Calculus I

Midterm Exam

Name: ______________________________

To receive full credit you need to show your work. That is, full credit is only given if you explain your thought process clearly and in detail.

This midterm exam contributes 30% to your final grade in Calculus I.

PROBLEM	POINTS
1	(5)
2	(5)
3	(5)
4	(5)
5	(5)
6	(6)
7	(6)
8	(6)
9	(7)
TOTAL	(50)

1. (5 points) Find the domains and draw the graphs of the following functions.

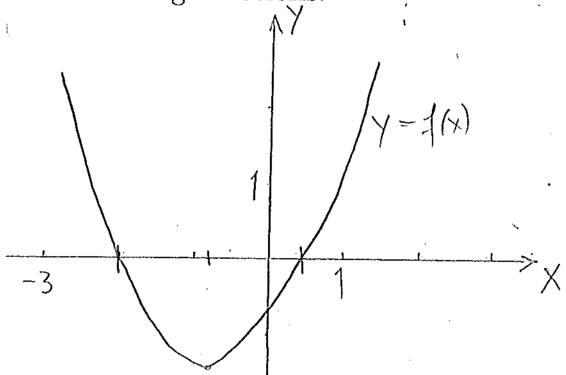
(a) $f(x) = x^2 + \frac{3}{2}x - 1$, $f(x)$ is a polynomial, so domain of f is all real numbers

$$0 = \left(x+\frac{3}{4}\right)^2 - \frac{9}{16} - 1 = \left(x+\frac{3}{4}\right)^2 - \frac{25}{16}$$

$$x+\frac{3}{4} = \pm\frac{5}{4} \Rightarrow x = -\frac{3}{4} \pm \frac{5}{4} = \begin{cases} \frac{1}{2} \\ -2 \end{cases}$$

$\Rightarrow f(x)=0$ when $x=\frac{1}{2}$ or $x=-2$,

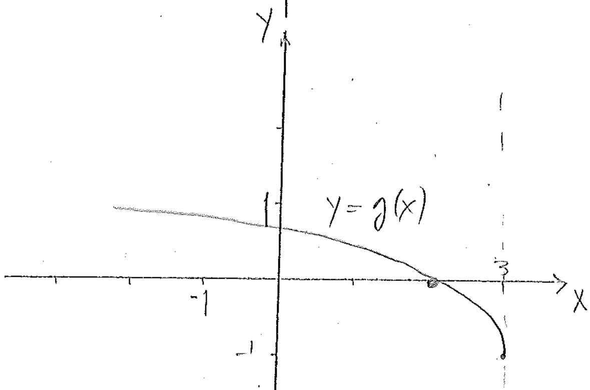
min. value of f is $-\frac{25}{16}$ (when $x=-\frac{3}{4}$)



(b) $g(x) = \sqrt{-x+3} - 1$

$\sqrt{x}$ is only defined for $x \geq 0$

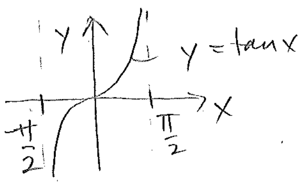
$\Rightarrow$ domain of g is $(-\infty, 3]$

