Exercise 1

i) One has

$$(X+X')\times Y = \begin{pmatrix} (x_2+x_2')y_3 - (x_3+x_3')y_2 \\ (x_3+x_3')y_1 - (x_1+x_1')y_3 \\ (x_1+x_1')y_2 - (x_2+x_2')y_1 \end{pmatrix}$$

$$= \begin{pmatrix} x_2y_3 - x_3y_2 + x_2'y_3 - x_3'y_2 \\ x_3y_1 - x_1y_3 + x_3'y_1 - x_1'y_3 \\ x_1y_2 - x_2y_1 + x_1'y_2 - x_2'y_1 \end{pmatrix} = X\times Y + X'\times Y.$$

Also $Y\times X = \begin{pmatrix} y_2x_3 - y_3x_2 \\ y_3x_1 - y_1x_3 \\ y_1x_2 - y_2x_1 \end{pmatrix} = \begin{pmatrix} -(x_2y_3 - x_3y_2) \\ -(x_3y_1 - x_1y_3) \\ -(x_1y_2 - x_2y_1) \end{pmatrix} = -X\times Y.$

ii) Let us compute $X\cdot(X\times Y) = (x_1,x_2,x_3)\cdot\begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$

$$= x_1x_2y_3 - x_1x_3y_2 + x_2x_3y_1 - x_2x_1y_3 + x_3x_1y_2 - x_3x_2y_1 = 0$$

A similar computation holds for $Y\cdot(X\times Y)$.
Since the scalar product is 0, it means that X and $X\times Y$, or Y and $X\times Y$, are perpendicular.

iii) $\|X\times Y\|^2 = (x_2y_3 - x_3y_2)^2 + (x_3y_1 - x_1y_3)^2 + (x_1y_2 - x_2y_1)^2$

$$\begin{aligned}
&= x_2^2y_3^2 + x_3^2y_2^2 - 2x_2x_3y_2y_3 \\
&\quad + x_3^2y_1^2 + x_1^2y_3^2 - 2x_1x_3y_1y_3 \\
&\quad + x_1^2y_2^2 + x_2^2y_1^2 - 2x_1x_2y_1y_2 \\
&= x_1^2(y_1^2+y_2^2+y_3^2) - x_1^2y_1^2 - 2x_2x_3y_2y_3 \\
&\quad + x_2^2(y_1^2+y_2^2+y_3^2) - x_2^2y_2^2 - 2x_1x_3y_1y_3 \\
&\quad + x_3^2(y_1^2+y_2^2+y_3^2) - x_3^2y_3^2 - 2x_1x_2y_1y_2 \\
&= (x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2) - (x_1y_1 + x_2y_2 - x_3y_3)^2 \\
&= \|X\|^2\,\|Y\|^2 - (X\cdot Y)^2.
\end{aligned}$$

iv) Since $X \cdot Y = \| X \| \| Y \| \cos \theta$, it follows that

$$(X \cdot Y)^2 = \|X\|^2 \|Y\|^2 \cos^2 \theta \quad , \text{ and}$$

$$\| X \|^2 \| Y \|^2 - (X \cdot Y)^2 = \| X \|^2 \| Y \|^2 (1 - \cos^2 \theta)$$

$$= \| X \|^2 \| Y \| \sin^2 \theta ,$$

and by taking the square root ;

$$\| X \times Y \| = \| X \| \| Y \| \sqrt{\sin^2 \theta}$$

$$= \| X \| \| Y \| \; | \sin \theta | .$$

v) More generally :

$$\sum_{1 \leq i < j \leq n} (x_i y_j - x_j y_i)^2 = \sum_{1 \leq i < j \leq n} (x_i^2 y_j^2 + x_j^2 y_i^2 - 2 x_i x_j y_i y_j)$$

$$= \sum_{1 \leq i < j \leq n} (x_i^2 y_j^2 + x_j^2 y_i^2) - 2 \sum_{1 \leq i < j \leq n} x_i x_j y_i y_j$$

$$= \sum_{i,j=1}^{n} x_i^2 y_j^2 - \sum_{i=1}^{n} x_i^2 y_i^2 - 2 \sum_{1 \leq i < j \leq n} x_i x_j y_i y_j$$

$$= \sum_{i=1}^{n} x_i^2 \left(\sum_{j=1}^{n} y_j^2 \right) - \left[\sum_{i=1}^{n} x_i^2 y_i^2 + 2 \sum_{1 \leq i < j \leq n} x_i x_j y_i y_j \right]$$

$$= \| X \|^2 \| Y \|^2 - (X \cdot Y)^2 .$$