科学研究費助成事業「新学術領域研究（研究領域提案型）」平成29-33年度
次世代物質探索のための離散幾何学

筑波大学
University of Tsukuba

Topological band theory for classical diffusion phenomena and the game theory

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(U Tsukuba)

Poster presentation
2021/09/28 @ zoom

Collaborators

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Yasuhiro Hatsugai

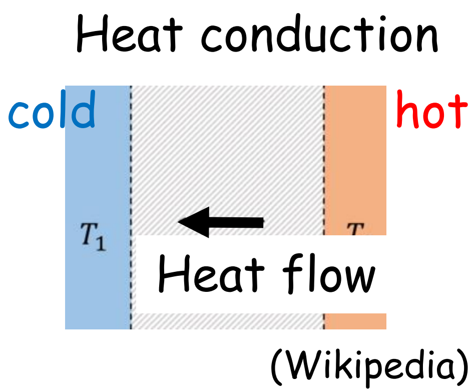
筑波大学
University of Tsukuba

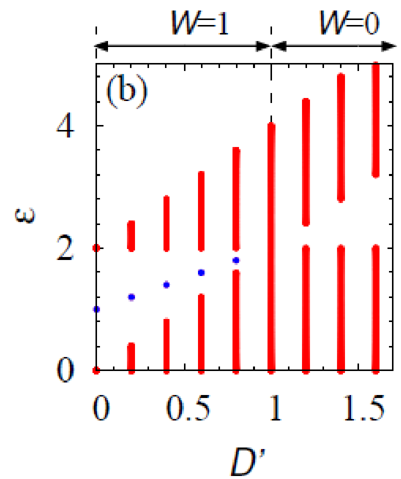
References

- TY-Hatsugai, Sci. Rep. **11**, 888 (2020)
- TY-Mizoguchi-Hatsugai, PRE **104**, 025003 (2021)

Main message: New platform of topological edge modes

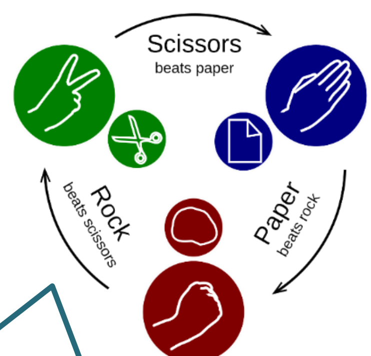
Classical diffusion phenomena

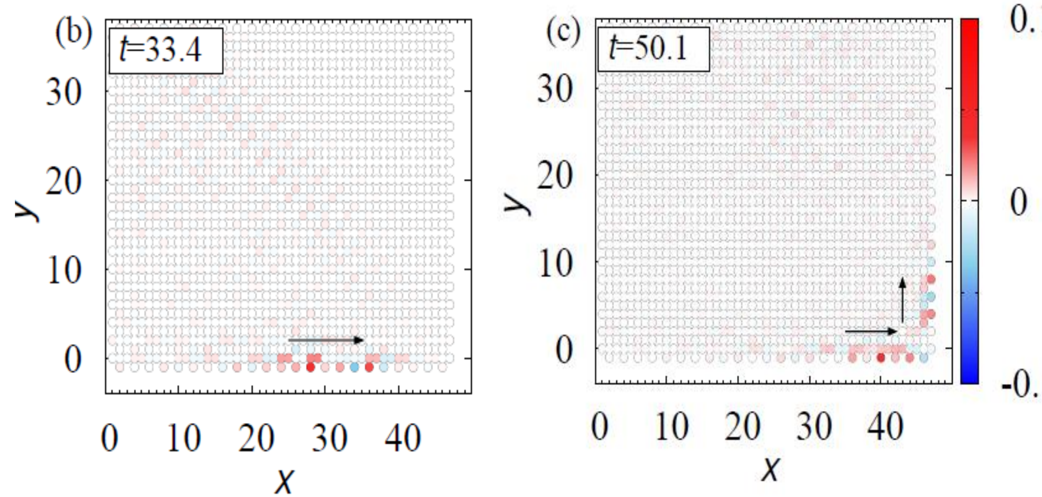




TY-Hatsugai, Sci. Rep (2020)

Evolutionary game theory





- bacterium (biology)
- human behaviors (social science)

TY-Mizoguchi-Hatsugai, PRE (2021)

Outline

1. Warm up: topological insulators

2. Motivation

3. Topological edge modes in classical diffusion phenomena

4. Chiral edge state in evolutionary game theory

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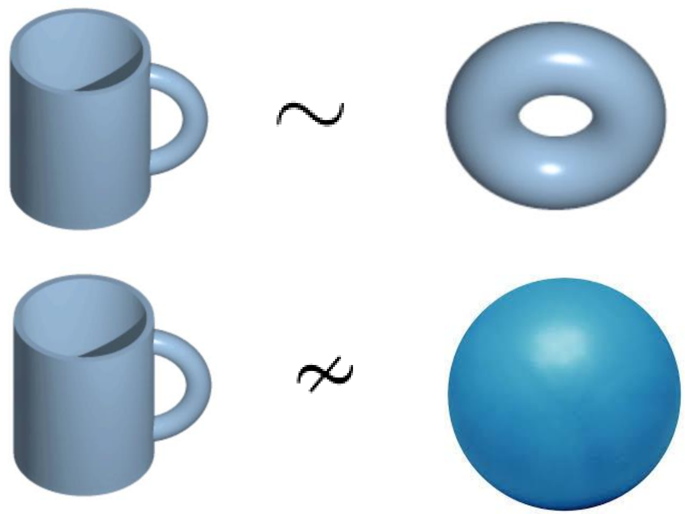
Warm up: topological insulators

Topology (simple example)



:)

(wikipedia)

# of holes does not change under continuous deformation

cut or glue

Topological insulators:

Topo. # and # of edge states do not change under continuous deformation

gap-closing in the bulk

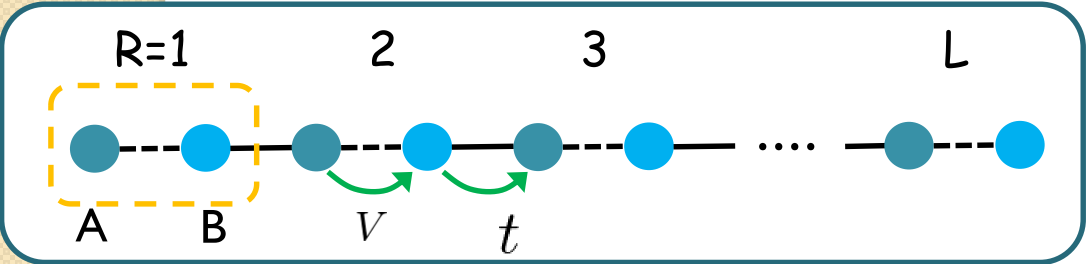
Warm up: topological insulators

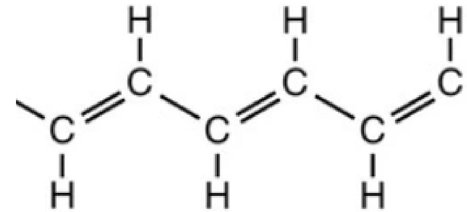
Topological insulators:

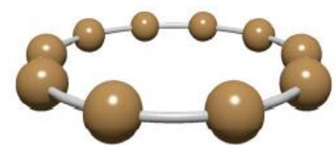
Topo. # and # of edge states do not change under continuous deformation

gap-closing in the bulk

Topological insulator in 1D
(Su Schrieffer Heeger model)





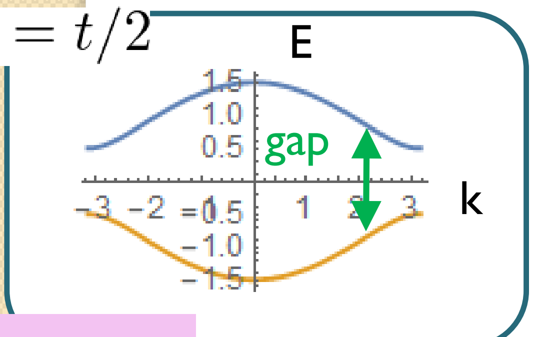


$$h(k) = d_1(k)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d_2(k)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

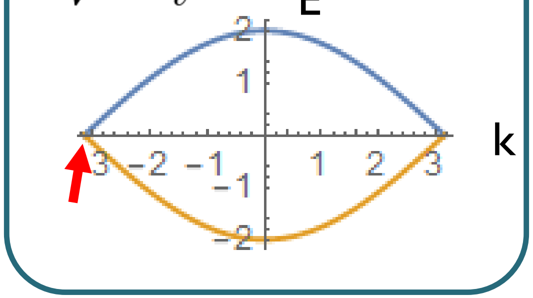
$$d_1(k) = V + t\cos k \qquad d_2(k) = t\sin k$$

