Construction and deformation of discrete surfaces via integrable transformations

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In this poster we introduce two topics related to integrable transformations:
(1) Discrete Weierstrass-type representations (based on joint work with M.Pember and D. Polly **[PPY]**)
(2) Discrete minimal Darboux transformations (based on ongoing project with Y. Jikumaru **[JY]**)

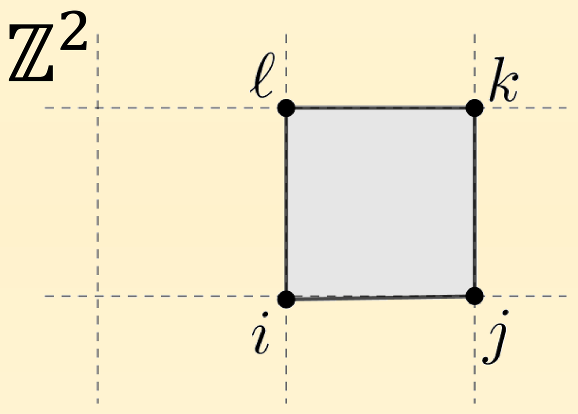
1. Discrete Weierstrass-type representations

Previous Works.

Discrete minimal surfaces (Bobenko, Pinkall)

(1) Choose a discrete holomorphic function $g\colon \mathbb{Z}^2 \to \mathbb{C}$ satisfying

$$cr(g_i, g_j, g_k, g_\ell) := \frac{(g_i - g_j)(g_k - g_\ell)}{(g_j - g_k)(g_\ell - g_i)} \equiv -1.$$



(2) Take the inverse image $\phi^{-1} \circ g \in \mathbb{S}^2$ of stereographic projection ϕ.

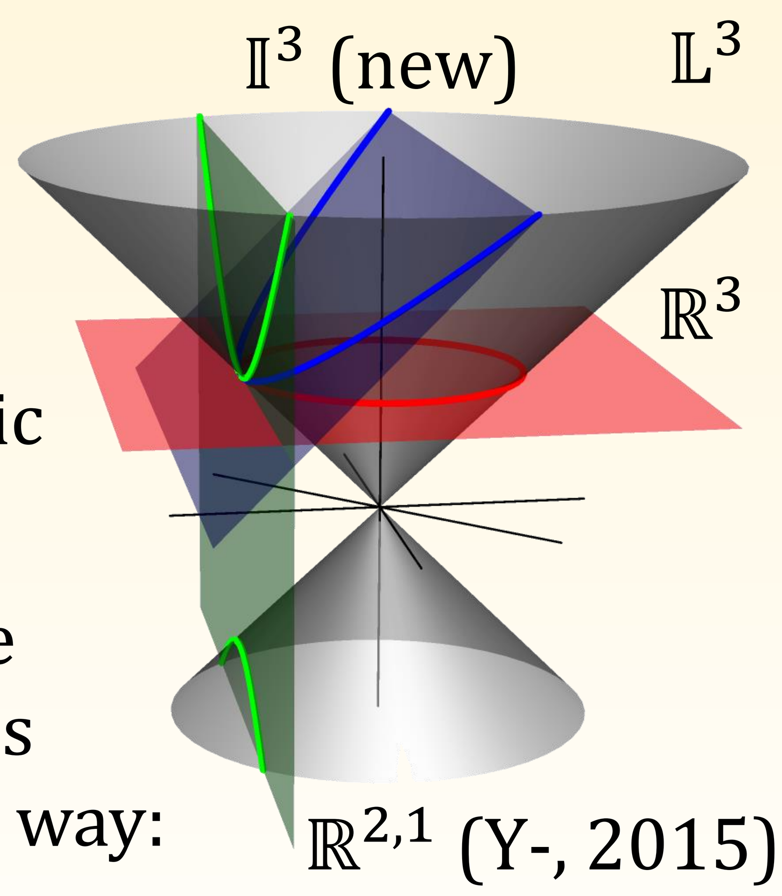
(3) Take the Christoffel transform of $\phi^{-1} \circ g$, and we have a discrete minimal surface in $\mathbb{R}^3$.

Observations.

View all the known results in Minkowski 4-space $\mathbb{R}^{3,1}$.

For all cases, the holomorphic Gauss map lies in $P(\mathbb{L}^3)$.

SO constructions for discrete zero mean curvature surfaces can be given in the following way:



Theorem ([PPY]).

For a fixed $\mathfrak{p} \in \mathbb{R}^{3,1} \setminus \{\mathbf{0}\}$, any discrete zero mean curvature surface f in $\mathfrak{p}^\perp$

$$(\mathfrak{p}^\perp := \{X \in \mathbb{R}^{3,1} | \langle X, \mathfrak{p} \rangle = -1\})$$

can be described in the following way:

1. Choose a discrete holomorphic function g, and projectivize the holomorphic lightlike Gauss map

$$G := (2\mathrm{Re}(g), 2\mathrm{Im}(g), -1 + |g|^2, 1 + |g|^2)^t \in \mathbb{L}^3$$

to a 3-dimensional hyperplane $\mathfrak{p}^\perp$.

