Exercise 1

1) There are n elements on a diagonal matrix, and they can be freely chosen. $\Rightarrow$ The dimension of the vector space of all diagonal matrices is $\underline{n}$.

2) For symmetric matrices, $a_{ij} = a_{ji}$. Thus one can choose freely the n elements a_{jj} on the diagonal, and choose freely the $\frac{n^2-n}{2}$ terms located above the diagonal. Together, it makes $n + \frac{n^2-n}{2} = \frac{n^2+n}{2} = \frac{1}{2}n(n+1)$ coefficients which can be freely choosen. Thus, the dimension for the vector space of all symmetric matrices is $\underline{\frac{1}{2}n(n+1)}$.

3) For skew-symmetric matrices, $a_{ij} = -a_{ji}$. In particular, $a_{jj} = 0$, i.e. the diagonal is filled with 0 only. One can thus freely choose only the $\frac{n^2-n}{2}$ terms which are located above the diagonal. Thus, the dimension of the vector space of all skew-symmetric matrices is $\underline{\frac{1}{2}n(n-1)}$.

Exercise 2

1) No, this set is not a vector space because $\mathbb{1}_n$ is invertible, but $\mathbb{1}_n + (-\mathbb{1}_n) = \mathbb{O}$ is not invertible. A vector space is stable for addition, and this is not true in this example.

2) No, this set is not a vector space because $\begin{pmatrix} 0 & 0 & \cdots & 1 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 1 & & & \end{pmatrix}$ are nilpotent, but $\begin{pmatrix} 0 & 0 & \cdots & 1 \\ \vdots & & 0 & \\ 1 & & & \end{pmatrix}$ is not, as easily checked.

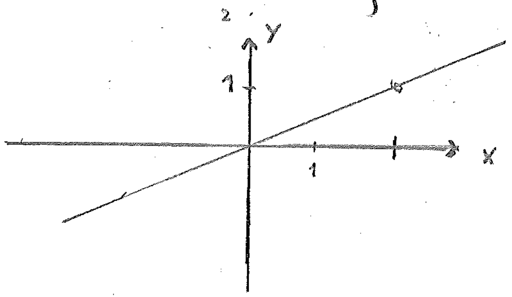
3) No, this is not a vector space, for the same reason as in 1) because $\mathbb{1}_n$ is upper-triangular and invertible.

⚠ On this page, all vectors are written in row instead of in column.

Exercise 3

Let $L := \{(x,y) \in \mathbb{R}^2 \mid x = 2\}$ and let us compute $F(L)$

with
$$F(L) = \{(u,v) \in \mathbb{R}^2 \mid (u,v) = F((x,y)) \text{ for some } (x,y) \in L\}$$
$$= \{(u,v) \in \mathbb{R}^2 \mid (u,v) = (xy, y) \text{ for } (x,y) \in L\}$$
$$= \{(u,v) \in \mathbb{R}^2 \mid u = xy \text{ and } v = y \text{ for } x = 2, y \text{ arbitrary}\}$$
$$= \{(u,v) \in \mathbb{R}^2 \mid u = 2y,\ v = y \text{ for } y \text{ arbitrary}\}$$
$$= \{(2y, y) \in \mathbb{R}^2 \mid y \text{ any element of } \mathbb{R}\}$$

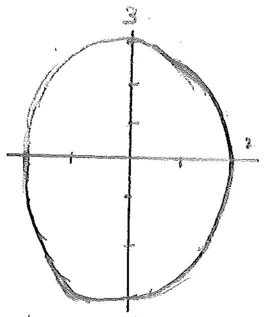


F(L)

Exercise 4

Let $S := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Then

$$F(S) = \{(u,v) \in \mathbb{R}^2 \mid (u,v) = F((x,y)) \text{ with } (x,y) \in S\}$$
$$= \{(u,v) \in \mathbb{R}^2 \mid (u,v) = (2x, 3y) \text{ with } (x,y) \in S\}$$
$$= \{(u,v) \in \mathbb{R}^2 \mid u = 2x,\ v = 3y \text{ and } x^2 + y^2 = 1\}$$
$$= \{(u,v) \in \mathbb{R}^2 \mid x = \tfrac{u}{2},\ y = \tfrac{v}{3} \text{ and } (\tfrac{u}{2})^2 + (\tfrac{v}{3})^2 = 1\}$$
$$= \{(u,v) \in \mathbb{R}^2 \mid (\tfrac{u}{2})^2 + (\tfrac{v}{3})^2 = 1\}$$



F(S)

Exercise 5

Since for any $Y \in V$, one has $Y = \lambda_1 X_1 + \lambda_2 X_2 + \dots \lambda_n X_n$ and since F is linear, it follows that

$$F(Y) = F(\lambda_1 X_1 + \dots + \lambda_n X_n) = F(\lambda_1 X_1) + \dots + F(\lambda_n X_n)$$
$$= \lambda_1 F(X_1) + \dots + \lambda_n F(X_n).$$

Thus, $F(Y)$ is known once $F(X_1), \dots, F(X_n)$ are known, and these n vector $F(X_1), \dots, F(X_n)$ entirely defines the linear map F.

If F is not linear this is not true. For example, if $F(x) = x^2$ for any $x \in \mathbb{R}$, Then $1 \in \mathbb{R}$ is a basis for $\mathbb{R}$ (the vector space $\mathbb{R} \equiv \mathbb{R}^1$) and $F(1) = 1$ but we don't have $4 = F(2) = F(2 \cdot 1) \neq 2\,F(1) = 2 \cdot 1 = 2$.

Exercise 6

Let V, W be vector spaces over the same field $\mathbb{F}$, and let $T : V \to W$ be linear. Set $\mathrm{Ker}(T) = \{X \in V \mid T(X) = 0\}$.

$\mathrm{Ker}(T)$ is a subspace of V since:

1) if $X, Y \in \mathrm{Ker}(T)$, then $T(X+Y) \overset{\text{linearity}}{=} T(X) + T(Y) = 0 + 0 = 0$,

   and thus $X + Y \in \mathrm{Ker}(T)$,

2) for any $\lambda \in \mathbb{F}$ and $X \in \mathrm{Ker}(T)$, one has

   $T(\lambda X) \overset{\text{linearity}}{=} \lambda T(X) = \lambda 0 = 0$,

   and thus $\lambda X \in \mathrm{Ker}(T)$.

The 2 conditions for being a subspace are thus checked.