Reflection Principles for Minimal and Maximal Surfaces

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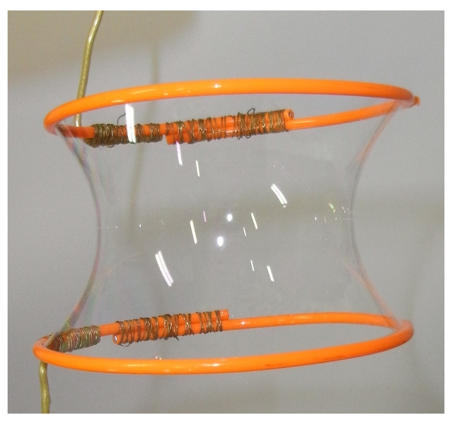
Minimal surfaces in Euclidean space

A *minimal surface* in Euclidean 3-space $\mathbb{E}^3 = (\mathbb{R}^3, ds_E^2 = dx_1^2 + dx_2^2 + dx_3^2)$ is a surface whose mean curvature H vanishes identically. Since the equation $H = 0$ is invariant under scalings, minimal surfaces appear in many situations with different scales:



• soap films • structure of diblock copolymers • black holes etc.

▹ **Minimal surface and harmonic function**: Mathematically, a minimal surface is described by using harmonic function (the function f satisfying the Laplace equation $\Delta f = 0$) as their coordinate functions.

$$X(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v)), \quad \Delta x_i = \frac{\partial^2 x_i}{\partial u^2} + \frac{\partial^2 x_i}{\partial v^2} = 0,$$

where Δ is the Laplacian of the surface metric $X^*ds_E^2 = f(du^2 + dv^2)$ for a positive valued function f.

Therefore, the classical **Schwarz reflection principle** by H.A. Schwarz (1870) in complex analysis yields that minimal surfaces have some kinds of symmetries.

(1) If a minimal surface X has a straight line segment L, the surface is invariant under the 180°-rotation with respect to L.

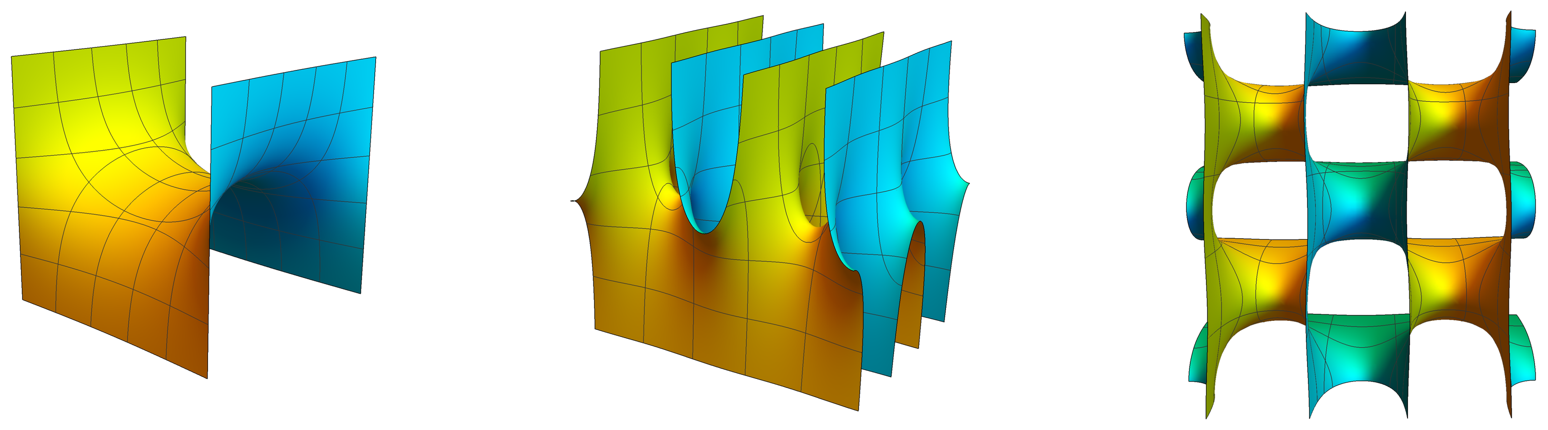


Fig. Fundamental piece of Scherk's minimal surface (left) and its extensions across the vertical lines (center and right).

(2) If a minimal surface X intersects a plane Π vertically, the surface is invariant under the reflection with respect to Π.

