XI Random process in continuous time

We'll consider $\{N_t\}_{t\geq 0}$ a continuous family of integer-valued r.v.

Example: About emails received in a letter box

Let N_t denote the # of ~~limit~~ emails received up to time t.

1) N_t is a r.v. taking values of $0, 1, 2, \cdots$

2) $N_0 = 0$

3) $N_s \leq N_t \Leftarrow s \leq t$

4) Independence: if $0 \leq s < t$ then the emails received between $(s, t]$ is indep. of the emails received before s

5) Arrival rate: $\exists \lambda > 0$ called arrival rate s.t. for h small enough

$$\mathbb{P}(N_{t+h} = n+1 \mid N_t = n) = \lambda h + o(h)$$
$$\mathbb{P}(N_{t+h} = n \mid N_t = n) = 1 - \lambda h + o(h)$$

not equivalent

For the probability of receiving 2 or more emails in $(t, t+h)$:

$$\mathbb{P}(N_{t+h} = n+2 \mid N_t = n) = 1 - \mathbb{P}(N_{t+h} \in \{n, n+1\} \mid N_t = n)$$
$$= 1 - (\lambda h + o(h)) - (1 - \lambda h + o(h)) = o(h) \text{ for } h \text{ small enough}$$

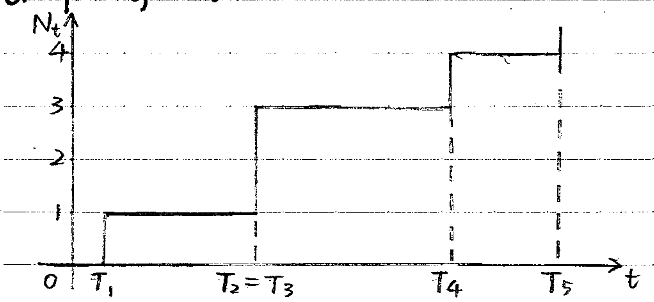
Def. A Poisson process with rate $\lambda > 0$

is a family of integer valued r.v. $\{N_t\}_{t\geq 0}$ satisfying conditions 1) ~ 5)

It can be used in models:

- the arrival of customers in a shop
- the clicks of Geiger counter for the detection of some particles (more realistic for long half-life of particles)

Graph of N_t



Let T_i be the time of arrival of the i^{th} email

$\Rightarrow T_i := \inf\{t \mid N_t = i\}$

No.

Date

¥ Then $0 \leq T_0 \leq T_1 \leq T_2 \leq \cdots$

and $\{T_i\}_{i\in\mathbb{N}}$ is a sequence of r.v. which determines

$$N_t = \max\{n \mid T_n \leq t\}$$

The sequence of $\{T_i\}_{i\in\mathbb{N}}$ is the inverse process of $\{N_t\}_{t\geq 0}$

Thm. For each $t>0$, the r.v. N_t has a Poisson distribution with parameter λt. That is for $\forall k \in \mathbb{N}$ and $t>0$

$$\mathbb{P}(N_t = k) = \frac{1}{k!}(\lambda t)^k e^{-\lambda t}$$

Corollary: $\mathbb{E}(N_t) = \lambda t$, $\mathrm{var}(N_t) = \lambda t$

Proof: Let $P_k(t) = \mathbb{P}(N_t = k)$

Consider $h>0$ small enough

Then by the partition thm,

$$\begin{aligned} P_k(t+h) = \mathbb{P}(N_{t+h}=k) &= \sum_{i=0}^{k} \mathbb{P}(N_{t+h}=k \mid N_t = i)\,\mathbb{P}(N_t = i) \\ &= \mathbb{P}(N_{t+h}=k \mid N_t = k) P_k(t) + \mathbb{P}(N_{t+h}=k \mid N_t = k-1) P_{k-1}(t) + o(h) \\ &= (1-\lambda h + o(h)) P_k(t) + (\lambda h + o(h)) P_{k-1}(t) + o(h) \end{aligned}$$

$$\Leftrightarrow P_k(t+h) - P_k(t) = \lambda h\,(P_{k-1}(t) - P_k(t)) + o(h)$$

$$\Leftrightarrow \frac{1}{h}(P_k(t+h) - P_k(t)) = \lambda\,(P_{k-1}(t) - P_k(t)) + o(1)$$

$$\xrightarrow{h\searrow 0} \left.\begin{aligned} P_k'(t) &= \lambda\,(P_{k-1}(t) - P_k(t)) \\ \text{For } k=0,\ P_0'(t) &= -\lambda\, P_0(t) \end{aligned}\right\} \text{system of difference-differencial equations}$$

Boundary condition:

$$P_k(0) = \begin{cases} 1, & \text{if } k=0 \\ 0, & \text{if } k \in \mathbb{N}_+ \end{cases}$$

3 ways of solving this system:

① Induction over k; ② Generating function; ③ Impection. □

$\downarrow$ Obtain one differential equation of generating function

Recall $T_i = \inf\{t \mid N_t = i\}$ and set $X_i := T_i - T_{i-1}$ (inter-arrival times)

Thm. In a Poisson process with parameter λ, $X_1, X_2, \cdots$ are independent random variables, each having the exponential distribution with parameter λ.

$$X(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1-e^{-\lambda t} & \text{if } t > 0 \end{cases}$$

Note that the X_i are independent (loss of memory)

Def. A positive ~~of~~ r.v. X has the loss of memory property if

$$\mathbb{P}(X > u+v \mid X \geqslant u) = \mathbb{P}(X > v) \quad \forall\, u, v \geqslant 0$$

Prop.