$$E(k) = \pm\sqrt{d_1^2(k) + d_2^2(k)}$$

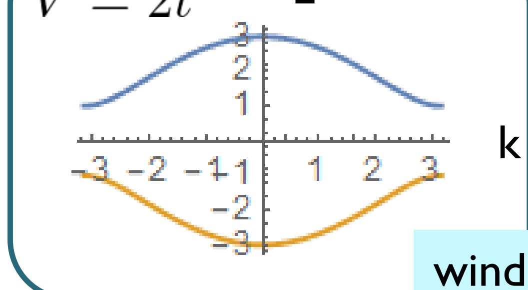
$V = t/2$



$V = t$



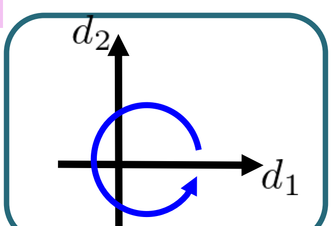
$V = 2t$

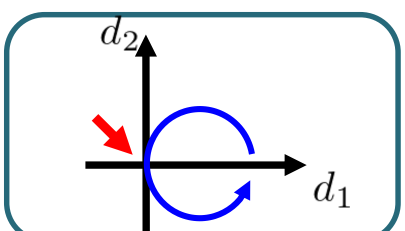


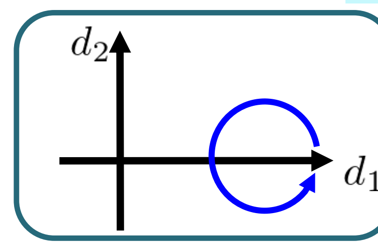
winding#=1

winding#=0

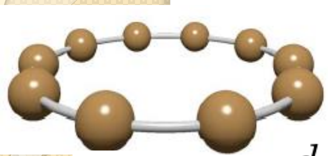
$$d_1 + id_2 = V + te^{ik}$$



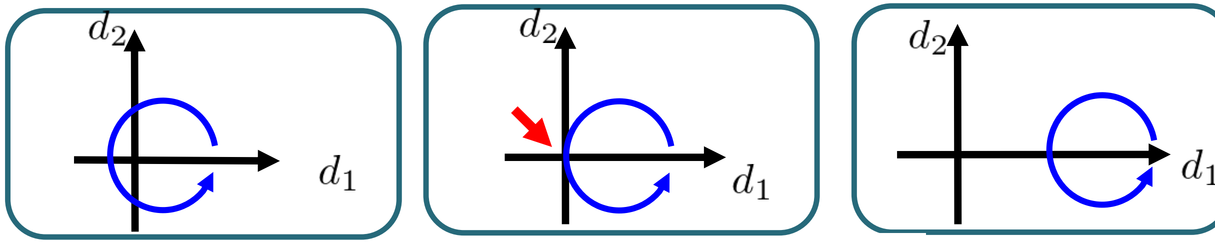


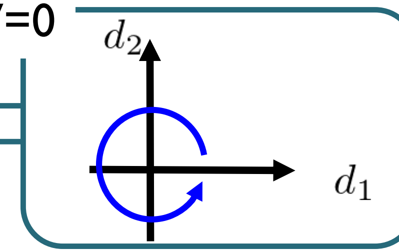


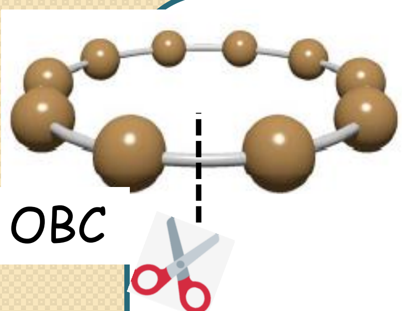
With chiral symmetry $\sigma_3 h(k) \sigma_3 = -h(k)$ winding # is quantized

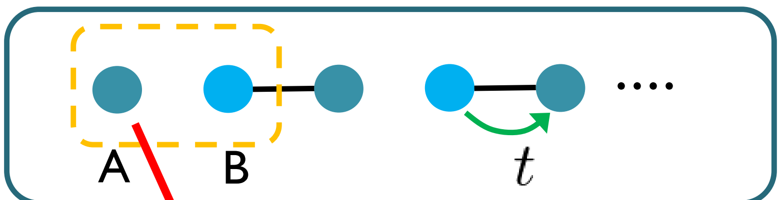


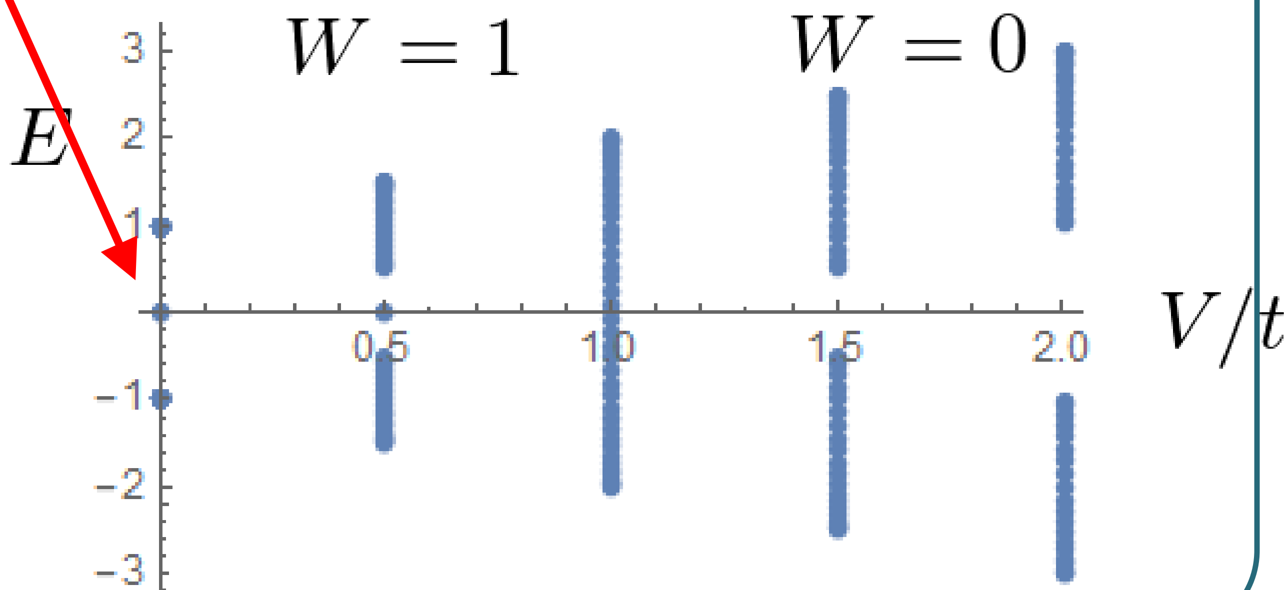
$$d_1 + id_2 = V + te^{ik}$$











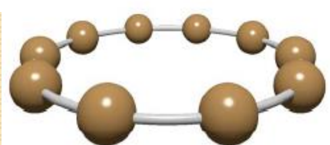
Short summary

Topological insulators (1D):

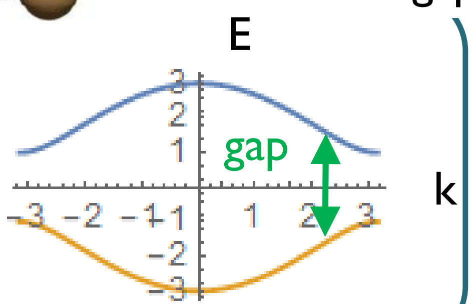
Winding number and # of edge states do not change under continuous deformation

