Homework 2

Exercise 1

Let $A = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

1) $\|A\|^2 = A \cdot A = \sum_{j=1}^{n} a_j^2 \geq 0$, and $\sum_{j=1}^{n} a_j^2 = 0$ if and only if $a_j = 0 \ \forall j = 1, \dots, n$, i.e. if and only if $A = (0, 0, \dots 0)$.

2) $\|\lambda A\|^2 = \|(\lambda a_1, \lambda a_2, \dots, \lambda a_n)\|^2 =$
$$= \sum_{j=1}^{n} (\lambda a_j)^2 = \lambda^2 \sum_{j=1}^{n} a_j^2 = \lambda^2 \|A\|^2.$$
Thus, $\|\lambda A\| = |\lambda| \, \|A\|$.

3) By choosing $\lambda = -1$, one directly gets the result from point 2).

Exercise 2

For any $A, B \in \mathbb{R}^n$ one has
$$\|A + B\|^2 = A^2 + 2 A \cdot B + B^2 = \|A\|^2 + 2 A \cdot B + \|B\|^2,$$
which directly implies the statement.

Exercise 3

1) $\|A\| = \sqrt{5}$, $\|B\| = \sqrt{2}$, $\dfrac{A \cdot B}{\|B\|^2} B = \left(\frac{3}{2}, -\frac{3}{2} \right)$.

2) $\|A\| = \sqrt{10}$, $\|B\| = 4$, $\dfrac{A \cdot B}{\|B\|^2} B = (0, 3)$.

3) $\|A\| = \sqrt{30}$, $\|B\| = \sqrt{3}$, $\dfrac{A \cdot B}{\|B\|^2} B = \left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3} \right)$.

Exercise 4

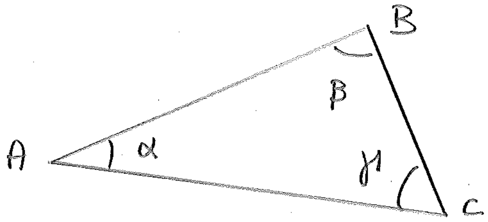
1) $\|A\| = \sqrt{5}$, $\|B\| = \sqrt{34}$

$$\cos\theta = \frac{A \cdot B}{\|A\|\,\|B\|} = \frac{11}{\sqrt{170}}.$$

2) $\|A\| = \sqrt{14}$, $\|B\| = \sqrt{35}$

$$\cos\theta = \frac{A \cdot B}{\|A\|\,\|B\|} = \frac{10}{\sqrt{2\cdot 7}\cdot\sqrt{7\cdot 5}} = \frac{10}{7\sqrt{10}} = \frac{\sqrt{10}}{7}.$$

Exercise 5



$B - A = (-1, -2, -6) \Rightarrow \|\overrightarrow{AB}\| = \|B - A\| = \sqrt{41}$

$C - A = (1, -3, -5) \Rightarrow \|\overrightarrow{AC}\| = \|C - A\| = \sqrt{35}$

$C - B = (2, -1, 1) \Rightarrow \|\overrightarrow{BC}\| = \|C - B\| = \sqrt{6}$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = \overrightarrow{O(B-A)} \cdot \overrightarrow{O(C-A)} = (B-A)\cdot(C-A) = 35$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = \overrightarrow{O(A-B)} \cdot \overrightarrow{O(C-B)} = 6$

$\overrightarrow{CA} \cdot \overrightarrow{CB} = \overrightarrow{O(A-C)} \cdot \overrightarrow{O(B-C)} = 0.$

Thus, $\cos\alpha = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\|\overrightarrow{AB}\|\,\|\overrightarrow{AC}\|} = \dfrac{35}{\sqrt{35}\cdot\sqrt{41}} = \sqrt{\dfrac{35}{41}}$

$$\cos\beta = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\|\overrightarrow{BA}\|\,\|\overrightarrow{BC}\|} = \frac{6}{\sqrt{41}\,\sqrt{6}} = \sqrt{\frac{6}{41}}$$

and $\cos\gamma = 0 \Rightarrow \gamma = 90^\circ$.

Exercise 6

By assumption, $A_j \cdot A_k = 0$ if $j \neq k$ but $A_j \cdot A_j \neq 0$ $\forall j$. Thus, consider $\sum_{j=1}^{n} c_j A_j = \mathbf{0}$ and take the scalar product of this equality with A_k. One gets

$$\left(\sum_{j=1}^{n} c_j A_j \right) \cdot A_k = \mathbf{0} \cdot A_k$$

$$\Longleftrightarrow \sum_{j=1}^{n} c_j A_j \cdot A_k = 0$$

$$\Longleftrightarrow c_k \, A_k \cdot A_k = 0 \qquad \Longrightarrow c_k = 0 .$$

Since k is arbitrary, one gets the result, i.e. all c_k must be equal to 0.

Exercise 7

1) $B - A = (-5, -2, 3)$ and $L = \{ (1,3,-1) + t(-5,-2,3) \mid t \in \mathbb{R} \}$
$= \{ (1-5t, 3-2t, -1+3t) \mid t \in \mathbb{R} \}$.

2) $B - A = (-1, -1, 4)$ and

$$L = \{ (-1,5,3) + t(-1,-1,4) \mid t \in \mathbb{R} \}$$

$$= \{ (-1-t, 5-t, 3+4t) \mid t \in \mathbb{R} \}.$$

Exercise 8

Solution 1: midpoint $= \frac{P+Q}{2} = \frac{1}{2}(P+Q)$.

Solution 2: the midpoint is obtained for $t = \frac{1}{2}$ in $L = \{ P + t(Q-P) \mid t \in [0,1] \}$, i.e. $P + \frac{1}{2}(Q-P) = \frac{1}{2}(P+Q)$.

One checks that $\| P - \frac{P+Q}{2} \| = \| Q - \frac{P+Q}{2} \|$.