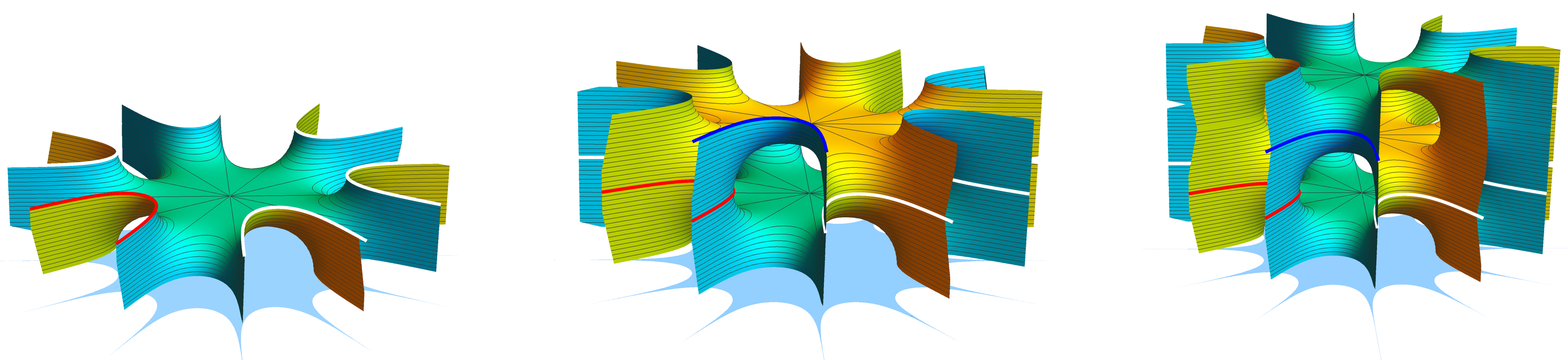


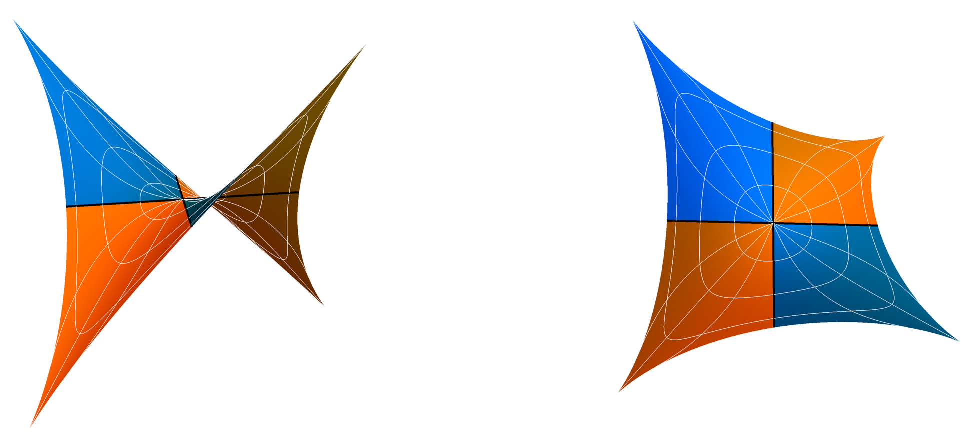
Fig. Construction of a *saddle tower* minimal surface by Karcher 1988. In the left figure, the boundary curves are lying on a plane, then by repeating planar reflections we can obtain a tower-like surface.

Maximal surfaces in spacetime

A *maximal surface* in the Lorentz-Minkowski 3-space $\mathbb{L}^3 = (\mathbb{R}^3, ds_L^2 = dx_1^2 + dx_2^2 - dx_3^2)$ is a surface whose induced metric is Riemannian and mean curvature H vanishes identically.

▹ **Reflection principles for maximal surfaces**: By definition, maximal surfaces in $\mathbb{L}^3$ are quite similar to minimal surfaces in $\mathbb{E}^3$, and hence the following reflection principles also hold for these surfaces:

(3) If a maximal surface X has a spacelike straight line segment L, the surface is invariant under the 180°-rotation with respect to L. Here, the line L is called spacelike if $L = \mathrm{span}(\vec{v})$ where $\langle \vec{v}, \vec{v} \rangle_L > 0$.



Maximal surfaces (continued)

(4) If a maximal surface X intersects a timelike plane Π vertically, the surface is invariant under the reflection with respect to Π. Here, Π is called timelike if $\Pi = \mathrm{span}(\vec{v}, \vec{w})$ where $\langle \vec{v}, \vec{v} \rangle_L > 0$, $\langle \vec{w}, \vec{w} \rangle_L < 0$.