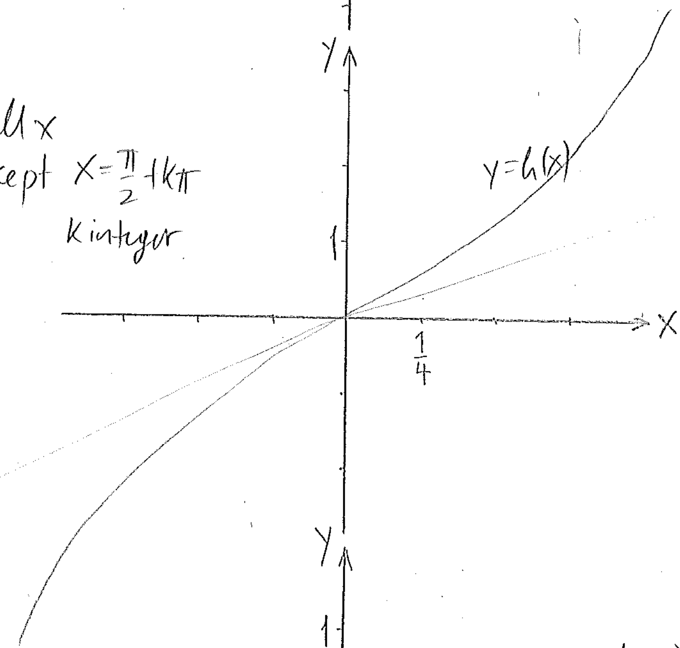


(c) $h(x) = \tan(x/\pi)$; domain of $\tan x$: all x except $x=\frac{\pi}{2}+k\pi$, k integer

$\Rightarrow h(x)$ is defined for all x except $\frac{x}{\pi} = \frac{\pi}{2} + k\pi$, k integer

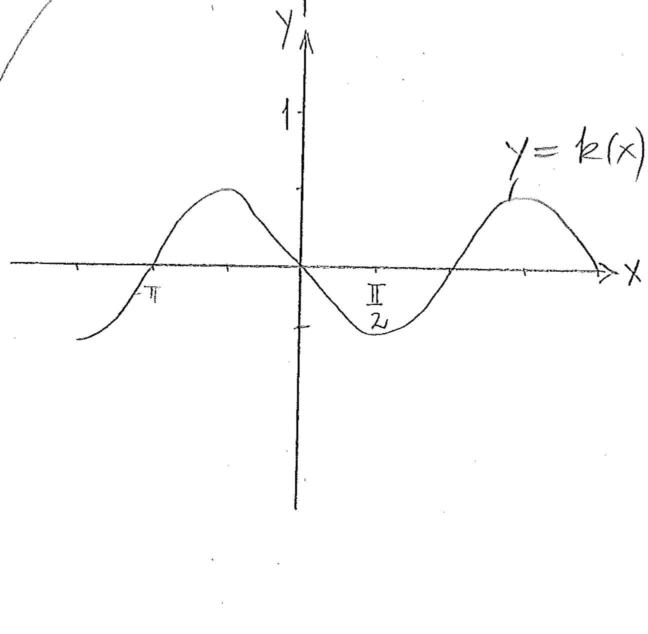
$\Rightarrow$ domain: all x except $\frac{\pi^2}{2} + k\pi^2$ for all integer k





(d) $k(x) = -\frac{1}{2}\sin(x)$

domain is all real numbers since the sin-function is defined everywhere.



2. (5 points)

(a) Find $\lim_{x \to 0} \frac{\cos x}{x}$.

$\lim\limits_{x \to 0^+} \frac{\cos x}{x} = +\infty$, $\lim\limits_{x \to 0^-} \frac{\cos x}{x} = -\infty$ $\Rightarrow$ $\lim\limits_{x \to 0} \frac{\cos x}{x}$ DNE.

since $|\cos x| \le 1$
and $|x|^{-1}$ becomes arbitrarily large

(b) Find $\lim_{x \to 5^-} \frac{2x-10}{|x-5|}$. $= \lim\limits_{x \to 5^-} \frac{2x-10}{5-x} = -2 \lim\limits_{x \to 5^-} \frac{x-5}{x-5} = 1 \cdot (-2)$

$x < 5 \Rightarrow x-5 < 0$

(c) Find $\lim_{x \to -\infty} \frac{x^3+x^2-1}{7-x^3} = \lim\limits_{x \to -\infty} \frac{\cancel{x^3}\left(1+\frac{1}{x}-\frac{1}{x^3}\right)}{\cancel{x^3}\left(\frac{7}{x^3}-1\right)} = -1$

(d) Find $\lim_{x \to 0} \sin(3x)/x$. $= \lim\limits_{x \to 0} 3 \cdot \frac{\sin(3x)}{3x} = 3 \cdot \lim\limits_{y \to 0} \frac{\sin(y)}{y} = 3 \cdot 1$

