Exercise 1

a) $\begin{pmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{pmatrix} \sim \begin{pmatrix} 0 & -9 & 26 \\ 0 & 9 & -26 \\ 1 & 2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 \\ 0 & 1 & -26/9 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 7/9 \\ 0 & 1 & -26/9 \\ 0 & 0 & 0 \end{pmatrix}.$

b) $\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

c) $\begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & -2 & 3/2 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -7/2 & 5/2 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

d) $\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1/3 \\ 0 & 0 & 0 & -2 & 1/3 \end{pmatrix}.$

$\sim \begin{pmatrix} 1 & 2 & 0 & 0 & 4/3 \\ 0 & 0 & 1 & -2 & 1/3 \\ 0 & 0 & 0 & 1 & -1/6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 & 4/3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/6 \end{pmatrix}.$

Exercise 2

a) Consider $\begin{pmatrix} 1 & 1 & -2 & 5 \\ 2 & 3 & 4 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -2 & 5 \\ 0 & 1 & 8 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -10 & 13 \\ 0 & 1 & 8 & -8 \end{pmatrix}.$

Thus the solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfies $\begin{cases} x - 10z = 13 \\ y + 8z = -8 \end{cases}$, and

the most general solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 + 10z \\ -8 - 8z \\ z \end{pmatrix}$ with z arbitrary.

b) Consider $\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$

Thus, the general solution is

$$\begin{cases} x_1 = -x_4 \\ x_2 = x_4 \\ x_3 = -x_4 \\ x_4 \text{ arbitrary.} \end{cases}$$

Again, the last column is not necessary.

c) Consider

$$\begin{pmatrix} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus, the general solution is

$$\begin{cases} x_1 = -2x_2 \\ x_2 \text{ arbitrary} \\ x_3 = x_4 = x_5 = 0 \end{cases}.$$

Exercise 3

Consider the polynomial $P(x) = a + bx + cx^2 + dx^3$, with a, b, c, d unknowns. By assumption one has

$P(0) = a = -1$ because $(0,-1)$ is on the curve,

$P(1) = a + b + c + d = -1$ because $(1,-1)$ is on the curve,

$P(-1) = a - b + c - d = -5$ " $(-1,-5)$ " " " ",

$P(2) = a + 2b + 4c + 8d = 1$ " $(2,1)$ " " " ".

Thus, the system for the three unknowns b, c, d leads to

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & -1 & -4 \\ 2 & 4 & 8 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & -4 \\ 0 & 2 & 6 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 6 & 6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$ One gets

$b = 1$, $c = -2$ and $d = 1$.

Thus, $P(x) = -1 + x - 2x^2 + x^3$.

Exercise 4

By considering the augmented matrix one has

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & k & 4 & 6 \\ 1 & 2 & k+2 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & k-2 & 1 & 2 \\ 0 & 0 & k-1 & 2 \end{pmatrix} \quad (*).$$

1) If $k=1$, then $(*) \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ and this leads to a system with no solution since $0x+0y+0z=2$ has no solution.

2) If $k=2$, then $(*) \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

and the corresponding system is

$\begin{cases} x = -2y - 2 \\ y \text{ arbitrary} \\ z = 2 \end{cases}$. Thus, there are infinitely many solutions.

3) If $k \notin \{1, 2\}$, then $(*) \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1/k-1 & 2/k-1 \\ 0 & 0 & 1 & 2/k-1 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}$ with $*$ an expression depending on k.

Thus the corresponding system has a unique solution.