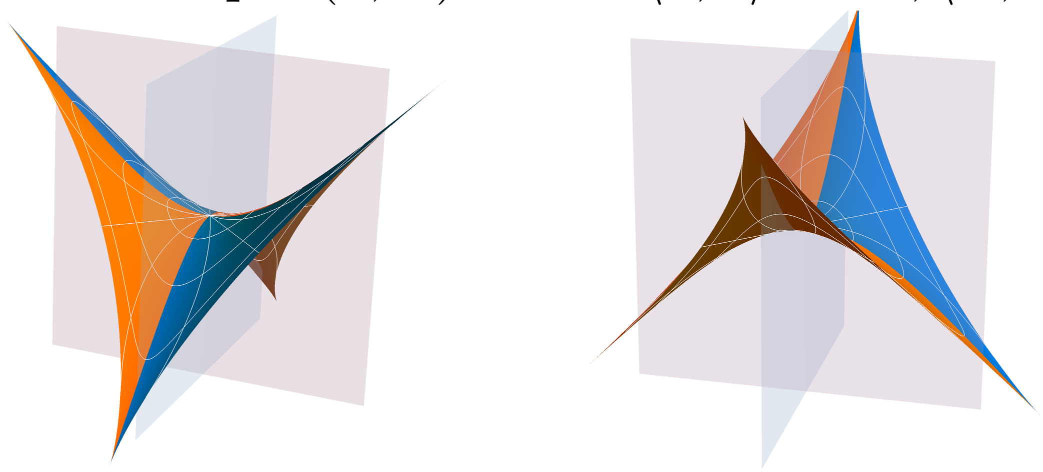
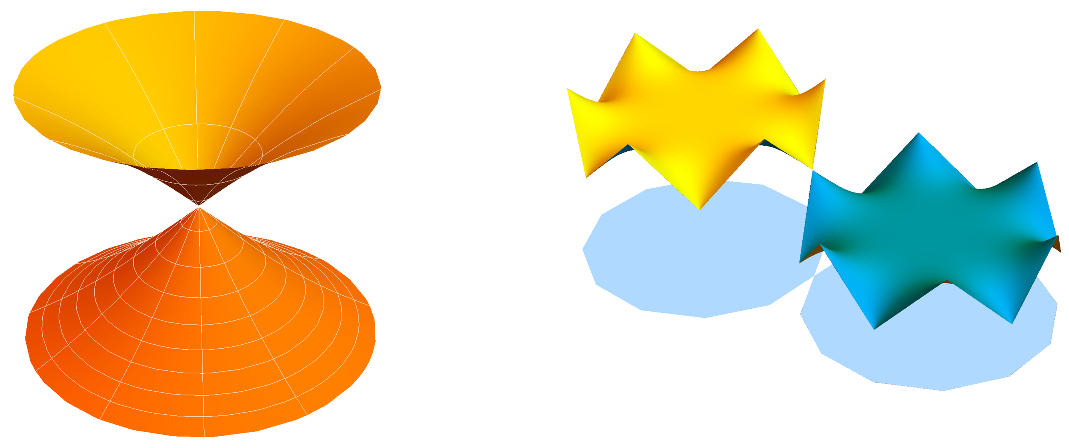


Fig. The maxmal Enneper surface from different point of views.

(5) If a maximal surface $X = X(u,v)$ shrinks to a point $P \in \mathbb{L}^3$ by the relation $X(u,0) = \{P\}$, then X is invariant under the point symmetry with respect to P. This P is called a *shrinking singularity*.



Question and main theorem

In the Minkowski spacetime of special relativity, a lightlike line/geodesic in $\mathbb{L}^3$ is the path of a light ray, along which $ds_L^2 = 0$. Hence, we can use neither Riemannian geometry nor Lorentzian geometry on this kind of lines, so the following question was difficult and remained to be still open.

Does a reflection principle hold for lightlike line segments?

As an answer to this question, we obtain the following theorem.

Theorem 1. *If a maximal surface X has a lightlike line segment L which connects two shrinking singularities as in the sense (5), then X is invariant the point symmetry with respect to midpoint of L.*