gap-closing in the bulk

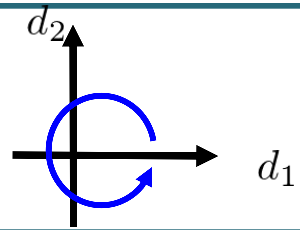
PBC



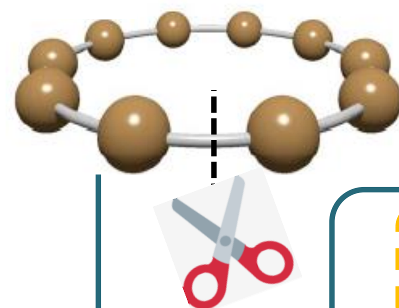
Bulk-gap

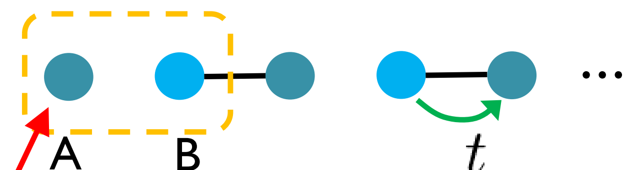


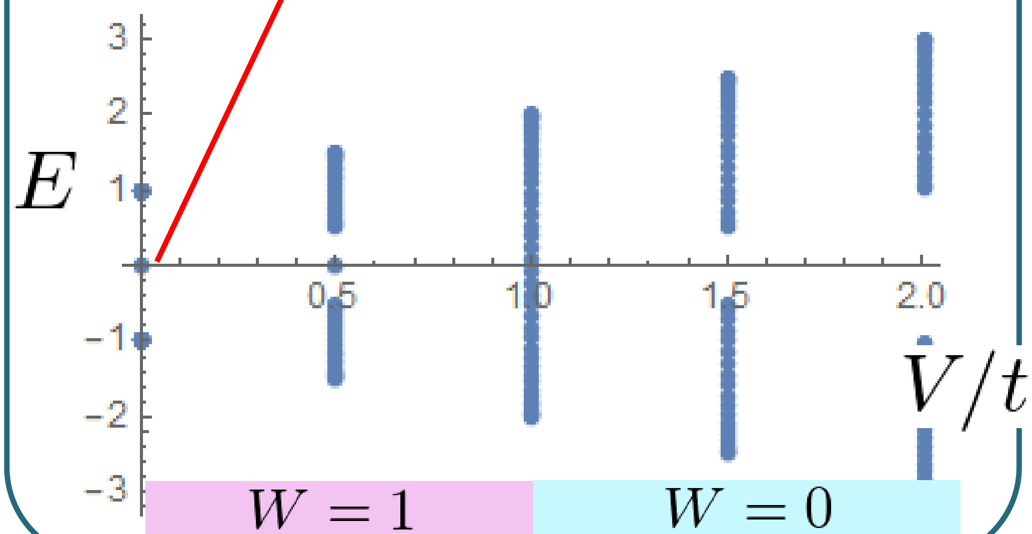
winding#=1



OBC





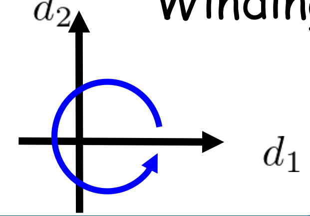


Warm up: topological insulators

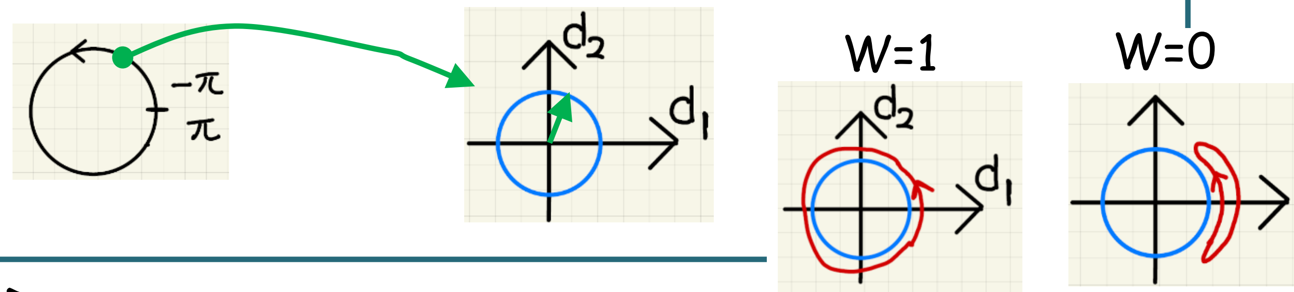
1D → 2D

Topological insulator 1D

Winding#

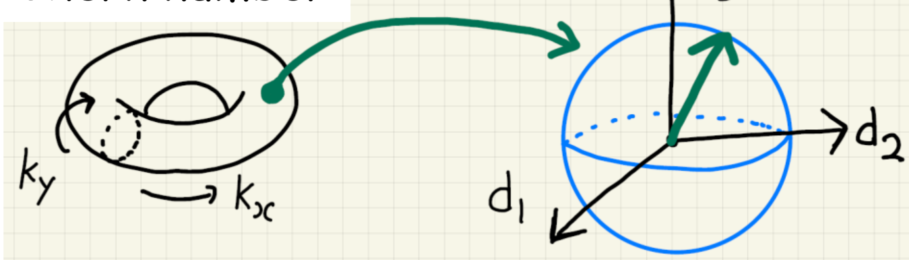


$$-\pi \le k < \pi \longrightarrow \vec{d}(k)/|d(k)|$$

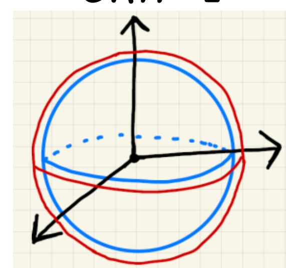


Topological insulator 2D

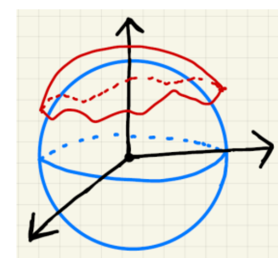
Chern number



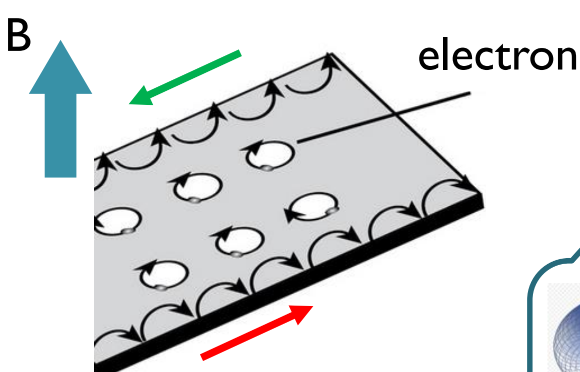
Ch#=1

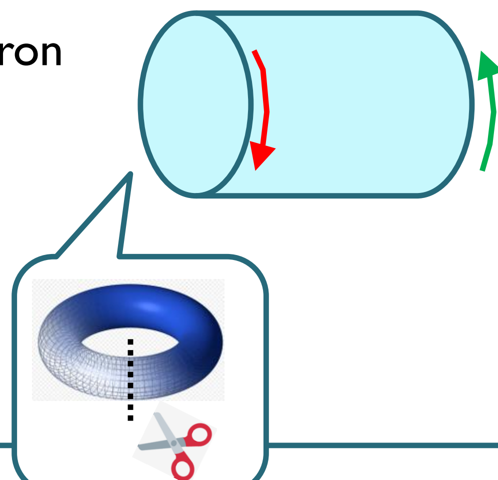


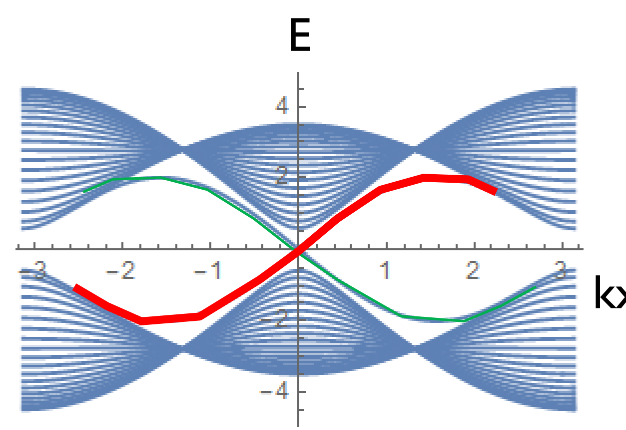
Ch#=0



Chern insulator (integer quantum Hall system)





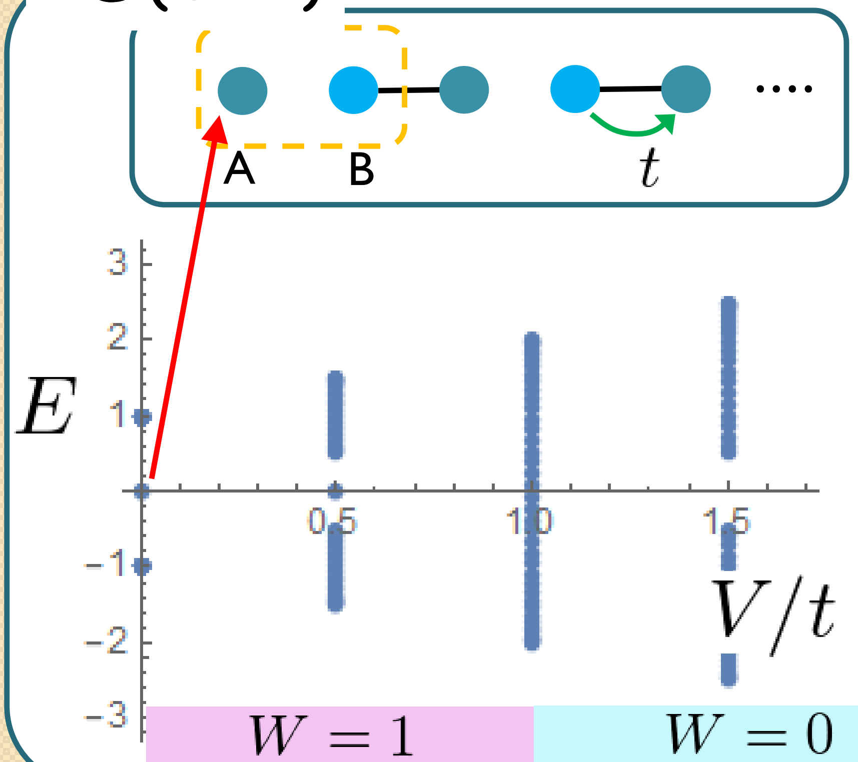


Topological insulators:

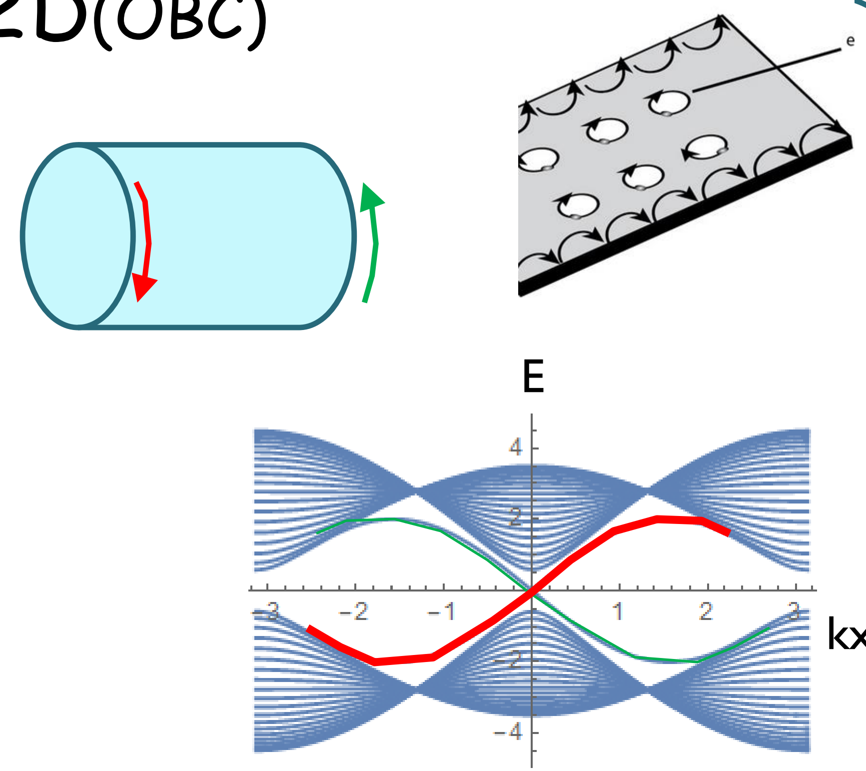
Topo. # and # of edge states do not change under continuous deformation

gap-closing for PBC

1D(OBC)



2D(OBC)



Outline

1. Warm up: topological insulators

2. Motivation

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4. Chiral edge state in evolutionary game theory

Motivation

Topological edge modes beyond quantum mechanics

Y. Hatsugai PRL(1993)

Integer quantum Hall effect

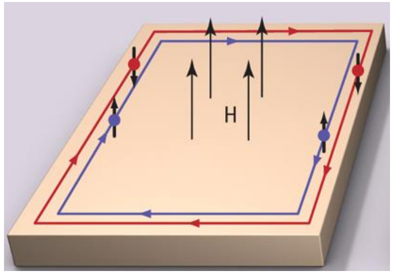
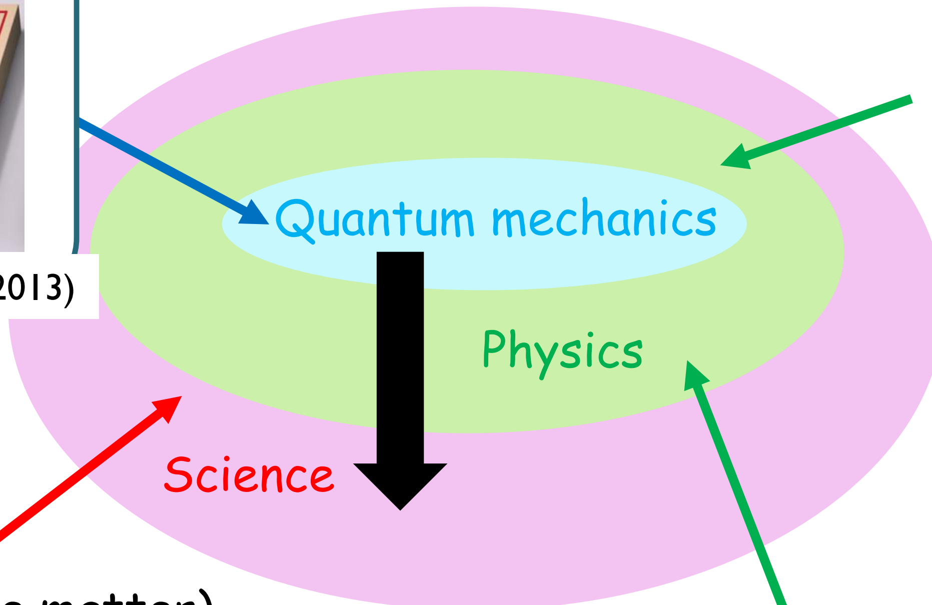
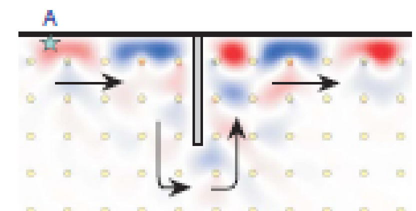


Fig. from S. Oh Science (2013)



Photonic crystals

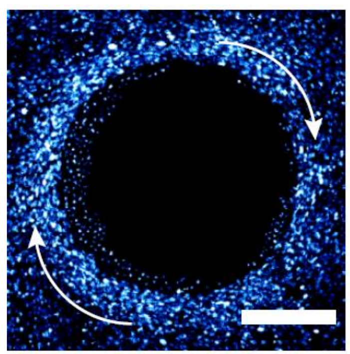
- Raghu-Haldane PRA (2008)