(e) Show that $\lim_{x \to 0} x \sin(1/x) = 0$

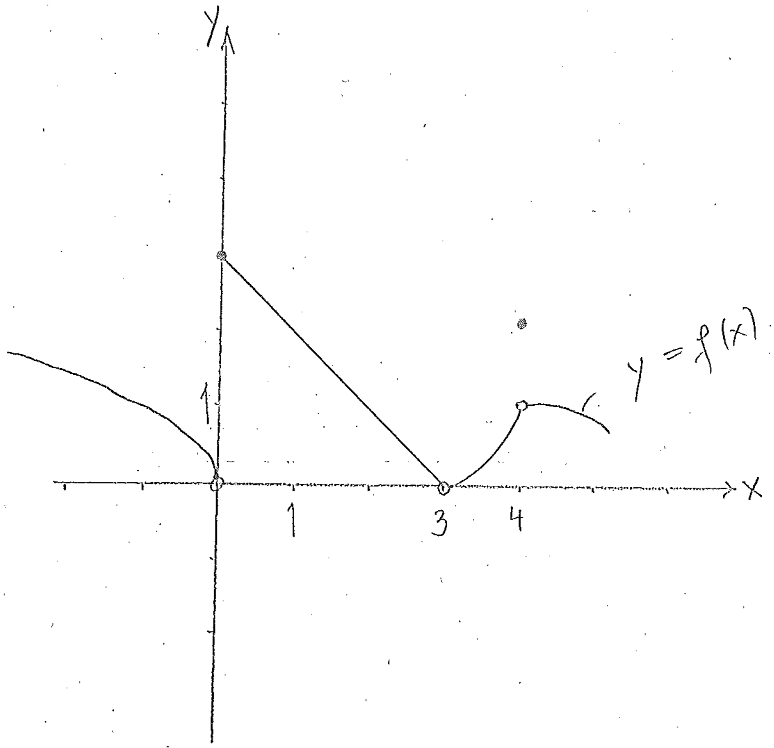
For instance: let $\epsilon > 0$, choose $\delta = \epsilon$ then if $|x| = |x-0| < \delta = \epsilon$, then $|\underbrace{x \sin\left(\frac{1}{x}\right)}_{1 \cdot 1 \le 1}| \le |x| < \epsilon$. Hence by definition $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

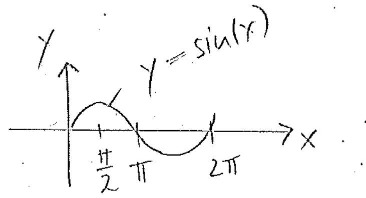
(Can also be shown using the squeeze theorem)

3. (5 points) Let

$$f(x) = \begin{cases} \sqrt{-x} & x < 0 \\ 3 - x & 0 \le x < 3 \\ (x-3)^2 & 3 < x < 4 \\ 2 & x = 4 \\ \sin(\pi x/8) & x > 4 \end{cases}.$$

(a) Sketch the graph of f on the interval $[-2, 5]$.





(b) Where is f continuous? Where is f discontinuous?

f is cont. on $(-\infty, 0)$ since $\sqrt{-x}$ is cont. there

— " — $(1, 3)$ since linear fcts. are cont.

— " — $(3, 4)$ since polynomials are cont.

— " — $(4, 5)$ since sin(x) is continuous.

f is disc. at $x = 0$ since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

— " — $x = 3$ since f(x) is not defined at 3

— " — $x = 4$ since $f(4) \neq \lim_{x \to 4} f(x)$.

4. (5 points) Find $f'(x)$ for the following functions by using the definition of the derivative. To receive credit you may **not** use the rules of differentiation.

(a) $f(x) = 2x + 7$

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)+7-2x-7}{h} = \lim_{h \to 0} \frac{2h}{h} = \underline{2}$$

(b) $f(x) = \sqrt{x^2+3}$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{(x+h)^2+3} - \sqrt{x^2+3}}{h} = \lim_{h \to 0} \frac{(x+h)^2+3-x^2-3}{h \cdot \left(\sqrt{(x+h)^2+3} + \sqrt{x^2+3}\right)}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h}+h^{\cancel{2}}}{\cancel{h}\left(\sqrt{(x+h)^2+3} + \sqrt{x^2+3}\right)} = \frac{2x}{2\sqrt{x^2+3}} = \underline{\underline{\frac{x}{\sqrt{x^2+3}}}}$$

(c) $f(x) = 1/x$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x-x-h}{h(x+h)(x)} = \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(x+h)(x)} = \underline{\underline{-\frac{1}{x^2}}}$$

5. (5 points) Compute the derivative of f in the cases below using the rules of differentiation.

