Nagoya University, G30 program — Autumn 2014

Mathematics Tutorial I (linear algebra part) — Instructor : Serge Richard

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Homework 13

Exercise 1 *Determine the dimension of*

1. *The vector space of all $n \times n$ diagonal matrices,*
2. *The vector space of all $n \times n$ symmetric matrices,*
3. *The vector space of all $n \times n$ skew-symmetric matrices.*

Exercise 2 *Are the following sets vector spaces (justify your answers):*

1. *The set of invertible $n \times n$ matrices,*
2. *The set of $n \times n$ nilpotent matrices,*
3. *The set of $n \times n$ upper-triangular and invertible matrices.*

Exercise 3 *Let $\mathrm{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by $\mathrm{F}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} xy \\ y \end{smallmatrix}\right)$ for any $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^2$. Describe the image by F of the line $\{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^2 \mid x = 2\}$.*

Exercise 4 *Let $\mathrm{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by $\mathrm{F}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2x \\ 3y \end{smallmatrix}\right)$ for any $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^2$. Describe the image by F of the points lying on the unit circle centered at $\mathbf{0}$, i.e. $\left\{ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \right\}$.*

Exercise 5 *Let V be a vector space of dimension n, and let $\{X_1, \dots, X_n\}$ be a basis for V. Let F be a linear map from V into itself. Show that F is uniquely defined if one knows $\mathrm{F}(X_j)$ for $j \in \{1, \dots, n\}$. Is it also true if F is an arbitrary map from V into itself ?*

Exercise 6 *Let V, W be vector spaces over the same field, and let $\mathrm{T} : V \to W$ be a linear map. Show that the following set is a subspace of V:*

$$\{X \in V \mid \mathrm{T}(X) = \mathbf{0}\}.$$

This subspace is called the kernel of T.

Exercise 7 *A* doubly stochastic matrix *is a $n \times n$ matrix $\mathcal{A} = (a_{jk})$ such that $a_{jk} \in [0, 1]$ and such that the sum of the elements of each line is equal to 1, as well as the sum of the elements of each column.*

(i) Show that the product of two doubly stochastic matrices is still a doubly stochastic matrix,

(ii) Show that the set of all doubly stochastic matrices is a convex set.