Homework 5

Exercise 1

$$\mathcal{A}+\mathcal{B} = \begin{pmatrix} 0 & 7 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathcal{A}-2\mathcal{B} = \begin{pmatrix} 3 & -8 & 7 \\ -3 & -2 & 4 \end{pmatrix}$$

$${}^t\mathcal{A} = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{pmatrix}.$$

Exercise 2

Set $\mathcal{A} = (a_{ij})$, $\mathcal{B} = (b_{ij})$, $C = (c_{ij})$.

i.a) $$[\mathcal{A}(\mathcal{B}+C)]_{ij} = \sum_{k=1}^{n} a_{ik}(\mathcal{B}+C)_{kj} = \sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj})$$
$$= \sum_{k=1}^{n}(a_{ik}b_{kj} + a_{ik}c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$
$$= (\mathcal{A}\mathcal{B})_{ij} + (\mathcal{A}C)_{ij}.$$

i.b) $$((\lambda\mathcal{A})\mathcal{B})_{ij} = \sum_{k=1}^{n}(\lambda\mathcal{A})_{ik}b_{kj} = \sum_{k=1}^{n}\lambda a_{ik}b_{kj} = \lambda(\mathcal{A}\mathcal{B})_{ij}$$
$$= \sum_{k=1}^{n} a_{ik}(\lambda b_{kj}) = \sum_{k=1}^{n} a_{ik}(\lambda\mathcal{B})_{kj} = (\mathcal{A}(\lambda\mathcal{B}))_{ij}.$$

ii) $$((\mathcal{A}\mathcal{B})C)_{ij} = \sum_{k=1}^{p}(\mathcal{A}\mathcal{B})_{ik}c_{kj} = \sum_{k=1}^{p}\left(\sum_{\ell=1}^{n} a_{i\ell}b_{\ell k}\right)c_{kj}$$
$$= \sum_{k=1}^{p}\sum_{\ell=1}^{n} a_{i\ell}b_{\ell k}c_{kj} \underset{\uparrow \text{ permutation of sums}}{=} \sum_{\ell=1}^{n}\sum_{k=1}^{p} a_{i\ell}b_{\ell k}c_{kj}$$
$$= \sum_{\ell=1}^{n} \underset{\uparrow \text{ independent of } k}{a_{i\ell}} \sum_{k=1}^{p} b_{\ell k}c_{kj} = \sum_{\ell=1}^{n} a_{i\ell}(\mathcal{B}C)_{\ell j} = (\mathcal{A}(\mathcal{B}C))_{ij}.$$

iii) $$({}^t(\mathcal{A}\mathcal{B}))_{ij} = (\mathcal{A}\mathcal{B})_{ji} = \sum_{k=1}^{n} a_{jk}b_{ki} = \sum_{k=1}^{n} b_{ki}a_{jk}$$
$$= \sum_{k=1}^{n} {}^tb_{ik}\,{}^ta_{kj} = \sum_{k=1}^{n}({}^t\mathcal{B})_{ik}({}^t\mathcal{A})_{kj} = ({}^t\mathcal{B}\,{}^t\mathcal{A})_{ij}.$$

$$\Longrightarrow \quad {}^t(\mathcal{A}\mathcal{B}) = {}^t\mathcal{B}\,{}^t\mathcal{A}.$$

Exercise 3

Set $A = (a_{ij})$

Recall that $(\mathbb{1}_n)_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$. Then

$$(\mathbb{1}_n A)_{ij} = \sum_{k=1}^{n} (\mathbb{1}_n)_{ik}\, a_{kj} = a_{ij} = (A)_{ij}.$$

Similarly $(A\,\mathbb{1}_m)_{ij} = \sum_{k=1}^{m} a_{ik} (\mathbb{1}_m)_{kj} = a_{ij} = (A)_{ij}$.

Exercise 4

Set $A = (a_{ij})$

Let us show that $A + {}^tA$ is symmetric. Indeed one has:

$(A + {}^tA)_{ij} = a_{ij} + {}^ta_{ij}$, but $({}^t(A + {}^tA))_{ij} = (A + {}^tA)_{ji}$

$= a_{ji} + {}^ta_{ji} = {}^ta_{ij} + a_{ij} = a_{ij} + {}^ta_{ij}$. Since both expression are the same, one gets that $A + {}^tA$ is symmetric.

On the other hand: $(A - {}^tA)_{ij} = a_{ij} - {}^ta_{ij} = {}^ta_{ji} - a_{ji}$

$= -(a_{ji} - {}^ta_{ji}) = -(A - {}^tA)_{ji} = -({}^t(A - {}^tA))_{ij}$, which means that ${}^t(A - {}^tA) = -(A - {}^tA)$, or that $A - {}^tA$ is skew-symmetric.

Note that since $A = \dfrac{A + {}^tA}{2} + \dfrac{A - {}^tA}{2}$, any matrix is the sum of a symmetric matrix and a skew-symmetric matrix.

Exercise 5

1) Observe that $(\mathbb{1}_n - A)(\mathbb{1}_n + A) = \mathbb{1}_n - A^2 = \mathbb{1}_n$ since $A^2 = 0$, and that $(\mathbb{1}_n + A)(\mathbb{1}_n - A) = \mathbb{1}_n$ for the same reason. Thus, $\mathbb{1}_n - A$ is invertible, with inverse $\mathbb{1}_n + A$.

2) One has $A^2 + 2A + \mathbb{1} = 0 \iff -A^2 - 2A = \mathbb{1}_n$

$\iff A(-\mathbb{1}_n - A) = \mathbb{1}_n$, and also $(-\mathbb{1}_n - A)A = \mathbb{1}_n$.

Thus $(-\mathbb{1}_n - A)$ is the inverse of A, which is therefore invertible.