Z. Wang *et al.*, Nature (2009)

Biology (Active matter)

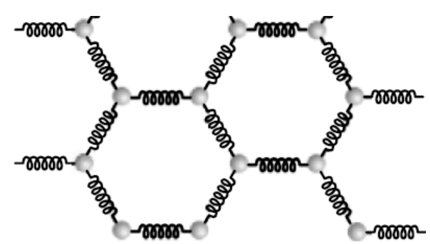
chiral modes
of cells



- L. Yamaguchi and K. Kawaguchi et al., arXiv (2020)

Classical mechanics

- Kariyado-Hatsugai Sci. Rep. (2015)



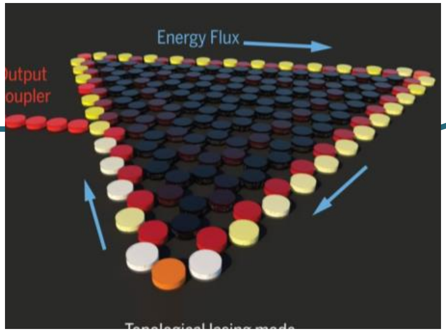
Motivation

Finding a new platform of chiral edge mods is significant

- Universal understanding is available

- New devices thanks to robust edge states

<Topological insulator laser>



G. Harari et al., Science (2018)

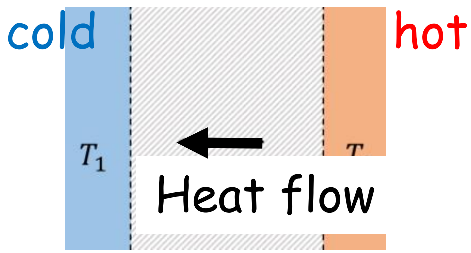
Question we address:

New platform of topological edge modes?

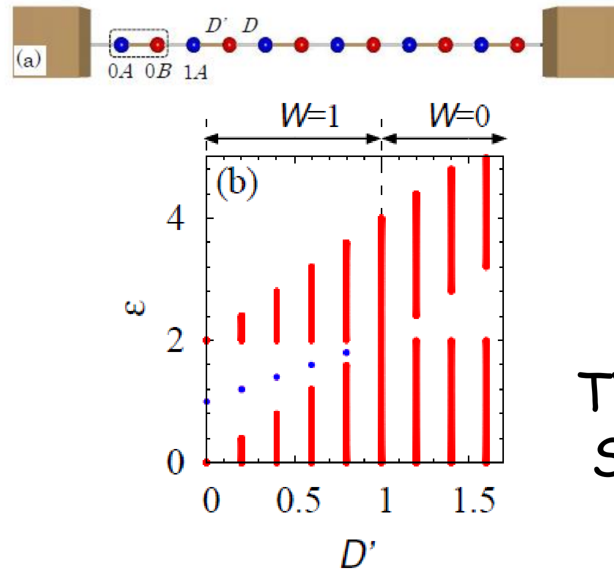
(In particular, chiral edge modes)

Part I: Classical diffusion phenomena: 1D edge state

Heat conduction



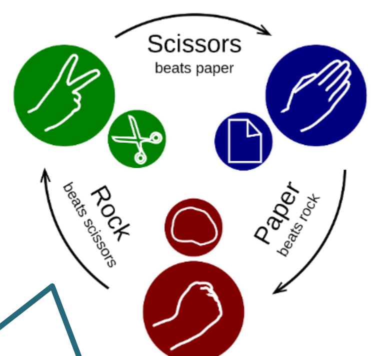
(Wikipedia)

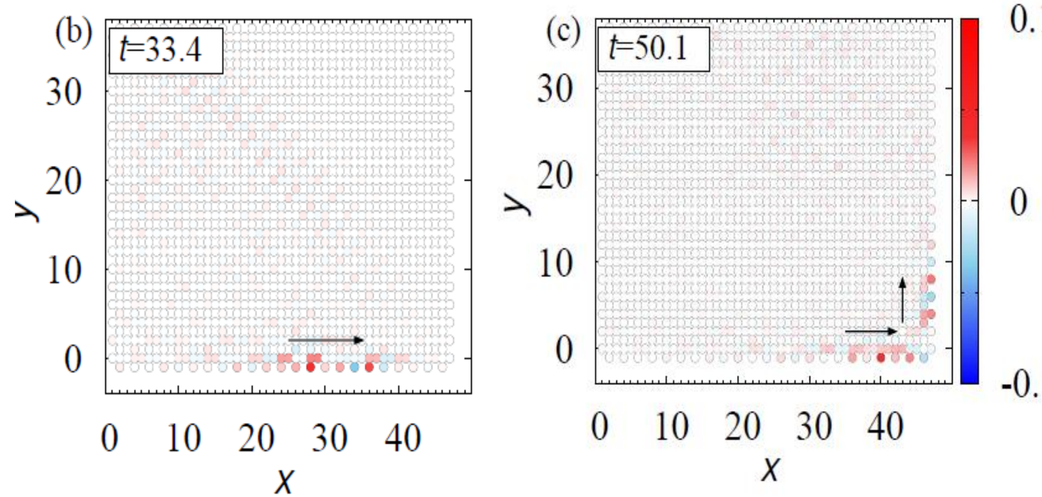


TY-Hatsugai, Sci. Rep. (2020)

Part II:

Evolutionary game theory: chiral edge state





- bacterium (biology)
- human behaviors (social science)

TY-Mizoguchi-Hatsugai, PRE (2021)

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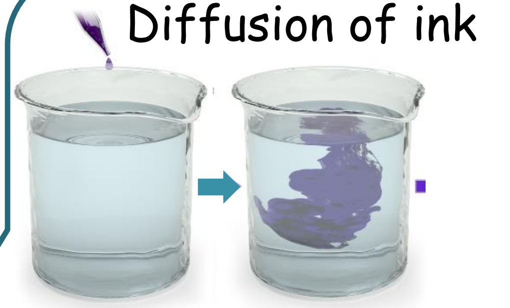
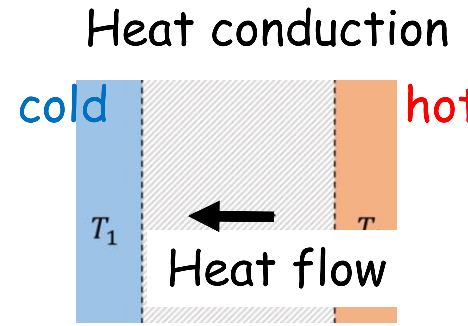
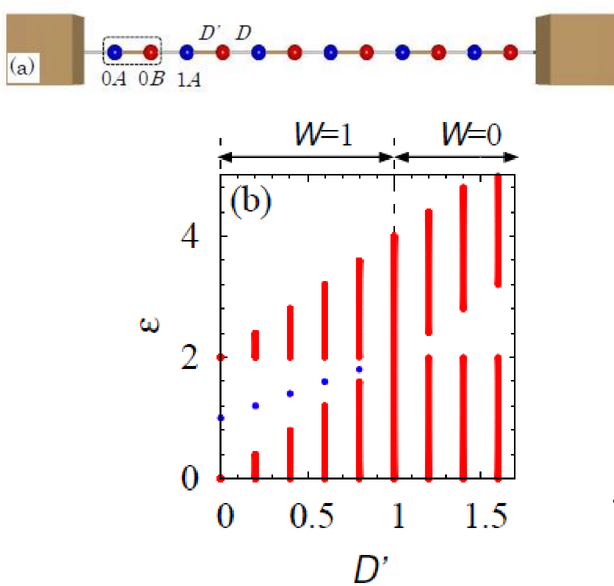
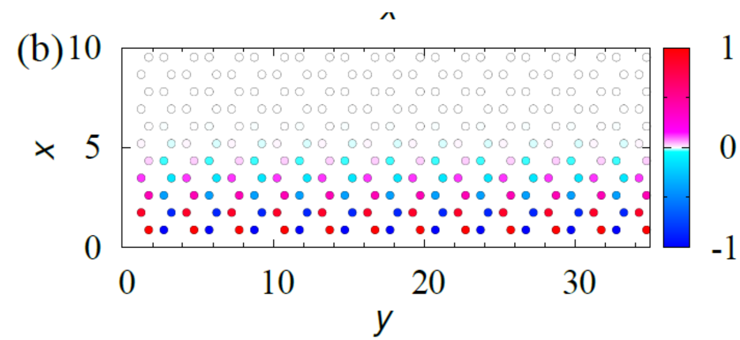
Main results





Classical diffusion phenomena:
a new platform of topological edge modes





TY-Hatsugai, Sci. Rep. **11**, 888 (2020)

Key idea

Generic diffusion: Fick's law

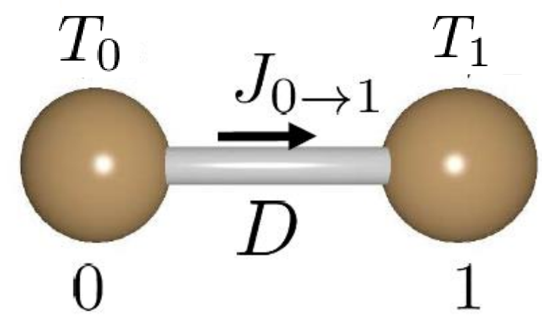
Heat conduction of two sites

- Newton's law

$$J_{0\to 1} = D(T_0 - T_1)$$

- continuity equation

$$\frac{\partial T_0}{\partial t} + J_{0\to 1} = 0$$

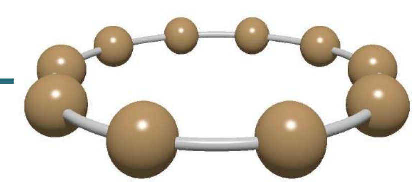


Discretized heat conduction equation (Building block)

$$\frac{\partial}{\partial t}\vec{T} = -D\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\vec{T}$$

$$\vec{T} = \begin{pmatrix} T_0 \\ T_1 \end{pmatrix}$$

Generic lattice



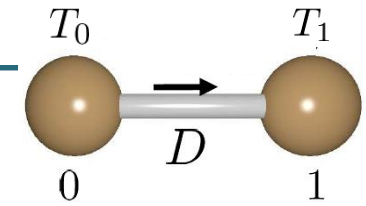
$$\frac{\partial}{\partial t}\vec{T} = -\hat{H}\vec{T}$$