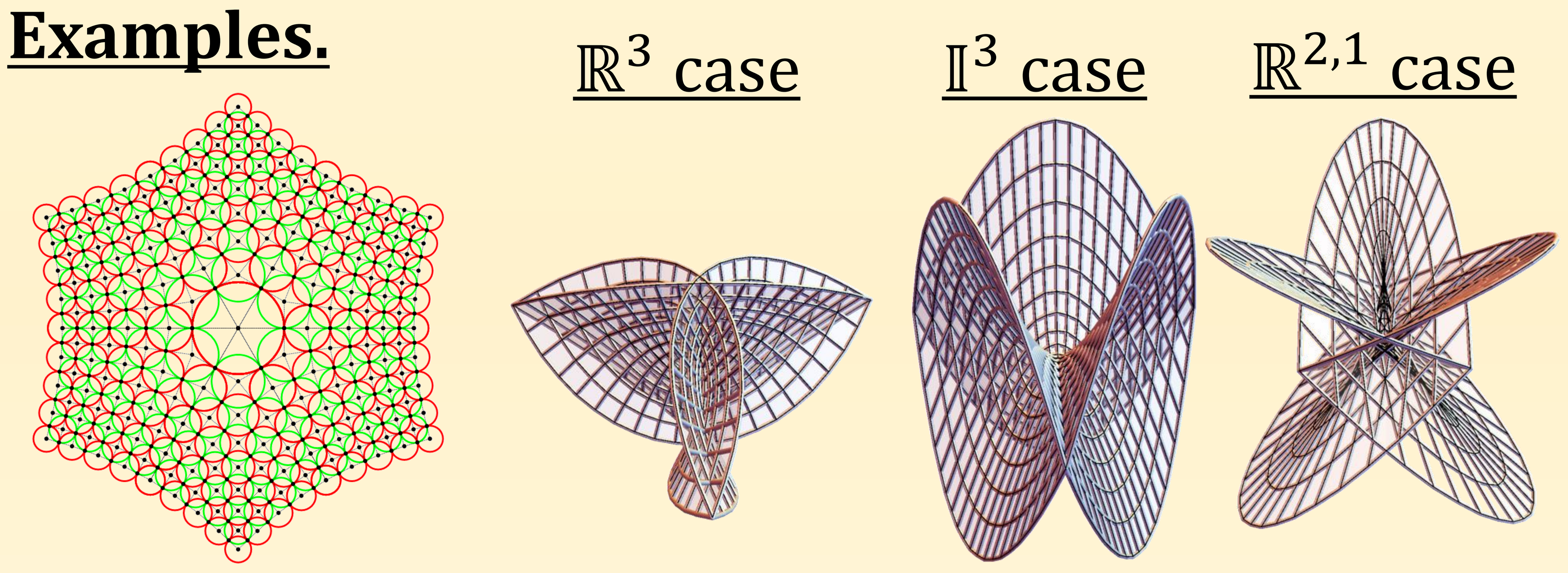
2. Take the Ω-dual transform of $\langle G \rangle \in P(\mathbb{L}^3)$, and we obtain a discrete zero mean curvature surface in $\mathfrak{p}^\perp$.

Furthermore, although we do not explain details, we are able to construct discrete surfaces in other space forms via the Ω-dual transformations.

Main Theorem ([PPY]).

All the known discrete Weierstrass-type representations can be viewed as certain applications of the Ω-dual transformation to a prescribed holomorphic lightlike Gauss map.

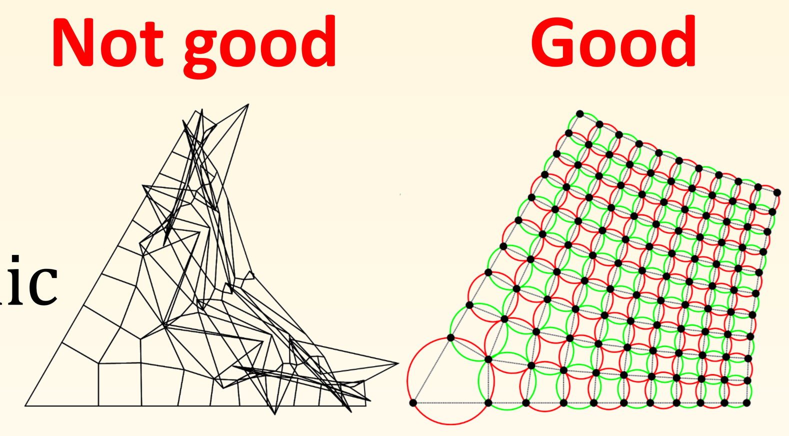
Examples.



2. Discrete minimal Darboux transformations (in progress)

Motivation.

In general, it is difficult to generate discrete holomorphic functions.



Theorem ([JY])

For any discrete isothermic surface f, the following diagram holds, where C, D_λ, G_μ denotes the Christoffel, Darboux, and Goursat transformation, respectively:

$$\begin{tikzcd}
G_\mu(f) \arrow[rrr, leftrightarrow, "C"] \arrow[ddd, leftrightarrow, "D_\lambda"'] & & & \mu(f^*) \arrow[ddd, leftrightarrow, "D_\lambda"] \\
& f \arrow[ul, "G_\mu"'] \arrow[r, leftrightarrow, "C"] \arrow[d, leftrightarrow, "D_\lambda"'] & f^* \arrow[ur, "\mu"] \arrow[d, leftrightarrow, "D_\lambda"] & \\
& \hat{f} \arrow[dl, "G_\mu"'] \arrow[r, leftrightarrow, "C"'] & \hat{f}^* \arrow[dr, "\mu"] & \\
G_\mu(\hat{f}) \arrow[rrr, leftrightarrow, "C"'] & & & \mu(\hat{f}^*)
\end{tikzcd}$$

Application ([JY])

If two discrete minimal surfaces are related by the Darboux transformations, the corresponding discrete holomorphic functions are also related by Darboux transformations.