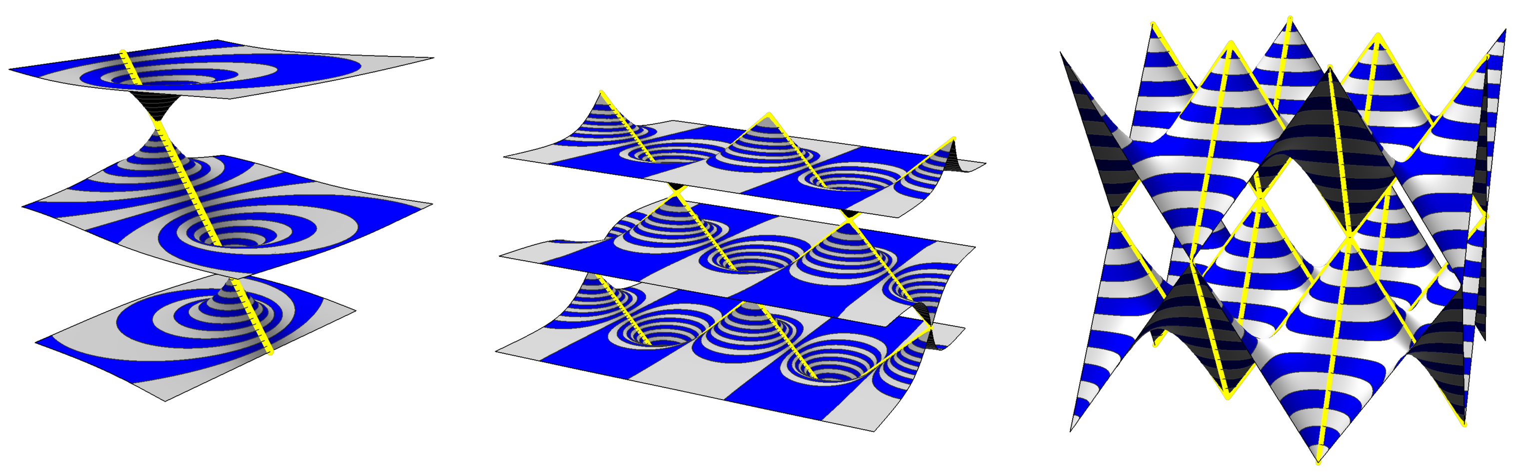
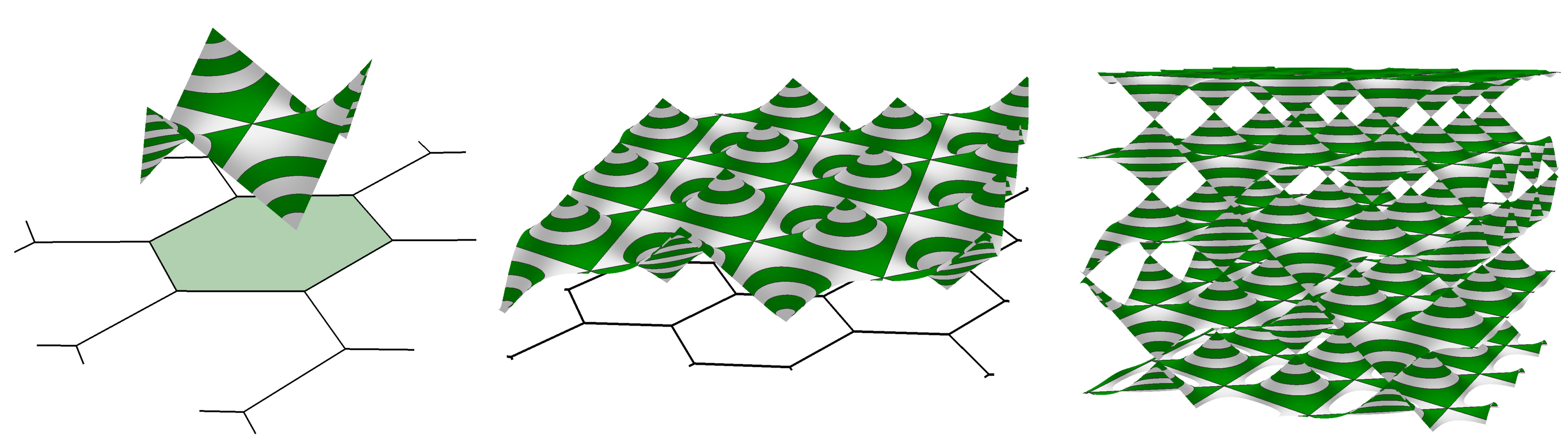


Fig. Periodic maximal surfaces with lightlike lines (yellow lines).



References

■ **S. Akamine and H. Fujino**, *Reflection principle for lightlike line segments on maximal surfaces*, Ann. Global Anal. Geom. **59** (2021), 93–108.
■ **L.J. Alías, R.M.B. Chaves, and P. Mira**, *Björling problem for maximal surfaces in Lorentz-Minkowski space*, Math. Proc. Cambridge Philos. Soc. **134** (2003), no. 2, 289–316.
■ **H. Karcher**, *Embedded minimal surfaces derived from Scherk's examples*, Manuscripta Math. **62** (1988), no. 1, 83–114.
■ **Y. W. Kim and S.-D. Yang**, *Prescribing singularities of maximal surfaces via a singular Björling representation formula*, J. Geom. Phys. **57** (2007), no. 11, 2167–2177.