## Exercise 1

1) i) If $X \in \lambda S$, then there exists $x \in S$ such that $X = \lambda x$. Similarly, if $Y \in S$, there exists $y \in S$ such that $Y = \lambda y$. Then, for any $t \in [0,1]$ one has

$$(1-t)X + tY = (1-t)\lambda x + t\lambda y = \lambda[(1-t)x + ty] \in \lambda S$$

since $(1-t)x + ty \in S$ because $S$ is convex.

ii) Let $X+Y$ and $X'+Y$ belong to $S+Y$, with $X, X' \in S$. Then for any $t \in [0,1]$ one has

$$(1-t)(X+Y) + t(X'+Y) = (1-t)X + tX' + (1-t+t)Y$$

$$= (1-t)X + tX' + Y \in S+Y$$

since $(1-t)X + tX' \in S$ because $S$ is convex.

In other words, a convex set remaim convex under a rescalling by $\lambda$, or under a shift by $Y$.

2) Let $S_1, S_2$ be convex sets, and let $X, Y \in S_1 \cap S_2$. Since $S_1$ is convex, for any $t \in [0,1]$ one has $(1-t)X + tY \in S_1$, and since $S_2$ is convex one has $(1-t)X + tY \in S_2$. Thus, $(1-t)X + tY$ belongs to $S_1$ and to $S_2$, and thus belongs to $S_1 \cap S_2$.

This means that $S_1 \cap S_2$ is convex.

## Exercise 2

1) Let us show first that the three vectors are linearly independent. Indeed one has

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \lambda_1 + \lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 0 \\ \lambda_3 = 0 \end{cases} \Leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0 .$$

2) Let us show that the three vectors generate $\mathbb{R}^3$.
Indeed for any $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ one has

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \lambda_1 + \lambda_3 = x \\ \lambda_2 + \lambda_3 = y \\ \lambda_3 = z \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = x - z \\ \lambda_2 = y - z \\ \lambda_3 = z \end{cases} .$$

Thus, any element of $\mathbb{R}^3$ can be obtained as a linear combination of the three vectors.

## Exercise 4

i) One has $\lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$ .

The coordinates of $Y$ in the basis $\{X_1, X_2\}$ are $(1, -1)$.

ii) One has $\lambda_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Leftrightarrow \begin{cases} \lambda_1 + \lambda_2 = 2 \\ -\lambda_1 + \lambda_2 = 1 \end{cases}$

$\Leftrightarrow \begin{cases} \lambda_1 = 1/2 \\ \lambda_2 = 3/2 \end{cases}$ . The coordinates of $Y$ in the basis $\{X_1, X_2\}$ are $(1/2, 3/2)$.

## Exercise 3

1) Assume $(ad-bc) \neq 0$ : One has

$$\lambda_1 \begin{pmatrix} a \\ b \end{pmatrix} + \lambda_2 \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} \lambda_1 a + \lambda_2 c = 0 \\ \lambda_1 b + \lambda_2 d = 0 \end{cases}$$

$$\iff \begin{cases} \lambda_1 (ad-bc) + \lambda_2 (cd-cd) = 0 \\ \lambda_1 (ab-ab) + \lambda_2 (cb-ad) = 0 \end{cases}$$

$$\iff \begin{cases} \lambda_1 (ad-bc) = 0 \\ \lambda_2 (ad-bc) = 0 \end{cases} \iff \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}$$

because of the assumption.

and thus $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ are linearly independent.

2) Assume $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}$ linearly independent :

Clearly $\begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ because $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not linearly independent of any other vector.

Similarly, one can not have $b = d = 0$ because $\begin{pmatrix} a \\ 0 \end{pmatrix}$ and $\begin{pmatrix} c \\ 0 \end{pmatrix}$ are not linearly independent. Thus, by considering

$d \begin{pmatrix} a \\ b \end{pmatrix} - b \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ad-bc \\ 0 \end{pmatrix}$, this must be different from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ since $(b,d) \neq (0,0)$ and $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}$ are linearly independent. One concludes that $(ad-bc) \neq 0$.

## Exercise 5

Consider $\lambda_1 X_1 + \dots + \lambda_n X_n = 0$ with $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.
By taking successively the scalar product of both sides of this equality with $X_j$, one gets

$$(\lambda_1 X_1 + \dots + \lambda_n X_n) \cdot X_j = 0 \cdot X_j$$

$$\iff \lambda_j X_j \cdot X_j = 0 \iff \lambda_j = 0 \text{ because } X_j \cdot X_j \neq 0.$$

Thus, $\lambda_j = 0 \ \forall j \in \{1, \dots, n\}$ and $X_1, \dots, X_n$ are linearly independent.

<u>Exercise 6</u>, see also ex 1 Homework 11.

i) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ are 2 linearly independent elements of $S_1$, and one can not find a third one. Thus, $\dim S_1 = 2$.

ii) $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is an element of $S_2$, and all other elements of $S_2$ are of the form $\lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Thus $\dim S_2 = 1$

iii) $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$ are 2 linearly independent elements of $S_3$, and one can not find a third one.
Thus $\dim S_3 = 2$.

<u>Exercise 7</u>, see also ex 1 Homework 9

The rank is easily deduced once the matrix is in the standard form.

a) rank 2

b) rank 3

c) rank 2

d) rank 3