(a) $f(x) = 7x^2 + \frac{3}{x}$

$f'(x) = 14 - \frac{3}{x^2}$ by power rule

(b) $g(x) = \sin(x) \cdot \cos(x)$

$g'(x) = \cos^2 x - \sin^2 x$ by product rule

(c) $f(x) = \frac{x}{\cos(x)+x}$

$f'(x) = \frac{\cos x + x + x \cdot \sin x - x}{(\cos(x) + x)^2}$ by quotient rule

(d) $f(x) = \tan(\sin(\cos(x)))$

$f'(x) = \sec^2(\sin(\cos(x))) \cdot \cos(\cos(x)) \cdot (-\sin(x))$ by chain rule

(e) $f(x) + x \cdot (f(x))^3 = 5$

$\Rightarrow f'(x) + (f(x))^3 + 3x \cdot (f(x))^2 \cdot f'(x) = 0$ using implicit differentiation

$\Rightarrow f'(x)\left(1 + 3x (f(x))^2\right) = -(f(x))^3$

$\Rightarrow f'(x) = \frac{-(f(x))^3}{1 + 3x \cdot (f(x))^2}$

6. (6 points) (a) Find the parabola with equation $y = ax^2 + bx$ whose tangent line at $(1, 1)$ has equation $y = 3x - 2$.

$y' = 2ax + b \Rightarrow y'(1) = 2a + b = 3$

$\Downarrow$

also $y(1) = 1 \Rightarrow 1 = a + b \Rightarrow b = 1 - a \Rightarrow 2a + 1 - a = 3 \Rightarrow a + 1 = 3$

$\Rightarrow a = 2$

$\Rightarrow b = -1$

check: $y = f'(1)(x-1) + f(1)$; $f'(1) = 2 \cdot 2 \cdot 1 - 1 = 3$

$f(1) = 2 \cdot 1^2 - 1 \cdot 1 = 1$

$\Rightarrow y = 3x - 3 + 1$

$\Rightarrow \underline{y = 3x - 2}$ ✓

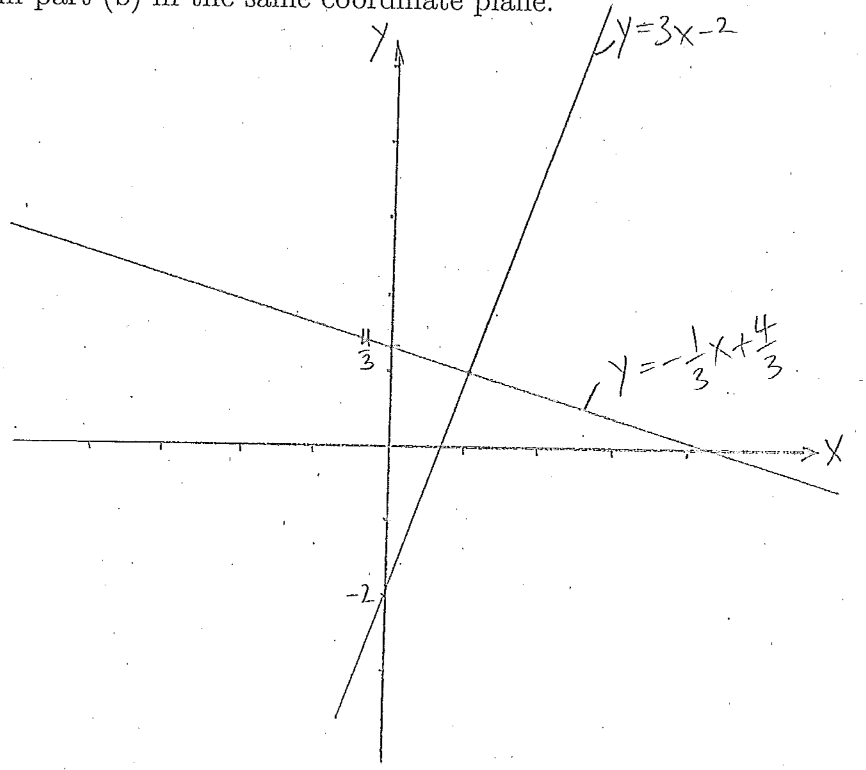
$\Rightarrow$ the parabola is $\underline{y = 2x^2 - x}$

