From [B G] Chapter 9

Set $B_R = \{x \in \mathbb{R}^d \mid \|x\| < R\}$

(ball of radius R ~~wi~~and centered at 0)

For $\mu \in M(\mathbb{R}^d)$, we set $\mu_R = \mu|_{B_R}$

and $\gamma := \lim_{R\to\infty} \frac{1}{\mathrm{vol}(B_R)} \mu_R * \widetilde{\mu_R}$ called natural autocorrelation measure

$=: \mu \circledast \tilde{\mu}$ called Eberlein convolution

⚠ The autocorrelation could depend on the ~~sa~~ shape used in the limitting process $\gamma_{Hef} = \gamma_{BG}$?

Asking that the autocorrelation doesn't depend on the shape is a slightly stronger assumption

About a Dirac measure at 0

Proposition : Let $\mu \in M(\mathbb{R}^d)$ with a natural autocorrelation

Then $\hat{\gamma}(\{0\}) = \lim_{n\to\infty} \lim_{R\to\infty} \frac{1}{\mathrm{vol}(B_R)} \int_{B_R} |(\delta_z * \mu)(h_n)|^2 dz \geq 0$

with $h_n = \frac{\gamma_{B_n}}{\mathrm{vol}(B_n)}$

Proof

Observe that for suitable function g

$$\mu * \tilde{\mu}(g * \tilde{g}) = \int dz \left|\int \mu(dx) g(x+z)\right|^2 = \int dz |\mu(g(\cdot + z))|^2$$

$$= \int dz |(\delta_z * \mu)(g)|^2$$

Thm

$$\underbrace{\gamma(h_r * \widetilde{h_r})}_{\text{is continuous}} \overset{\text{by def} + \uparrow}{=} \lim_{R\to\infty} \frac{1}{\mathrm{vol}(B_R)} \int_{\mathbb{R}^d} dz |(\delta_z * \mu_R)(h_r)|^2$$

with support in B_{2R} (closed) $=: \overline{B_{2R}}$

If we set $\phi_r = h_r * \widetilde{h_r}$, then

gamma function (!) Bessel function

$$\hat{\phi}_r(k) \underset{= \check{\phi}_r(k) =}{=} \left(\frac{\Gamma(1+d/2) J_{d/2}(2\pi|k|r)}{\pi|k|r}\right)^2 \geq 0$$

inverse Fourier transform

Note that $\hat{\phi}_r(0) = 1$ (need Taylor expantion of $\mathcal{J}_{d/2}$ near 0),

for fixed $k \in \mathbb{R}^d \setminus \{0\}$, $\lim_{r\to\infty} \hat{\phi}_n(k) = 0$

(we say ϕ_r converges to 0 as $r \to \infty$ pointwise on $\mathbb{R}^d \setminus \{0\}$)

Now $\hat{\phi}_r$ converges to $\chi_{\{0\}}$

$$\gamma(h_r * \tilde{h}_r) = \hat{\check{\gamma}}(h_r * \tilde{h}_r) = \hat{\gamma}(\check{\phi}_r) = \hat{\gamma}(\hat{\phi}_r)$$

$$\xrightarrow{\text{by Lebesgue D.C.T.}} \hat{\gamma}(\chi_{\{0\}}) := \hat{\gamma}(\{0\})$$

Let's ~~consider~~ Λ a point set $\subset \mathbb{R}^d$ and set $\Lambda_R = \Lambda|_{B_R}$

We define $\overline{\mathrm{dem}(\Lambda)} = \limsup_{r\to\infty} \frac{\mathrm{card}(\Lambda_r)}{\mathrm{vol}(B_r)} = \lim_{R\to\infty} \sup_{r > R} \frac{\mathrm{card}(\Lambda_r)}{\mathrm{vol}(B_r)}$

$$\underline{\mathrm{dem}(\Lambda)} = \liminf_{r\to\infty} \frac{\mathrm{card}(\Lambda_r)}{\mathrm{vol}(B_r)} = \lim_{R\to\infty} \inf_{r > R} \frac{\mathrm{card}(\Lambda_r)}{\mathrm{vol}(B_r)}$$

Always $0 \leq \underline{\mathrm{dem}(\Lambda)} \leq \overline{\mathrm{dem}(\Lambda)} \leq \infty$

If $\underline{\mathrm{dem}(\Lambda)} = \overline{\mathrm{dem}(\Lambda)} < \infty$ we call this quantity the density of Λ

and denote it by $\mathrm{dens}(\Lambda)$

Corollary

Let Λ be a locally finite point set ~~and support that~~

Let $\mu := \sum_{x\in\Lambda} \delta_x$ and suppose that it has a natural autocorrelation γ

Then $\hat{\gamma}(\{0\}) = (\mathrm{dens}(\Lambda))^2$

Proof

Existance of natural autocorrelation $\Rightarrow$ dens(Λ) exists

Then $|(\delta_z * \sum_{x\in\Lambda} \delta_x)(k_r)|^2 = \frac{1}{\mathrm{vol}(B_r)^2} \mathrm{card}(\Lambda \cap B_r(z))^2$ = density^2 inside $B_r(z)$ =: $(*)$ (ball of radius r and centered at z)

$\frac{1}{\mathrm{vol}(B_R)} \int_{B_R} dz\,(*)$ average density^2 in any ball of radius r