Matrix $\hat{H}$ mimics a tight-binding model

Example: 1D chain

Example: 2-sites to 3-sites

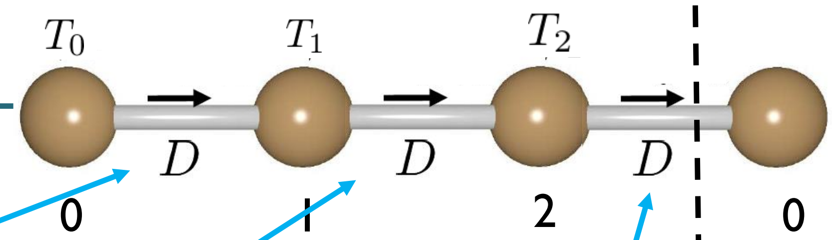
2-sites



$$\frac{\partial}{\partial t}\vec{T} = -D\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\vec{T}$$

$$\vec{T} = \begin{pmatrix} T_0 \\ T_1 \end{pmatrix}$$

3-sites

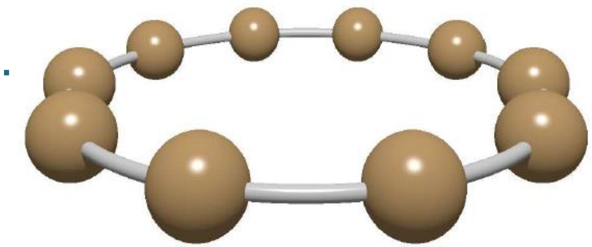


$$\partial_t\vec{T}(t) = -D[\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}]\vec{T}(t).$$

$$\Rightarrow \frac{\partial}{\partial t}\vec{T}(t) = -D\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\vec{T}(t).$$

Example: L-site chain

Discretized heat conduction equation (L-sites)



$$\frac{\partial}{\partial t}\vec{T} = -\hat{H}\vec{T}$$

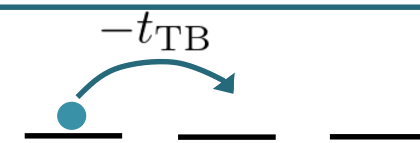
$$\hat{H} = D\begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

correspondence

$$t_{\mathrm{TB}} = D$$

$$\mu_{\mathrm{TB}} = -2D$$

Tight-binding model
(quantum)



μ_{TB} :on-site potential

$$H_{\mathrm{TB}} = t_{\mathrm{TB}}\begin{pmatrix} 0 & -1 & 0 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix} - \mu_{\mathrm{TB}}\mathbb{1}$$

Matrix $\hat{H}$ mimics a tight-binding model !

Example: L-site chain

~Reproducing the ordinary heat conduction equation~

Real-space

$$\frac{\partial}{\partial t}\vec{T} = -\hat{H}\vec{T}$$

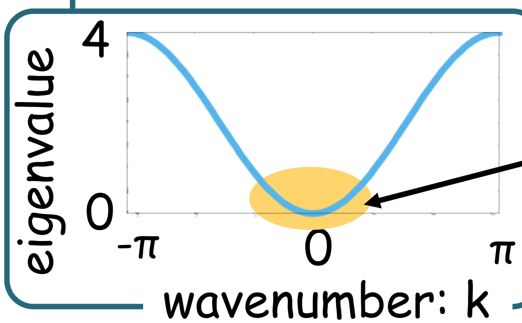
Fourier transformation

k-space

$$\frac{\partial}{\partial t}\vec{T}_k = -D(2-2\cos k)\vec{T}_k$$

$k \to 0$

$$\frac{\partial}{\partial t}\vec{T}_k = -Dk^2\vec{T}_k$$



$k \leftrightarrow i\partial_x$

Ordinary heat conduction eq:

$$\frac{\partial}{\partial t}\vec{T}_x = D\frac{\partial^2}{\partial x^2}\vec{T}_x$$

Key idea (short summary)

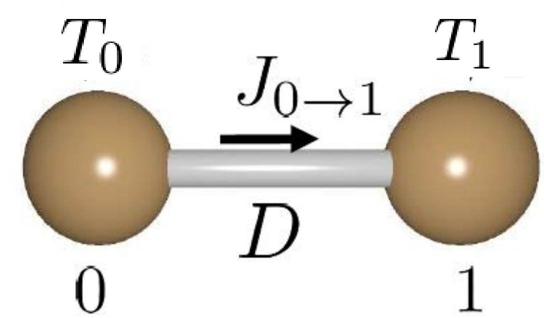
Heat conduction of two sites

- Newton's law

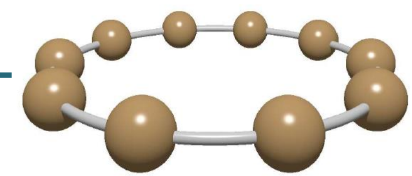
$$J_{0\to1} = D(T_0 - T_1)$$

- continuity equation

$$\frac{\partial T_0}{\partial t} + J_{0\to1} = 0$$



Generic lattice

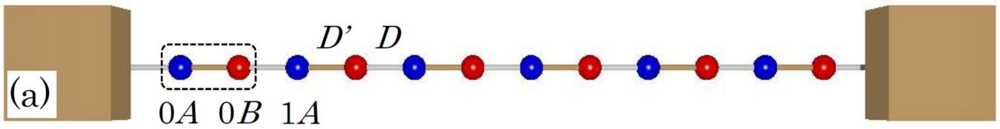


$$\frac{\partial}{\partial t}\vec{T} = -\hat{H}\vec{T}$$

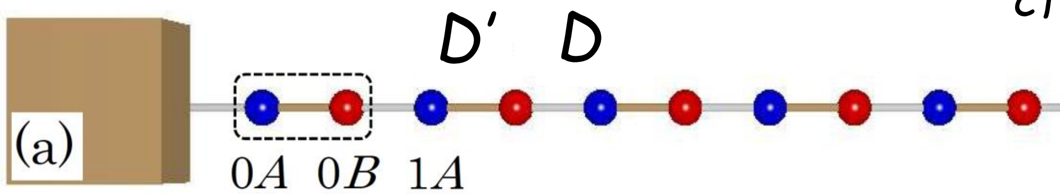
Matrix $\hat{H}$ mimics a tight-binding model

Topological edge modes of heat conduction

~ Su-Schrieffer-Heeger model ~



Counter-part of Su-Schrieffer-Heeger model



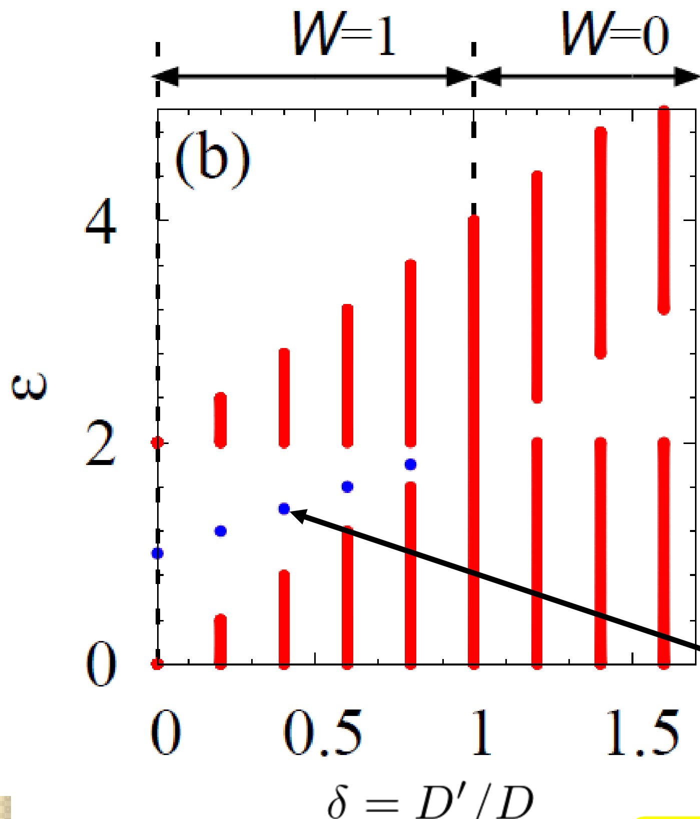
cf. SSH model (quantum)

$$h_{\rm Q}(k) = \begin{pmatrix} 0 & V + te^{-ik} \\ \text{c.c.} & 0 \end{pmatrix}$$

$$\frac{\partial \vec{T}_k}{\partial t} = -h_{\rm SSH}(k)\vec{T}_k$$

$\delta = D'/D$

$$\hat{h}_{\rm SSH}(k_x) = D \begin{pmatrix} 1+\delta & \delta + e^{ik_x} \\ \delta + e^{-ik_x} & 1+\delta \end{pmatrix}$$



Winding number

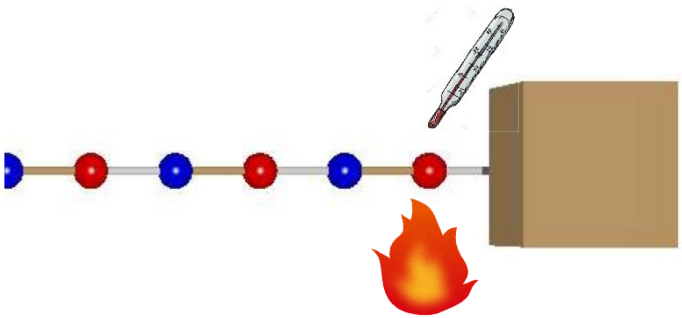
$$W = -\int_{-\pi}^{\pi} \frac{dk_x}{4\pi i} \mathrm{tr}[\sigma_3 \hat{h}'^{-1}_{\rm SSH}(k_x) \partial_{k_x} \hat{h}'_{\rm SSH}(k_x)].$$

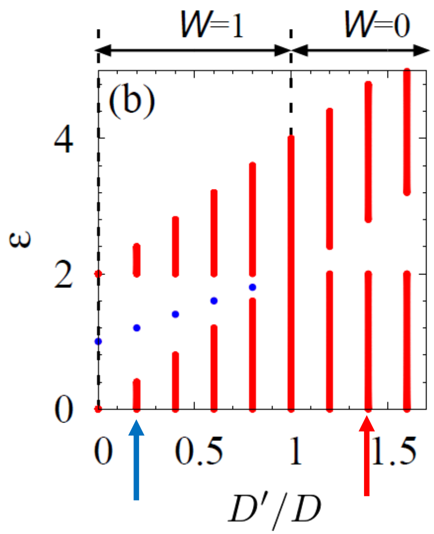
Edge states emerge
due to topology in the bulk

Bulk-edge correspondence of heat conduction

Q:

How to experimentally access the edge modes?

