Nagoya University, G30 program Fall 2015

Mathematics Tutorial Ia (calculus) Instructor : Serge Richard

---

Homework 13

Exercise 1 *Prove the following statement (the ration test) :*
Let $\{a_j\}_{j=1}^{\infty}$ be a sequence with positive terms only. Assume that there exists $c \in (0,1)$ and $N \in \mathbb{N}$ such that

$$\frac{a_{j+1}}{a_j} \leq c \qquad \forall j \geq N.$$

Then the corresponding series $a_1 + a_2 + a_3 + \ldots$ is convergent.

Exercise 2 *Show that the series with generic term* $a_j = \frac{j}{3^j}$ *is convergent, in other terms show that*

$$\sum_{j=1}^{\infty} \frac{j}{3^j} < \infty.$$

Exercise 3 *For* $f : [1, \infty) \to \mathbb{R}_+$ *decreasing, prove the following statement (the integral test) :*
The series $f(1) + f(2) + f(3) + \ldots$ is convergent if and only if $\lim_{M\to\infty} \int_1^M f(x)\,\mathrm{d}x$ is convergent, *or in a more simple form show that*

$$\sum_{j=1}^{\infty} f(j) < \infty \quad \Longleftrightarrow \quad \int_1^{\infty} f(x)\,\mathrm{d}x < \infty.$$

Exercise 4 *For any* $\varepsilon > 0$ *show that the series with generic term* $a_j = j^{-1-\varepsilon}$ *is convergent.*

Exercise 5 *Prove the following statement:*
Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of real numbers satisfying

(i) $\lim_{j\to\infty} a_j = 0$,

(ii) $a_j\, a_{j+1} \leq 0$ for all $j \in \mathbb{N}$,

(iii) $|a_{j+1}| \leq |a_j|$ for all $j \in \mathbb{N}$.

Then the corresponding series is convergent.