3, Measure L^∞, distribution and the Fourier Transform

+) $\hat{\phi}(k) = \int \phi(x) e^{-2\pi i \langle k, x \rangle} dx$

$\phi(x) = \int \hat{\phi}(k) e^{2\pi i \langle x, k \rangle} dk$

translate of ϕ by a

If $F[\phi(x)](k) = \hat{\phi}(k)$ and $T_a \phi(x) = \phi(x-a)$

then $\widehat{T_a \phi}(k) = e^{-2\pi i \langle a, k \rangle} \hat{\phi}(k)$

<u>Def</u>: A tempered distribution is a continuous linear functional on $\mathcal{S}$. The Fourier transform $\hat{T}$ of tempered distribution T is defined.

$$\hat{T}\phi := T\hat{\phi} \quad \forall \phi \in \mathcal{S}$$

$\hat{T}$ is again a tempered distribution.

<u>Def</u>: A measure μ is called tempered if μ defines a tempered distribution T_μ:

$$\mu(\phi) = T_\mu \phi := \int d\mu\, \phi \quad \forall \phi \in \mathcal{S}.$$

<u>Thm</u>: If $\int (1+|x|)^{-k} |\mu|(dx) < \infty$ (slowly increasing), then for some $k \in \mathbb{N}$

μ is tempered.

<u>Remark</u>: Every translation bounded measure is slowly increasing and therefore tempered.

<u>Thm</u>: If $\mu(\phi * \tilde{\phi}) \geq 0 \ \forall \phi \in \mathcal{S}$ (positive definite)

then $\hat{\mu}$ is a positive measure.

<u>Remark</u>: The Fourier transform of a tempered measure is a tempered distribution; but it may not be a measure.

<u>Remark</u>: Every autocorrelation is a positive definite; hence the Fourier transform of an autocorrelation is a positive measure