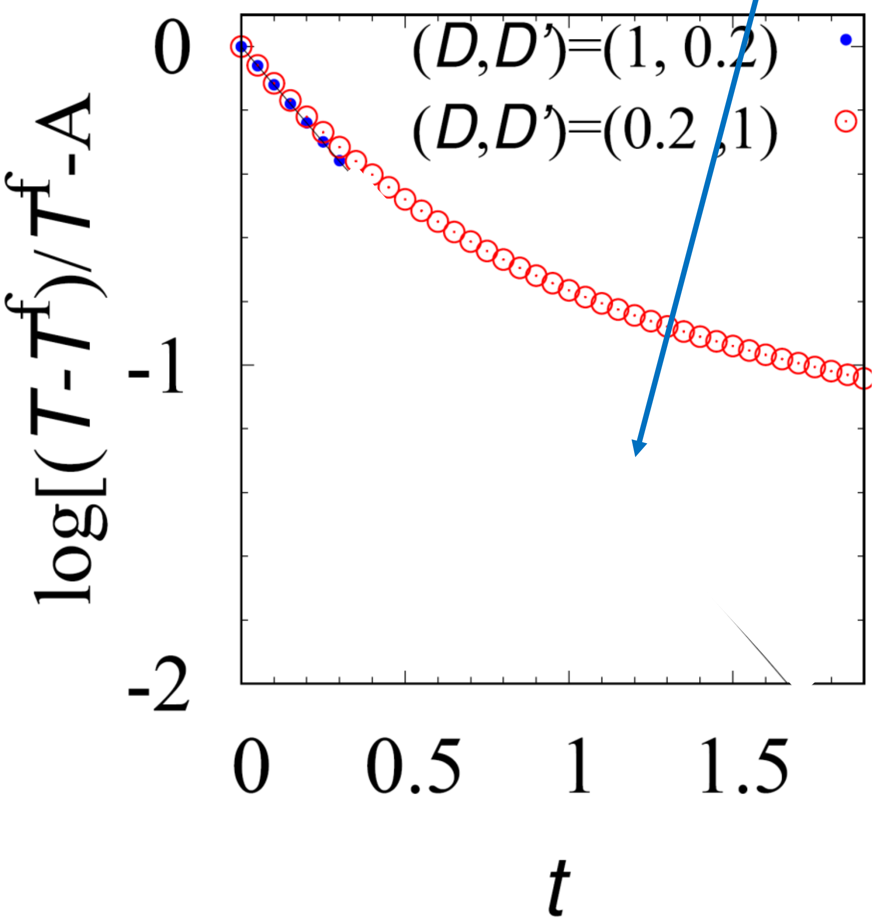




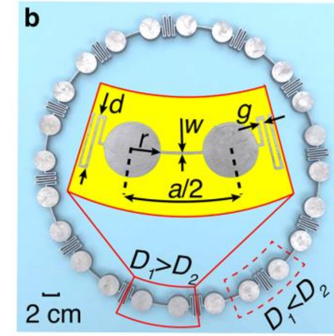
A

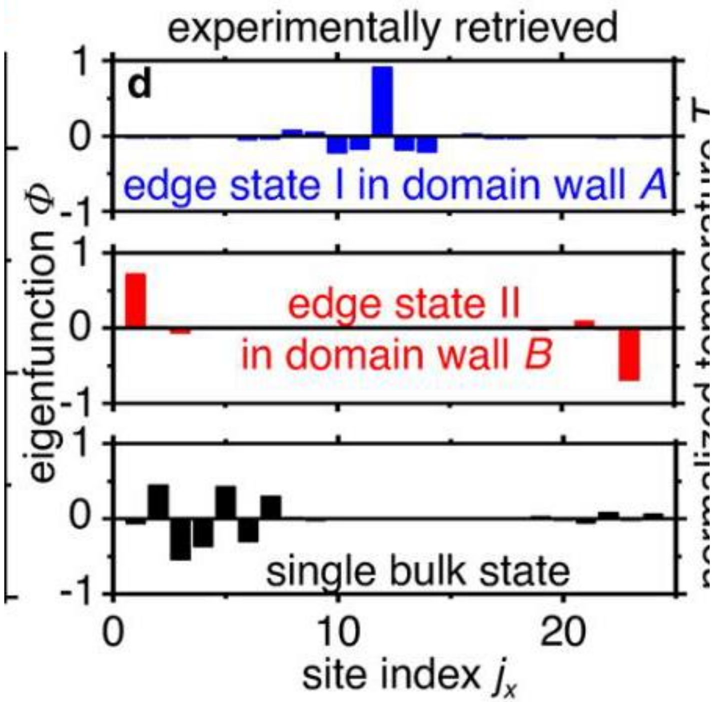
$T_{\mathrm{edge}} \sim e^{-(D+D')t}$

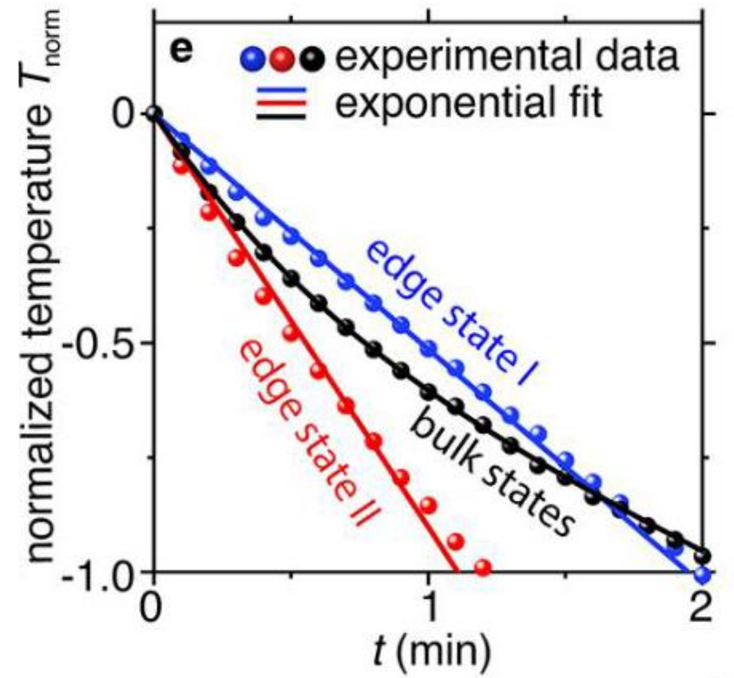
exponential decay governed by edge states



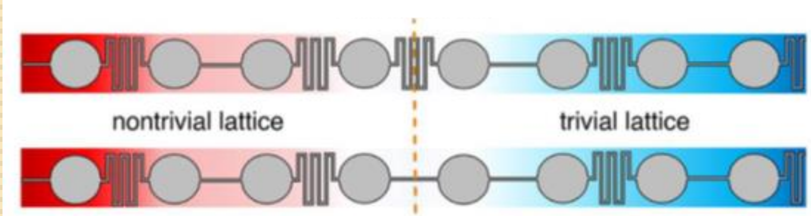
Topological edge modes are experimentally observed







Domain wall A



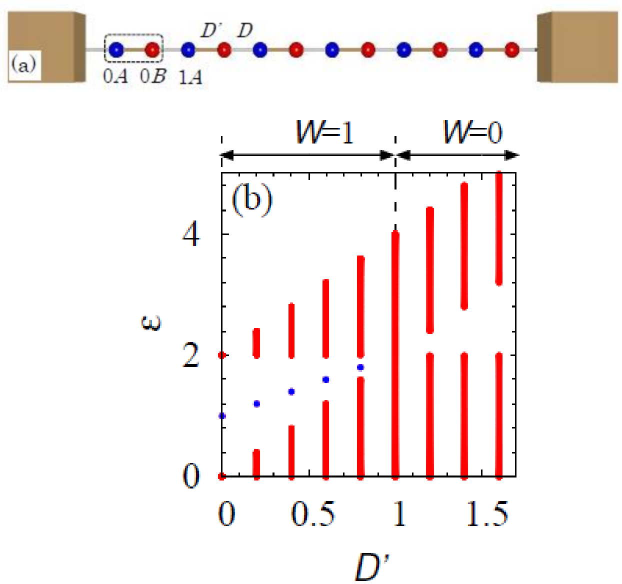
Domain wall B

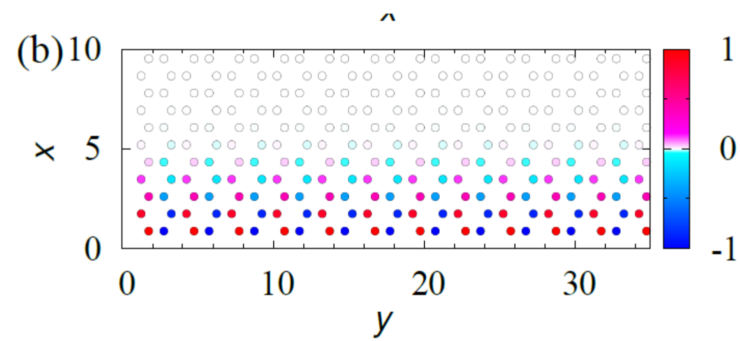
M. Qi, et al. arXiv:2107.05231

H. Hu, et al., arXiv:2107.05811

Summary of part I

Classical diffusion phenomena:
a new platform of topological edge modes





TY-Hatsugai, Sci. Rep. **11**, 888 (2020)

Outline

1. Warm up: topological insulators

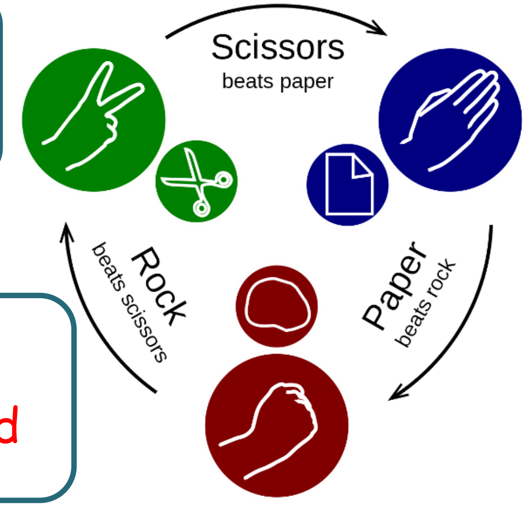
2. Motivation

3. Topological edge modes in classical diffusion phenomena

4. Chiral edge state in evolutionary game theory

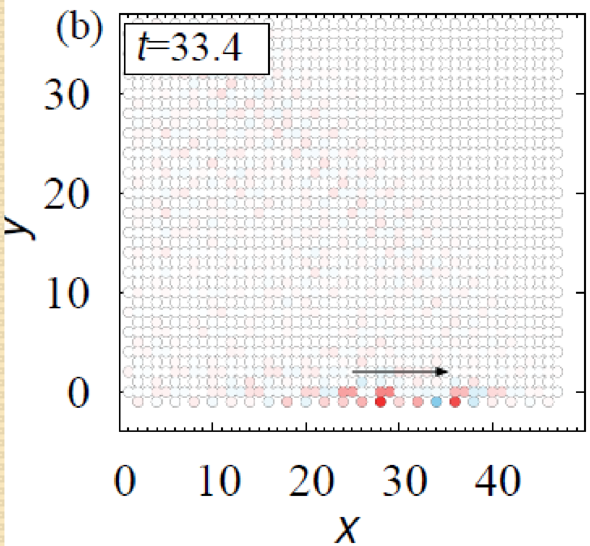
Main results

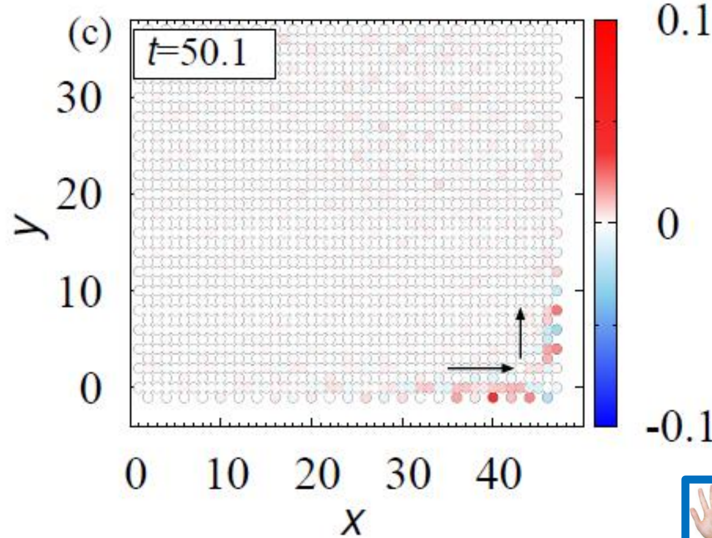
- bacterium (biology)
- human behaviors (social science)

