The Bannach-Tarski Theorem

Thm:- let $A, B \subset \mathbb{R}^3$ bounded subsets with non-empty interior, then $\exists n \in \mathbb{N}$ and $\{A_i\}_{i\leq n}$, $\{B_i\}_{i\leq n}$ partitions of A, B s.th A_i is congruent to B_i for all $i \leq n$

loosely speaking, one can find a partition of A into n subsets and rearrange them using only rigid transformation "rotation, translation" to get B.

Remark:- the theorem implies the existance of non measurable sets.

let $\mathcal{P}(\mathbb{R}^3)$ be the power set of $\mathbb{R}^3$, $(\mathbb{R}^3, \mathcal{P}(\mathbb{R}^3))$ is a measurable space

one defines $\mu : \mathcal{P}(\mathbb{R}^3) \longrightarrow \mathbb{R}$ to be the measure that corrosponds to volume

So if $A \in \mathcal{P}(\mathbb{R}^3)$, $B \in \mathcal{P}(\mathbb{R}^3)$, $\{A_i\}_{i\leq n}$, $\{B_i\}_{i\leq n}$ are partitions of A, B

let $g = \{ g_i : \mathbb{R}^3 \to \mathbb{R}^3 \mid g_i : x \mapsto x + y_i$ where y_i is a fixed vector in $\mathbb{R}^3 \}$

then g represents the set of all translations

let $SO_3 = \{ R_i \in M_3(\mathbb{R}) \mid R_i R_i^T = I_3, \det(R_i) = 1 \}$

then SO_3 is the set of all rotations

Since μ is the volume, then it should be invariant under the effect of g and SO_3

Based on B-T theorem, we can write B_i in the following way

$B_i = g_i R_i A_i$ "we just rotated and moved A_i to get B_i"

then $B = \biguplus_i^n B_i = \biguplus_i^n g R_i A_i$

now one has

$$\mu(B) = \mu\left(\biguplus_i^n g_i R_i A_i\right) = \sum_i^n \mu(g_i R_i A_i)$$

Since μ is invariant then

$\mu(B) = \sum_i^n \mu(A_i) = \mu(A)$ then we showed that the volume of A, B is the same. which is a contradiction. This implies that the set A_i, B_i are non-measurable with the measure μ for ~~all~~ some $i \leq n$

The B-T thm is valid in $\mathbb{R}^3$ and in higher dimentions ($\mathbb{R}^n$) but it fails when $n=1,2$. however, we can show a similar result ~~with~~ by partitioning the sets into infinite number of sets.

we do a proof of a similar result in $\mathbb{R}^2$.

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let $S \subset \mathbb{R}^2$ be the unit circle centered at the origin.

let $\mathcal{P} = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in M_2(\mathbb{R}) \,\middle|\, \theta = 2\pi n \text{ where } n \in \mathbb{Q} \cap [0,1) \right\}$

is the set of all rotations by a rational multiple of 2π

now because the number of elements in $\mathcal{P}$ depends on the allowed values for n and because $\mathbb{Q} \cap [0,1)$ is countable then $\mathcal{P}$ is also countable. hence, one can numerate the elements of $\mathcal{P}$.

So, let $\mathcal{P} = \{\mathcal{P}_i\}_{i \in \mathbb{N}}$

let $P \in S$ then one define

$\mathcal{P}(P) := \{\mathcal{P}_i(P) \mid \mathcal{P}_i \in \mathcal{P}\}$

~~clear~~ clearly if $q, q' \in \mathcal{P}(P)$ then one can always find $\varphi \in \mathcal{P}$ s.th $\varphi(q) = q'$. then $\mathcal{P}(P)$ form an equivalant class.

Applying $\mathcal{P}$ to all points in S partitions S into an uncountable number of equivalent classes.

By the axiom of choice, one picks up one element from each equivalent class to form a set. we call it the choice set M.

we define

$\mathcal{P}_i(M) := \{\mathcal{P}_i(P) \mid P \in M\}$

$\mathcal{P}(M) := \biguplus_i \mathcal{P}_i(M)$

clearly $\mathcal{P}(M) = S$

remark:- Let $A, B \in \mathcal{S}$ then one can always find $C \in \mathcal{S}$ s.th $A = CB$

One define ψ_{2i}, ψ_{2i-1} s.th

$$\psi_{2i} \mathcal{S}_{2i} = \mathcal{S}_i \quad , \quad \psi_{2i-1} \mathcal{S}_{2i-1} = \mathcal{S}_i$$

then, we can define a bijection between the even numbered elements of $\mathcal{S}$ and $\mathcal{S}$ itself. hence, we have:-

$$\mathcal{S} = \{\psi_{2i} \mathcal{S}_{2i}\}_{i \in \mathbb{N}} = \{\psi_{2i-1} \mathcal{S}_{2i-1}\}_{i \in \mathbb{N}}$$

then

$$S = \biguplus_i \mathcal{S}_i(M) = \left[\biguplus_i \mathcal{S}_{2i}(M)\right] \uplus \left[\biguplus_i \mathcal{S}_{2i-1}(M)\right]$$

$$= \biguplus_i \psi_{2i} \mathcal{S}_{2i}(M) = \biguplus_i \psi_{2i-1} \mathcal{S}_{2i-1}(M)$$

Q.E.D.

Some remarks:-

In the case of the circle in $\mathbb{R}^2$, we had to divide it into countable infinite number of subsets. When we go to $\mathbb{R}^3$ and higher dimensions where B-T thm holds, one can get the same paradoxical results with finite number of pieces. "actually five pieces are enough for balls in $\mathbb{R}^3$"

$\Rightarrow$ the paradox is actually a result of the complexity of the group of rotations. "note that, we used the concept of cardinality of the set of rational rotations to prove the result for circles." so the thm is actually about the complexity of SO_n

" $SO_n := \{A \in M_n(\mathbb{R}) \mid AA^T = I_n , \det(A) = 1\}$ "

general notions:-

In a more general settings, we have:-

let X be a set, $\mathcal{P}(X)$ is the powerset, then $(X, \mathcal{P}(X))$ is a measurable space. then let μ be a measure on $(X, \mathcal{P}(X))$ then $(X, \mathcal{P}(X), \mu)$ is ~~a measurab~~ measure space.

now let $G = \{ g : \mathcal{P}(X) \longrightarrow \mathcal{P}(X) \mid \mu(g(P)) = \mu(P) \ \forall P \in \mathcal{P}(X), \forall g \in G$ s.th P is measurable $\}$

now suppose $E \in \mathcal{P}(X)$. we say E is G-paradoxical "paradoxical with respect to G" if for some m, n there exist $g_1, \dots, g_m \in G$ and $h_1, \dots, h_n \in G$ and pair wise disjoint $A_1, \dots, A_m$ and $B_1, \dots, B_n \subseteq E$

s.th $E = \bigcup g_i(A_i) = \bigcup h_j(B_j)$ and we say that A_i, B_j are non-measurable by the measure μ for some i, j

* note that in that definition, $\{A_i\}_i^m \cup \{B_j\}_j^n$ may not cover all of E

* note that $\{ g_i(A_i) \}$ may not be pair wise disjoint, similarly for $\{h_j(B_j)\}$

The B-T thm shows that any ball in $\mathbb{R}^3$ is paradoxical w.r.t. SO_3 "rotation"

The example we showed proves that circles in $\mathbb{R}^2$ are paradoxical w.r.t SO_2 "rotation"

The results arise from the complexity of the set G and the freedom allowed by the measurable space.