Homework 7

Exercise 1

Let $A = (a_{ij})$, $B = (b_{ij})$ be upper-triangular $n \times n$ matrices, i.e. $a_{ij} = 0$ whenever $i > j$ and $b_{ij} = 0$ whenever $i > j$.

Then, for $i > j$ one has

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \underbrace{\sum_{k=1}^{i-1} a_{ik} b_{kj}}_{= 0 \text{ because } i > k} + \underbrace{\sum_{k=i}^{n} a_{ik} b_{kj}}_{= 0 \text{ because } k \geq i > j \text{ (by assumption)}}$$

$$= 0 + 0 = 0$$

Thus $(AB)_{ij} = 0$ whenever $i > j$, meaning that AB is upper-triangular.

Exercise 2

Recall that A, B are similar if $B = UAU^{-1}$ for some invertible matrix U.

1) If A is nilpotent, there exists $m \in \mathbb{N}$ such that $A^m = 0$. Then

$$B^m = (UAU^{-1})^m = (UA\underbrace{U^{-1})(U}_{= 1_n}AU^{-1}) \dots (UAU^{-1})$$

$$= UA^mU^{-1} = U0U^{-1} = 0.$$

2) Since $A = U^{-1}BU$, if B is nilpotent, there exists $m \in \mathbb{N}$ such that $B^m = 0$, and then

$$A^m = (U^{-1}BU)^m = U^{-1}B^mU = U^{-1}0U = 0.$$

Thus one has shown that A nilpotent $\Leftrightarrow$ B nilpotent.

Exercise 3

1) The row s of A is at the row r of $I_{rs}A$, and there are 0 everywhere else.

2) The row r and s of A are interchanged in $(I_{rs} + I_{sr})A$, and there are 0 everywhere else.

3) The row r and s of A are interchanged in the matrix $(1\!\!1_m + I_{rs} + I_{sr} - I_{rr} - I_{ss})A$, the other rows remain in place.

4) c times the rows s of A is added to the row r of A in $(1\!\!1_m + c\,I_{rs})A$, the other rows remain in place.

Exercise 4

a) $\begin{cases} 2x - 3y + 4z = 0 \\ 3x + y + z = 0 \end{cases} \iff \begin{cases} 3(2x - 3y + 4z) - 2(3x + y + z) = 0 \\ 2x - 3y + 4z = 0 \end{cases}$

$\iff \begin{cases} -11y + 10z = 0 \\ 2x = 3y - 4z \end{cases} \iff \begin{cases} 10z = 11y \\ 2x = 3y - 4z \end{cases}$

One possible solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 10 \\ 11 \end{pmatrix}$, but there are an infinite number of different solutions.

b) $\begin{cases} 2x+y+4z+w=0 \\ -3x+2y-3z+w=0 \\ x+y+z=0 \end{cases} \iff \begin{cases} x=-y-z \\ w=3x-2y+3z \\ 2x+y+4z+w-(-3x+2y-3z-w)=0 \end{cases}$

$\iff \begin{cases} x=-y-z \\ w=3x-2y+3z \\ 5x-y+7z=0 \end{cases} \iff \begin{cases} x=-y-z \\ w=3(-y-z)-2y+3z \\ 5(-y-z)-y+7z=0 \end{cases}$

$\iff \begin{cases} x=-y-z \\ w=-5y \\ -6y+2z=0 \end{cases} \iff \begin{cases} z=3y \\ x=-y-3y=-4y \\ w=-5y \end{cases}$

A possible solution is $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 3 \\ -5 \end{pmatrix}$, but the most general solution is $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -4y \\ y \\ 3y \\ -5y \end{pmatrix}$ with y arbitrary (but different from 0, otherwise the solution is trivial).

c) $\begin{cases} -2x+3y+z+4w=0 \\ x+y+2z+3w=0 \\ 2x+y+z-2w=0 \end{cases}$

$\iff \begin{cases} -2x+3y+z+4w+(2x+y+z-2w)=0 \\ -2x+3y+z+4w+2(x+y+z+3w)=0 \\ x=-y-2z-3w \end{cases}$

$\iff \begin{cases} 4y+2z+2w=0 \\ 5y+5z+10w=0 \\ x=-y-2z-3w \end{cases} \iff \begin{cases} 5(4y+2z+2w)-4(5y+5z+10w)=0 \\ y=-z-2w \\ x=-(-z-2w)-2z-3w \end{cases}$

$\iff \begin{cases} -10z-30w=0 \\ y=-z-2w \\ x=-z-w \end{cases} \iff \begin{cases} z=-3w \\ y=-(-3w)-2w=w \\ x=+3w-w=2w \end{cases}$

One solution is $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \\ 1 \end{pmatrix}$ but the general solution is $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2w \\ w \\ -3w \\ w \end{pmatrix}$ for any $w \in \mathbb{R}^*$.