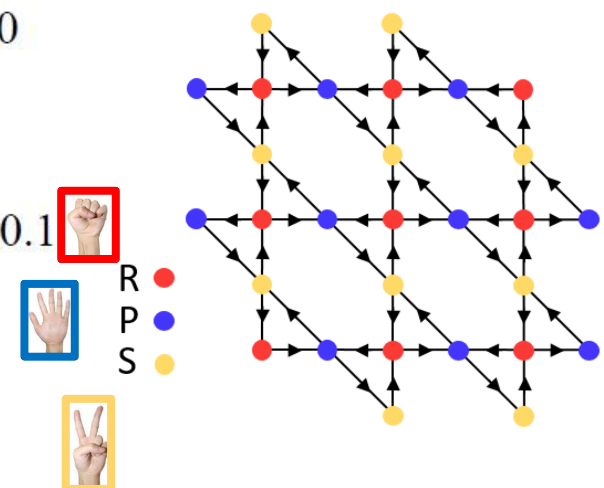


(from Wikipedia)

When players play a type of rock-paper-scissors game, a chiral edge mode is observed







cf: (Kitaev chain)

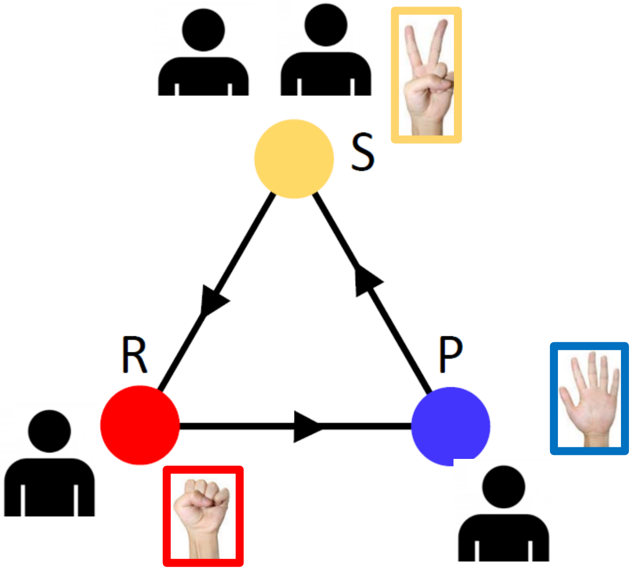
J. Knebel et al., PRL(2020)

TY-Mizoguchi-Hatsugai, PRE (2021)

Warm up:

(じゃんけん)

game theory of rock-paper-scissors



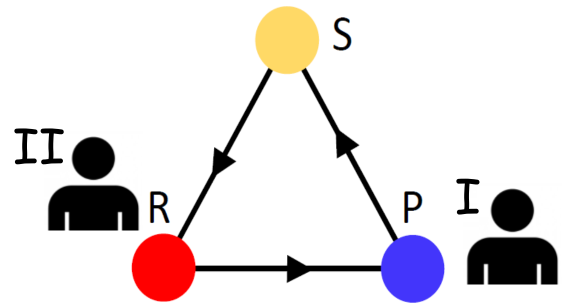
Game theory of rock-paper-scissors

(じゃんけん)

Opponent (Player II)

Player I		 Rock	 Paper	 Scissors
	Rock	Draw	Lose	Win
	Paper	Win	Draw	Lose
	Scissors	Lose	Win	Draw

Win/Lose for player I



Arrows:
flow of payoffs

payoff matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{array}{l} R \\ P \\ S \end{array}$$

2-players → many players

Expectation value of payoff

$$e_1^T A x$$

Player choosing "R"

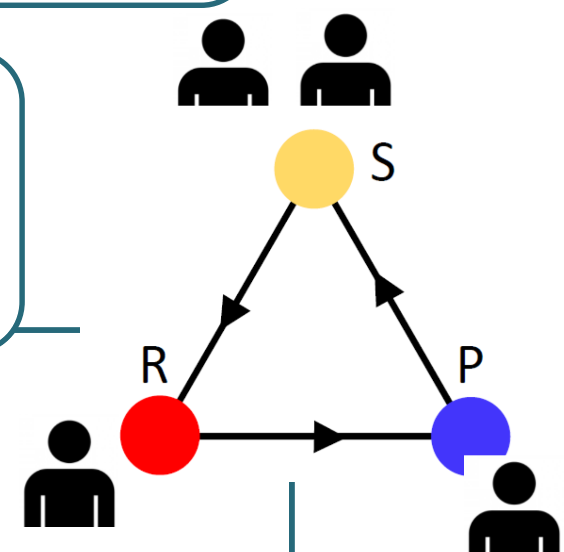
density of players

$$\boldsymbol{x} = \begin{array}{c} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{array} \begin{array}{c} \text{R} \\ \text{P} \\ \text{S} \end{array}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

payoff matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{array}{c} \text{R} \\ \text{P} \\ \text{S} \end{array}$$



Example (Expectation value)

R	$e_1^T A x = \frac{1}{4}$
P	$e_2^T A x = -\frac{1}{4}$
S	$e_3^T A x = 0$

$$\boldsymbol{x} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} \begin{array}{c} \text{R} \\ \text{P} \\ \text{S} \end{array}$$

$$A\boldsymbol{x} = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 0 \end{pmatrix}$$

Arrows:
flow of payoffs

Repeating the RPS game

Players move to obtain higher payoff

e.g., bacterium

Time-evolution of the population

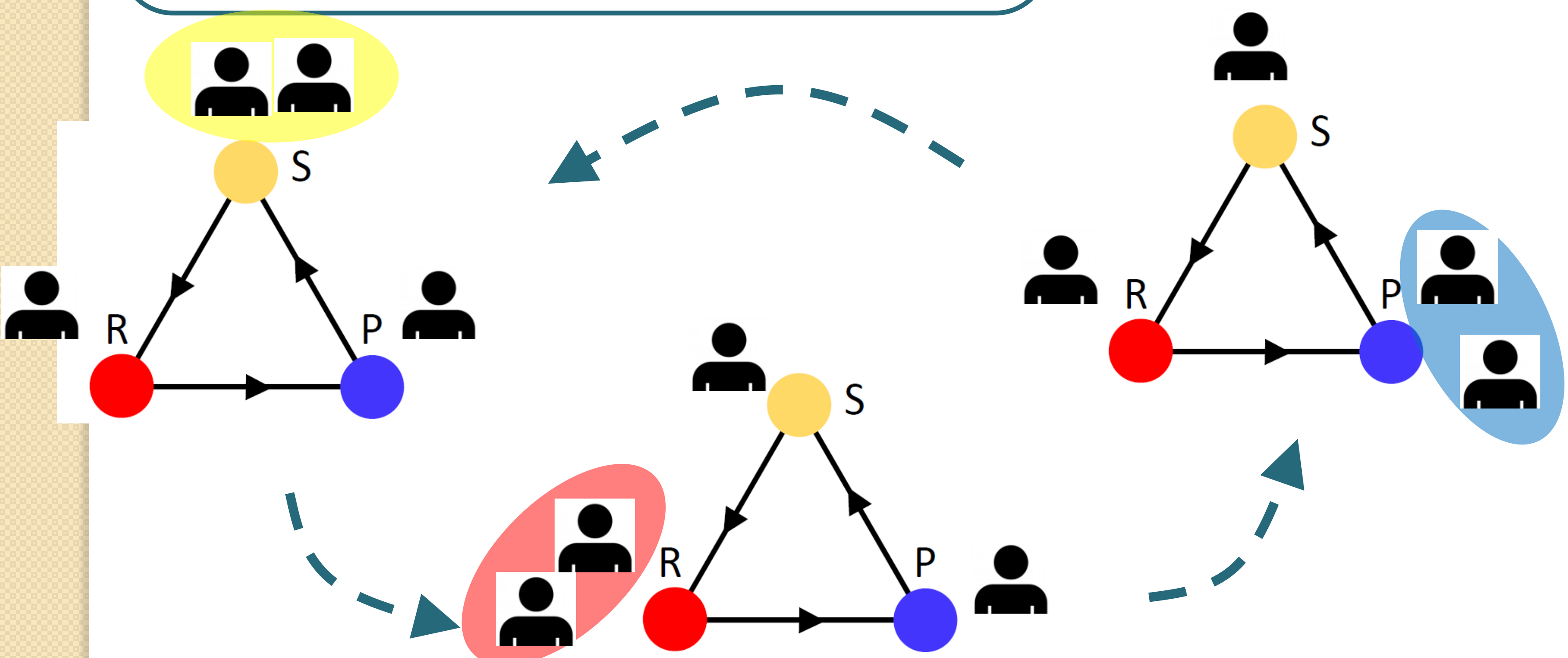
$$\partial_t x_I = x_I \boxed{\boldsymbol{e}_I^T A \boldsymbol{x}}$$

