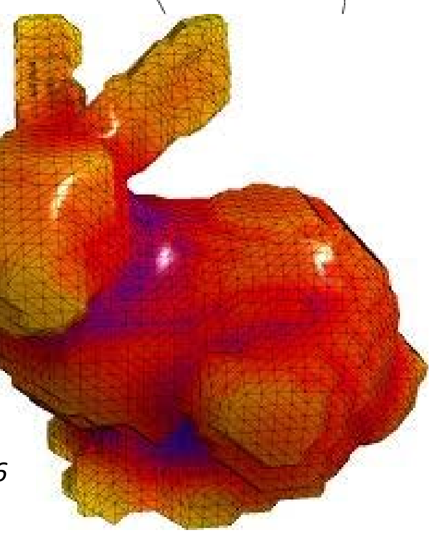
Curvature

HOW TO MEASURE THE CURVATURE BY THE INTRINSIC METHOD



Given a curved surface, make a parallel transport of an unit vector (red arrow) from A to B, C, and back to A again, along the triangle ABC. "Parallel"roughly means "Keep the same angle to the geodesic in question".

Thus, starting from 1, the vector comes back as 6. Notice the change of its direction! This change in comparison to the area of the triangle shows the curvature in this location (can be expressed in terms of Riemann curvature tensor). The curvature of a sphere, for example, can be recovered by this method.



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Interactive Curvature Tensor Visualization on Digital Surfaces*

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Abstract. Interactive visualization is a very convenient tool to explore complex scientific data or to try different parameter settings for a given processing algorithm. In this article, we present a tool to efficiently analyze the curvature tensor on the boundary of potentially large and dynamic digital objects (mean and Gaussian curvatures, principal curvatures, principal directions and normal vector field). More precisely, we combine a fully parallel pipeline on GPU to extract an adaptive triangulated isosurface of the digital object, with a curvature tensor estimation at each surface point based on integral invariants. Integral invariants being parametrized by a given ball radius, our proposal allows to explore interactively different radii and thus select the appropriate scale at which the computation is performed and visualized.

Appendix XII

From Wikipedia: *Riemann curvature tensor*

Formally [edit]

When a vector in a Euclidean space is parallel transported around a loop, it will again point in the initial direction after returning to its original position. However, this property does not hold in the general case. The Riemann curvature tensor directly measures the failure of this in a general Riemannian manifold. This failure is known as the non-holonomy of the manifold.

Let x_t be a curve in a Riemannian manifold M. Denote by $\tau_{x_t} : T_{x_0}M \to T_{x_t}M$ the parallel transport map along x_t. The parallel transport maps are related to the covariant derivative by

$$\nabla_{\dot{x}_0} Y = \lim_{h \to 0} \frac{1}{h} \left(Y_{x_0} - \tau_{x_h}^{-1}(Y_{x_h}) \right) = \frac{d}{dt} (\tau_{x_t} Y)\Big|_{t=0}$$

for each vector field Y defined along the curve.

Suppose that X and Y are a pair of commuting vector fields. Each of these fields generates a one-parameter group of diffeomorphisms in a neighborhood of x_0. Denote by τ_{tX} and τ_{tY}, respectively, the parallel transports along the flows of X and Y for time t. Parallel transport of a vector $Z \in T_{x_0}M$ around the quadrilateral with sides tY, sX, $-tY$, $-sX$ is given by

$$\tau_{sX}^{-1} \tau_{tY}^{-1} \tau_{sX} \tau_{tY} Z.$$

This measures the failure of parallel transport to return Z to its original position in the tangent space $T_{x_0}M$. Shrinking the loop by sending $s, t \to 0$ gives the infinitesimal description of this deviation:

$$\frac{d}{ds} \frac{d}{dt} \tau_{sX}^{-1} \tau_{tY}^{-1} \tau_{sX} \tau_{tY} Z \Big|_{s=t=0} = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) Z = R(X,Y)Z$$

where R is the Riemann curvature tensor.

From Wikipedia: *Constant curvature*

The Riemannian manifolds of constant curvature can be classified into the following three cases:

- elliptic geometry – constant positive sectional curvature
- Euclidean geometry – constant vanishing sectional curvature
- hyperbolic geometry – constant negative sectional curvature.

From Wikipedia: *Ricci curvature*

Direct geometric meaning [edit]

Near any point p in a Riemannian manifold (M,g), one can define preferred local coordinates, called geodesic normal coordinates. These are adapted to the metric so that geodesics through p correspond to straight lines through the origin, in such a manner that the geodesic distance from p corresponds to the Euclidean distance from the origin. In these coordinates, the metric tensor is well-approximated by the Euclidean metric, in the precise sense that

$$g_{ij} = \delta_{ij} + O\left(|x|^2\right).$$

In fact, by taking the Taylor expansion of the metric applied to a Jacobi field along a radial geodesic in the normal coordinate system, one has

$$g_{ij} = \delta_{ij} - \tfrac{1}{3} R_{ikjl} x^k x^l + O\left(|x|^3\right).$$

In these coordinates, the metric volume element then has the following expansion at p:

$$d\mu_g = \left[1 - \tfrac{1}{6} R_{jk} x^j x^k + O\left(|x|^3\right)\right] d\mu_{\text{Euclidean}},$$

which follows by expanding the square root of the determinant of the metric.