(b) Find an equation for the line normal to the parabola (found in part (a)) at the point $(1, 1)$.

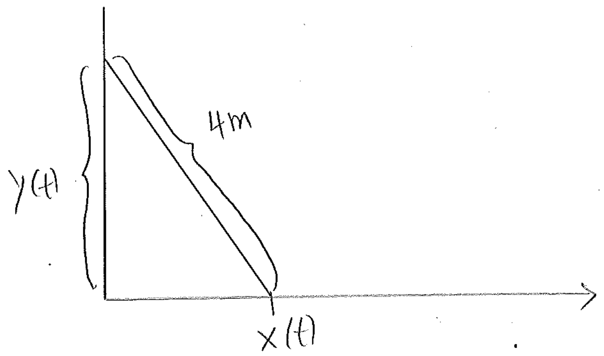
$(y - 1) = -\frac{1}{3}(x - 1)$ line is perpendicular to $y = 3x - 2$ and goes through $(1,1)$

$\Rightarrow y = -\frac{1}{3}x + \frac{1}{3} + 1 = \underline{-\frac{1}{3}x + \frac{4}{3}}$

(c) Draw the graph of the tangent line given in part (a) and the graph of the normal line in part (b) in the same coordinate plane.



7. (6 points) A ladder 4 m long is leaning against the side of a building. If the foot of the ladder is pulled away from the building at a constant rate of 0.2 m per second, how fast is the area of the triangle formed by the ladder, the building and the ground changing (in square meter per second) at the instant when the top of the ladder is 3 meter above the ground.



$y(t_0) = 3m$

$\frac{dx}{dt} = 0.2m$

$x(t_0) = \sqrt{16-9} = \sqrt{7}$

$A(t) = \frac{x(t) \cdot y(t)}{2}$, $(x(t))^2 + (y(t))^2 = 16 \Rightarrow x(t_0)\, x'(t_0) + y(t_0) \cdot y'(t_0) = 0$

$\Downarrow$

$\sqrt{7} \cdot 0.2 + 3 \cdot y'(t_0) = 0$

$\Downarrow$

$y'(t_0) = \frac{-\sqrt{7}}{3 \cdot 5}$

$\Rightarrow A'(t) = \frac{1}{2}\left(x'(t)\, y(t) + x(t) \cdot y'(t)\right)$

$\Rightarrow A'(t_0) = \frac{1}{2}\left(0.2 \cdot 3 + x(t_0) \cdot y'(t_0)\right)$

$= \frac{1}{2}\left(\frac{3}{5} + \sqrt{7} \cdot \frac{(-\sqrt{7})}{3 \cdot 5}\right) = \frac{1}{2}\left(\frac{3}{5} - \frac{7}{15}\right) = \frac{1}{15}\ m/s$

8. (6 points) (a) Find the linearization of $f(x) = \sqrt{x}$ for $a = 9$.

