University of Nagoya, G30 program Autumn 2013

Mathematics Tutorial I (linear algebra part) Instructor : Serge Richard

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Homework n° 5

Exercise 1.

1. Find some $\mathcal{A} \in M_2(\mathbb{R})$ such that $\mathcal{A}^2 = -\mathbb{I}_2$.
2. Determine all $\mathcal{A} \in M_2(\mathbb{R})$ such that $\mathcal{A}^2 = 0$.

Exercise 2. Let a, b be real numbers and let

$$\mathcal{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathcal{B} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

What is $\mathcal{AB}$? Compute $\mathcal{A}^2$ and $\mathcal{A}^3$. What is $\mathcal{A}^m$ for an arbitrary integer m, and how to prove it ?

Warning : From now on, points in $\mathbb{R}^n$ will always be denoted by column vector, *i.e.*

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

The set $\mathbb{R}^n$ will then be identified with the set $M_{n1}(\mathbb{R})$. With this convention, the row vector $(x_1, x_2, \ldots, x_n)$ corresponds to ${}^T X \in M_{1n}(\mathbb{R})$, and the scalar product of two column vectors $X \cdot Y$ corresponds to product of the matrices $({}^T X)Y \in M_{11}(\mathbb{R})$.

Exercise 3. One says that a matrix $\mathcal{A} \in M_n(\mathbb{R})$ is orthogonal if ${}^T\mathcal{A} = \mathcal{A}^{-1}$, or equivalently if ${}^T\mathcal{A}\mathcal{A} = \mathbb{I}_n$. Show that if $\mathcal{A} \in M_n(\mathbb{R})$ is an orthogonal matrix, then

1. $\|\mathcal{A}X\| = \|X\|$ for any $X \in \mathbb{R}^n$,
2. $(\mathcal{A}X) \cdot (\mathcal{A}Y) = X \cdot Y$ for any $X, Y \in \mathbb{R}^n$.

In other words, orthogonal matrices preserve lengths and angles between vectors of $\mathbb{R}^n$.

Exercise 4. A special type of 2×2 matrices represent rotations in the plane. For arbitrary $\theta \in \mathbb{R}$, consider the matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

1. Show that for arbitrary θ_1, θ_2 one has $R(\theta_1)\,R(\theta_2) = R(\theta_2)\,R(\theta_1)$,
2. Show that for arbitrary θ_1, θ_2 one has $R(\theta_1)\,R(\theta_2) = R(\theta_1 + \theta_2)$,
3. Show that for any θ, the matrix $R(\theta)$ has an inverse and write down this inverse.