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Ex 1: For a function $f \in \mathbb{R}$, if the limit:

$$c(x) := \lim_{L \to \infty} \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x+\xi)\,[f(\xi)]^{*}\, d\xi$$

exists for all x then c is called the autocorrelation function of f.

If $f \in L^1_{loc}$ (w.r.t L.m) and f is bounded, then $c\lambda$ is the autocorrelation measure of $f\lambda$.

Ex 2: Let X be a countable subset of $\mathbb{R}^d$ that is uniformly locally finite. Let δ_x denote the Dirac measure at x and consider the measure

$$\mu := \sum_{x \in X} \delta_x .$$

Let $A := \{x - y \mid x, y \in X\}$. Assume A is locally finite.

For $a \in A$ and positive L let $N_L(a)$ be the number of occurrence of a in the cube C_L:

$$N_L(a) := |\{x \in X \cap C_L \mid x - a \in X \cap C_L\}|$$

$2 \times$ the number of pairs in $X \cap C_L$, of which the difference is a.

Assume that for all $a \in A$ the limit:

$$n_a := \lim_{L \to \infty} L^{-d} N_L(a) \text{ exists and } n_a > 0$$

Then μ has a unique autocorrelation γ given by

$$\gamma := \sum_{a \in A} n_a \delta_a .$$

Proof:

+) Fourier transform: $\mathcal{F}[x \to f(x)](k) = \hat{f}(k) = \int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i \langle k, x\rangle} dx$

$\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product

+) Then, one wants to prove: $\hat{\bar{f}}(k) = \overline{\hat{f}}(k)$

Indeed: $\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \langle k, x\rangle} dx$.

$$\Rightarrow \hat{f}(-k) = \int_{-\infty}^{\infty} f(x)\, e^{2\pi i \langle k, x\rangle} dx$$

$$\Rightarrow \overline{\hat{f}(-k)} = \int_{-\infty}^{\infty} \overline{f(x)}\, e^{-2\pi i \langle k, x\rangle} dx = \hat{\bar{f}}(k) \quad (1)$$

$$\hat{\check{f}}(k) = \int_{-\infty}^{\infty} \check{f}(x)\, e^{-2\pi i \langle k, x\rangle} dx = \int_{-\infty}^{\infty} f(-x)\, e^{-2\pi i \langle k, x\rangle} dx$$

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$$= \int_{-\infty}^{\infty} f(x)\, e^{2\pi i \langle k, x\rangle} dx = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \langle -k, x\rangle} dx = \hat{f}(-k) \quad (2)$$

$$(1)\,(2) \Rightarrow \widehat{\check{f}}(k) = \overline{\hat{f}(k)}$$
$$\Rightarrow \hat{\bar{f}}(k) = \overline{\hat{f}(k)} \quad //$$

+) Convolution thm: $\widehat{f * h} = \hat{f} \cdot \hat{h}$ //

+) Fourier transform of the Dirac measure at x is given by:

$$\hat{\delta}_a(k) = e^{-2\pi i \langle a, k\rangle} \quad //$$

$$\left(\hat{\delta}_a(k) = \int_{-\infty}^{\infty} \delta_a(x)\, e^{-2\pi i \langle k, x\rangle} dx = e^{-2\pi i \langle k, a\rangle} \right)$$

main proof

$$\Rightarrow \begin{cases} \mu = \sum_{x \in X \cap C_L} \delta_x \\ \tilde{\mu} = \sum_{y \in X \cap C_L} \tilde{\delta}_y \end{cases}$$

$$\Rightarrow \mu * \tilde{\mu} = \left(\sum_{x \in X} \delta_x\right) * \left(\sum_{y \in X} \tilde{\delta}_y\right) = \sum_{x, y \in X} \delta_x * \tilde{\delta}_y$$

$$\hat{\tilde{\delta}}_y(k) = \overline{\hat{\delta}_y(k)} = \overline{e^{-2\pi i \langle y, k\rangle}} = e^{2\pi i \langle y, k\rangle} = e^{-2\pi i \langle -y, k\rangle}$$

$$\widehat{\delta_x * \tilde{\delta}_y}(k) = \hat{\delta}_x(k) \cdot \hat{\tilde{\delta}}_y(k) = e^{-2\pi i \langle x, k\rangle} \cdot e^{-2\pi i \langle -y, k\rangle}$$
$$= e^{-2\pi i \langle x - y, k\rangle} = \hat{\delta}_{x-y}(k)$$

$$\Rightarrow \delta_x * \tilde{\delta}_y = \delta_{x-y}$$

$$\Rightarrow \mu * \tilde{\mu} = \sum_{x, y \in X \cap C_L} \delta_{x-y} = \sum_{a \in A} N_L(a)\, \delta_a$$

$$\Rightarrow \gamma^L(f) = L^{-d} \sum_{a \in A \cap C_L} N_L(a)\, \delta_a(f) \xrightarrow{L \to \infty} \sum_{a \in A} n_a\, \delta_a(f) \quad \square$$

Note: later, X will be interpreted as a set of atomic position, A is the set of interatomic vectors. If X is the set of vertices of a tiling of $\mathbb{R}^d$ that is generated by projection method, all hypotheses on X and A are satisfied

Def: A measure μ is called translation bounded if $\forall$ compact set $K \subset \mathbb{R}^d$ there is a constant α_K s.t.:

$$\sup_{x \in \mathbb{R}^d} |\mu|(K + x) \leq \alpha_K \qquad (K + x := \{ z \in \mathbb{R}^d \mid z - x \in K \})$$

Ex: X in Ex 2. is translation bounded because it is uniformly locally finite