$L(x) = f'(a)(x-a) + f(a)$, $f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(a) = \frac{1}{6}$

$f(9) = 3$

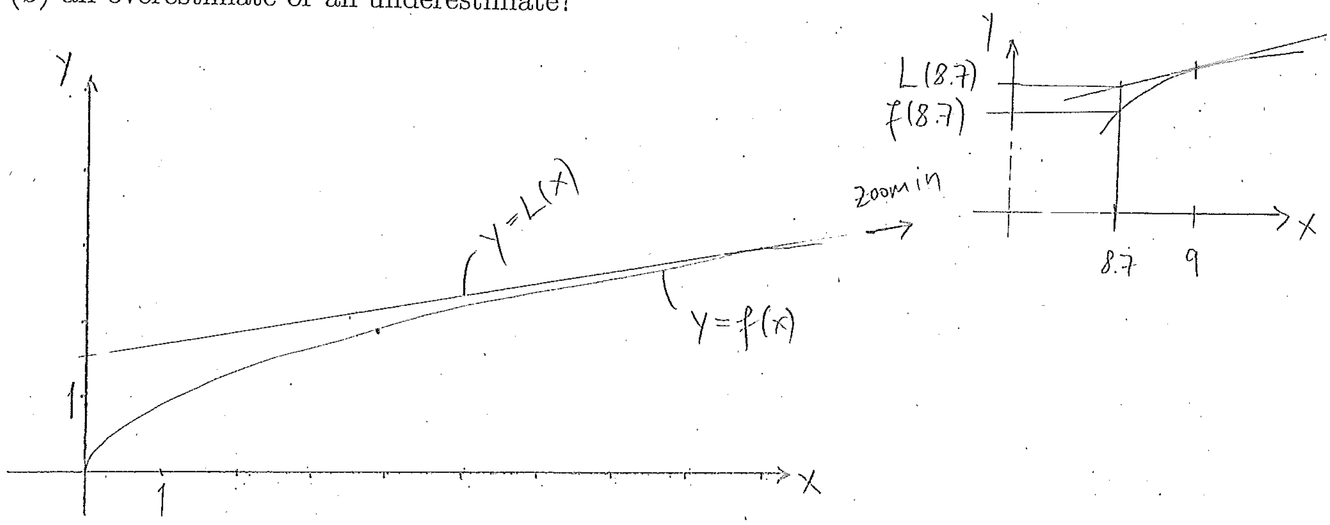
$\Rightarrow L(x) = \frac{1}{6}(x-9) + 3$

(b) Use (a) to approximate $\sqrt{8.7}$.

$\sqrt{8.7} \approx L(8.7) = \frac{1}{6}(-0.3) + 3 = \frac{-1}{6} \cdot \frac{3}{10} + 3 = \frac{-1}{20} + 3$

$= 2.95$

(c) Draw the graphs of f and its linearization to illustrate part (b). Is the approximation in (b) an overestimate or an underestimate?



the tangent line lies above the graph of $\sqrt{x}$, hence the approximation is an overestimate.

9. (7 points) Determine whether each of the following statements is true or false. If it is true, explain why. If it is false, give an example that disproves the statement.

(a) If f is a continuous function, then $f(4x) = 4f(x)$ for all x in the domain of f.

Let $f(x)=1$, then $f(4x)=1 \neq 4 = 4f(x)$.

Hence, the statement is false.

(b) If f and g are functions whose domains are all real numbers, then $f \circ g = g \circ f$.

Let $f(x)=1$, $g(x)=2$, both f and g have all real #'s as their domain. But $f(g(x))=1 \neq 2 = g(f(x))$.

Hence, the statement is false.

(c) If f is continuous at a, then f is differentiable at a.

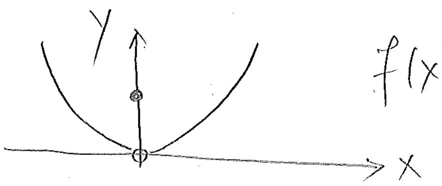
Consider $f(x)=|x|$. This is cont. everywhere but not diff. at 0, since

$$\lim_{h\to 0^+} \frac{f(h)}{h} = 1 \neq -1 = \lim_{h\to 0^-} \frac{f(h)}{|h|}.$$

The statement is false.

(d) If $f(x) > 1$ for all x and $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 1$.

False, for instance



$$f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

is a counterexample.

(e) If $y = y(x)$ is a twice differentiable function, then $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

False, let $y(x)=x \Rightarrow \left(\frac{dy}{dx}\right)^2 = (1)^2 = 1$

$$\frac{d^2y}{dx^2} = 0$$