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Autocorrelation of unbounded measure

1, Some definitions

<u>Def</u>: A characteristic function of a subset E of a set K is a function:

$$\chi_E : K \longrightarrow \{0, 1\}.$$
$$\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$$

<u>Def</u>: A simple function is a function:

$$\phi : K \longrightarrow \mathbb{R}$$
$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x), \quad a_i \in \mathbb{R}, \text{ sets } E_i \text{ are pairwise disjoint measurable sets.}$$

<u>Def</u>: Let $f : K \to \mathbb{R} \cup \{+\infty\}$ be a nonnegative measurable function. The integral of f w.r.t the measure μ is

$$\int_K f \, d\mu = \sup \left\{ \int_K \phi \, d\mu \,\middle|\, \phi \text{ is a simple function } 0 \leq \phi \leq f \right\}$$

with $\int_K \phi \, d\mu = \sum_{i=1}^{n} a_i \mu(E_i)$

One denotes: $\mu(f) := \int_K f \, d\mu$. One also can write $\int_K f \, \mu(dx)$ instead of $\int_K f \, d\mu$.

<u>Property</u>: A measure μ on $\mathbb{R}^d$ is linear functional on the space $\mathcal{K}$ of complex continuous function on $\mathbb{R}^d$ of compact support. Then for every subset K of $\mathbb{R}^d$, there is a constant a_K s.t

$$|\mu(f)| \leq a_K \|f\|_\infty, \text{ for all bounded complex function } f, \ supp(f) \subset K$$

$\|.\|_\infty$ denotes the supremum norm

<u>Def</u>: A measure is called positive if $\mu(f) \geq 0 \ \forall f \geq 0$

<u>Def</u>: For every measure μ, there is a smallest positive measure ϱ s.t. $|\mu(f)| \leq \varrho(|f|) \ \forall f \in \mathcal{K}$. The measure ϱ is called

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the absolute value of μ and will be denoted by $|\mu|$.

Def: A measure μ is called bounded if $|\mu|(\mathbb{R}^d)$ is finite, and unbounded otherwise.
($\Rightarrow$ $\mathcal{L}.m$ is unbounded).

Recall: Let μ and ν be 2 measures. Then the convolution of μ and ν is given by
$$\mu * \nu(f) := \int \mu(dx)\,\nu(dy)\,f(x+y)$$
It is well-defined if at least one measure has compact support.

Vague topology: A sequence of measure $\{\mu_n\}$ converge to μ in the vague topology if $\lim_{n\to\infty} \mu_n(f) = \mu(f)$ $\forall f \in \mathcal{K}$.

Def: For any function f, let $\check{f}(x) = f(-x)$ and $\tilde{f}(x) = (\check{f})^*$ ← complex conjugate
Similarly for a measure μ, let $\check{\mu}(f) = \mu(\check{f})$, $\mu^*(f) = \mu(f^*)$ and $\tilde{\mu} = (\check{\mu})^*$
For every $L > 0$, let C_L be the closed cube of side L center around the origin. The restriction of a measure μ is denoted μ_L
($\mu_L(g) := \mu(\chi_{C_L} g)$)
Let $\gamma^L := L^{-d}\mu_L * \tilde{\mu}_L$. Every vague limit point of γ^L is called an autocorrelation of μ

Remark: 1) Autocorrelation is a measure
2) A measure can have one autocorrelation, many autocorrelations or no autocorrelations.

Ex 1: Let f be a function on $\mathbb{R}$. f is the weight function of the measure μ_f
Then $\mu_f(g) := \int_K g\, f\, dx$ ← $\mathcal{L}$. measure
Assume $c(x) := \lim_{L\to\infty} \frac{1}{L}\int_{-L/2}^{L/2} f(x+y)\,(f(y))^*\,dy$ exists for all x.

$c(x)$ is the weight function of the measure γ_c
Then γ_c is the autocorrelation of μ_f