Nagoya University, G30 program Autumn 2013

Mathematics Tutorial I (linear algebra part) Instructor : Serge Richard

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Homework n° 13

Exercise 1. Determine which of the following maps are linear :

a) $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (x, z)$,

b) $F : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $F(X) = -X$,

c) $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(X) = X + (0, -1, 0)$,

d) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (2x, y - x)$,

e) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (y, x)$,

f) $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, y) = xy$.

Exercise 2. Determine the kernel and the range of the maps defined in the previous exercise.

Exercise 3. 1) Consider the subset of $\mathbb{R}^n$ consisting of all vectors $(x_1, \dots, x_n)$ such that $x_1 + x_2 + \cdots + x_n = 0$. Is it a subspace of $\mathbb{R}^n$? If so, what is its dimension ?

Exercise 4. Let $P : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be the map defined for any $\mathcal{A} \in M_n(\mathbb{R})$ by

$$P(\mathcal{A}) = \frac{1}{2}(\mathcal{A} + {}^T\!\mathcal{A}).$$

1. Show that P is a linear map.
2. Show that the kernel of P consists in the vector space of all skew-symmetric matrices.
3. Show that the range of P consists in the vector space of all symmetric matrices.
4. What is the dimension of the vector space of all symmetric matrices, and the dimension of the vector space of all skew-symmetric matrices ?

Exercise 5. Let $C^\infty(\mathbb{R})$ be the real functions on $\mathbb{R}$ which admit derivatives of all order. Let $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be the map which associates to any $f \in C^\infty(\mathbb{R})$ its derivative, *i.e.* $Df = f'$.

1. Is D a linear map ?
2. What is the kernel of D ?
3. What is the kernel of D^n, for any $n \in \mathbb{N}$, and what is the dimension of this vector space ?