A continuous positive r.v. has the loss of memory property iff X is the exponential distribution.

Proof: If $\lambda > 0$, $u, v \geqslant 0$, X is the exp. distribution,

$$\mathbb{P}(X > u+v \mid X > u) = \frac{\mathbb{P}(X > u+v \text{ and } X > u)}{\mathbb{P}(X > u)} = \frac{\mathbb{P}(X > u+v)}{\mathbb{P}(X > u)} = \frac{e^{-\lambda(u+v)}}{e^{-\lambda u}}$$

$$= e^{-\lambda v} = \mathbb{P}(X > v)$$

Conservely, set $G(u) := \mathbb{P}(X > u)\ \forall\, u \geqslant 0$

$$\left.\begin{aligned} \mathbb{P}(X > u+v \mid X > u) &= \frac{\mathbb{P}(X > u+v)}{\mathbb{P}(X > u)} = \frac{G(u+v)}{G(u)} \\ \xrightarrow{\text{By assumption}} &= \mathbb{P}(X > v) = G(v) \end{aligned}\right\} \Leftrightarrow G(u)G(v) = G(u+v)$$

Since G is non-$\nearrow$, the only solution is $G(u) = e^{-\lambda u}$ for $\lambda > 0$. $\square$

Essentially, Property 4) of Poisson process

$\Rightarrow$ X must have the loss of memory property

Remark

If $\{X_i\}_{i \in \mathbb{N}}$ is a family of ind. exp. dist. with parameter λ

We set $T_1 = X_1$, $T_2 = X_1 + X_2$, $\cdots$, $T_n = \sum_{j=1}^{n} X_j$, and $N_t = \max\{k \mid T_k \leq t\}$

Then $\{N_t\}_{t \geqslant 0}$ is a Poisson process with parameter λ.

Application to population growth:

Time $t=0$, $I \in \mathbb{N}$ of amoebas in a pond, each of them can divide at a random time with the rules: $\exists\, \lambda > 0$ birth rate s.t.

1) $\mathbb{P}(\text{division})$ in $(t, t+h)$ is $\lambda h + o(h)$
2) $\mathbb{P}(\text{no division})$ in $(t, t+h)$ is $1 - \lambda h + o(h)$
3) Each one is independent

Let M_t = # of amoebas at time t and set $P_k(t) = \mathbb{P}(M_t = k)$

$$\mathbb{P}(M_{t+h} = k \mid M_t = k) = \mathbb{P}(\text{no division}) \overset{*}{=} (1 - \lambda h + o(h))^k = 1 - k\lambda h + o(h)$$

for k small enough

$$\mathbb{P}(M_{t+h} = k+1 \mid M_t = k) = \mathbb{P}(1 \text{ division}) = \binom{k}{1}(\lambda h + o(h))(1 - \lambda h + o(h))^{k-1}$$
$$= k\lambda \cancel{t} h + o(h)$$

$$\mathbb{P}(M_{t+h} \geqslant k+2 \mid M_t = k) = o(h)$$

Thm. If $M_0 = I$ and $t > 0$

$$\mathbb{P}(M_t = k) = \binom{k-1}{I-1} e^{-\lambda I t}(1 - e^{-t})^{k-1} \quad \text{for } k \geqslant I$$

Application to population growth + death:

Division rate as before ; death rate with $\mu > 0$

For each amobas, for the interval $(t, t+h)$ one has

- death with prob. $\mathbb{P} = \mu h + o(h)$
- single division with $\mathbb{P} = \lambda h + o(h)$
- no change with prob. $\mathbb{P} = (1 - \mu h + o(h))(1 - \overset{\lambda}{\cancel{\mu}} h + o(h)) = 1 - (\mu + \lambda) h + o(h)$

Set M_t, $P_k(t)$ as before

$$\Rightarrow P_k'(t) = \lambda(k-1)P_{k-1}(t) - (\lambda + \mu) k P_k(t) + \mu(k+1) P_{k+1}(t)$$

Here the iteration method doesn't work because of

One needs a differential equation for the moment generatiang function, and we find $\underbrace{\mathbb{E}(e^{sM_t})}_{G(s,t)} = \cdots$