density of players

Expectation value of payoffs

payoff matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$



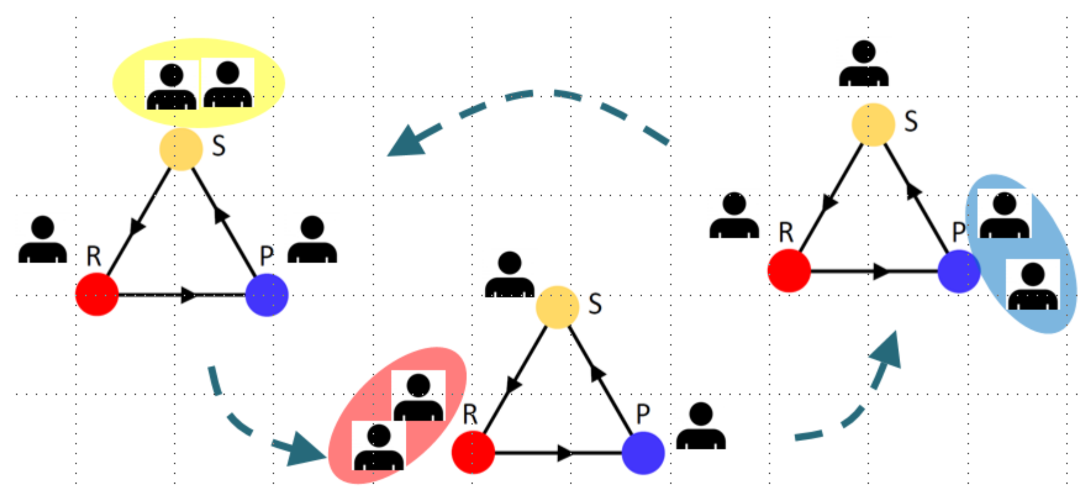
Short summary

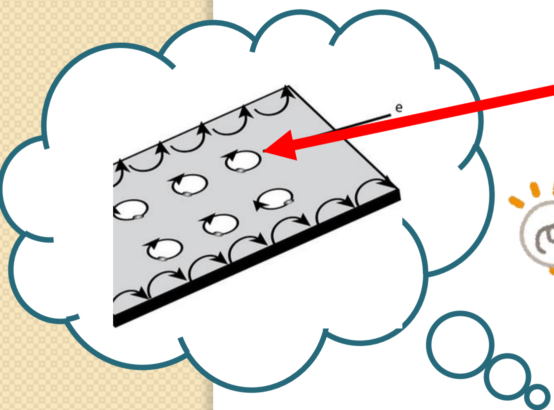
Time-evolution of the population

$$\partial_t x_I = x_I \mathbf{e}_I^T A \boldsymbol{x}$$

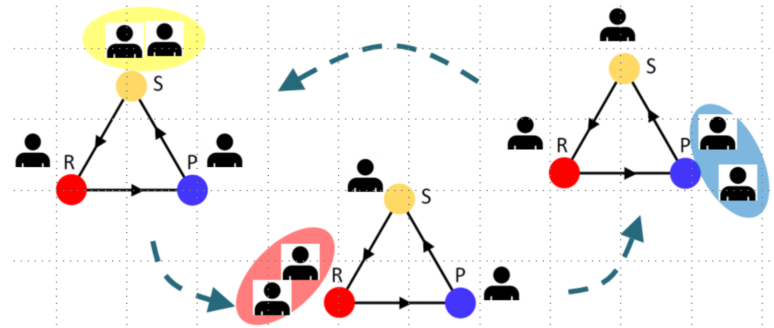
unit vector　　density of players

cycric motion of rock-paper-scissors

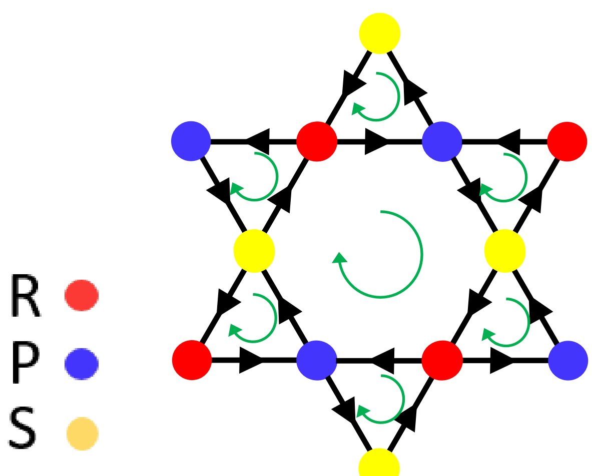


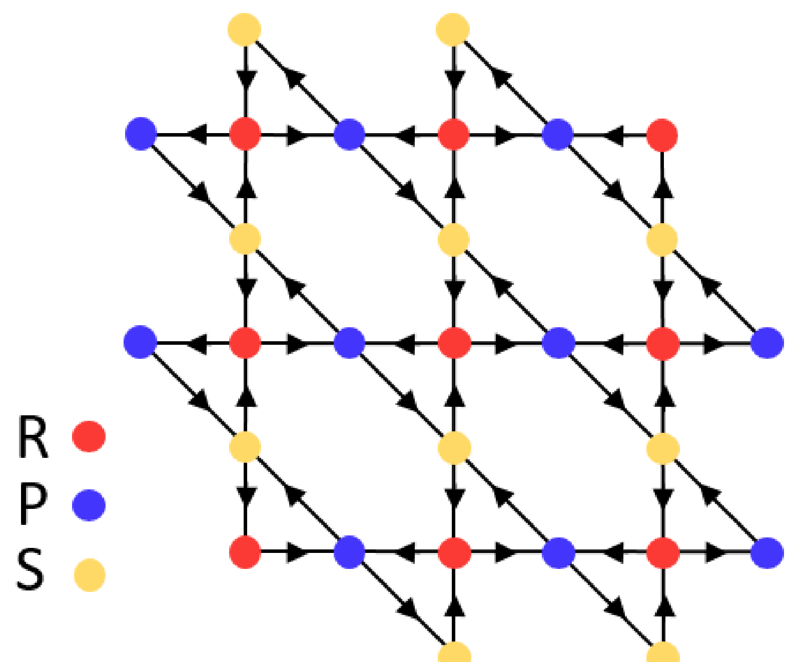


"cyclotron motion" of rock-paper-scissors!



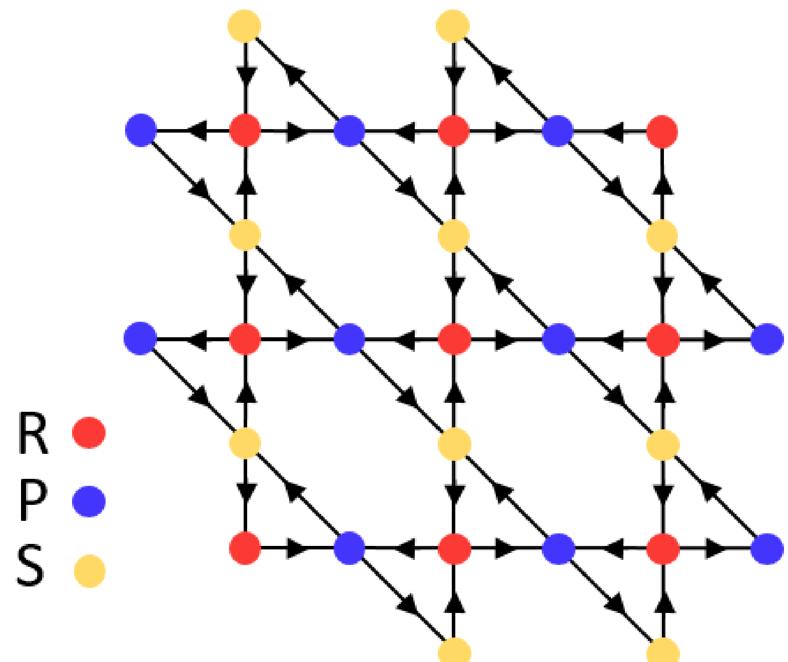
Let's arrange a network of rock-paper-scissors!





cf: (fermion model)
K. Ohgushi et al., PRB (2000)

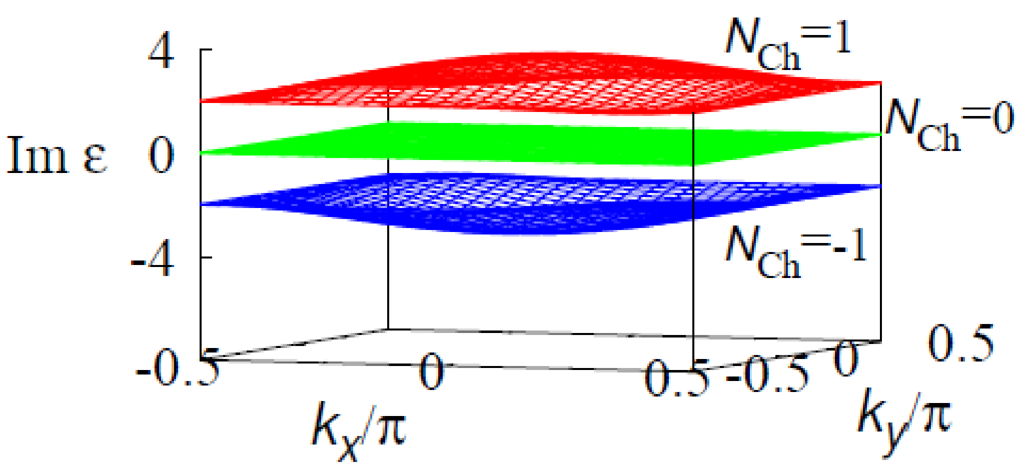
Kagome network of rock-paper-scissors



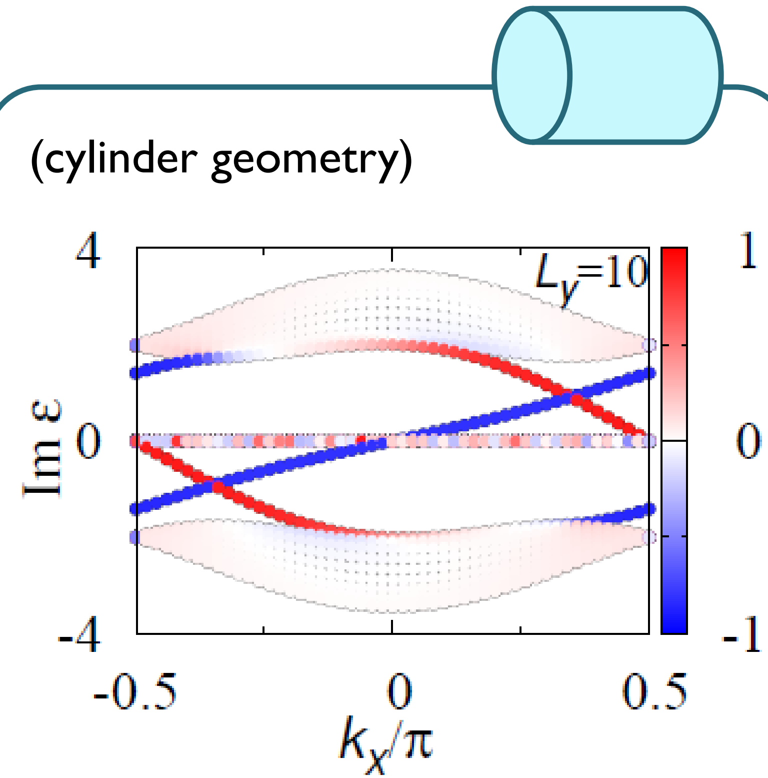
Topological properties

$$\partial_t x_I = x_I \boldsymbol{e}_I^T A \boldsymbol{x}$$

Band structure of payoff matrix *A* (PBC)



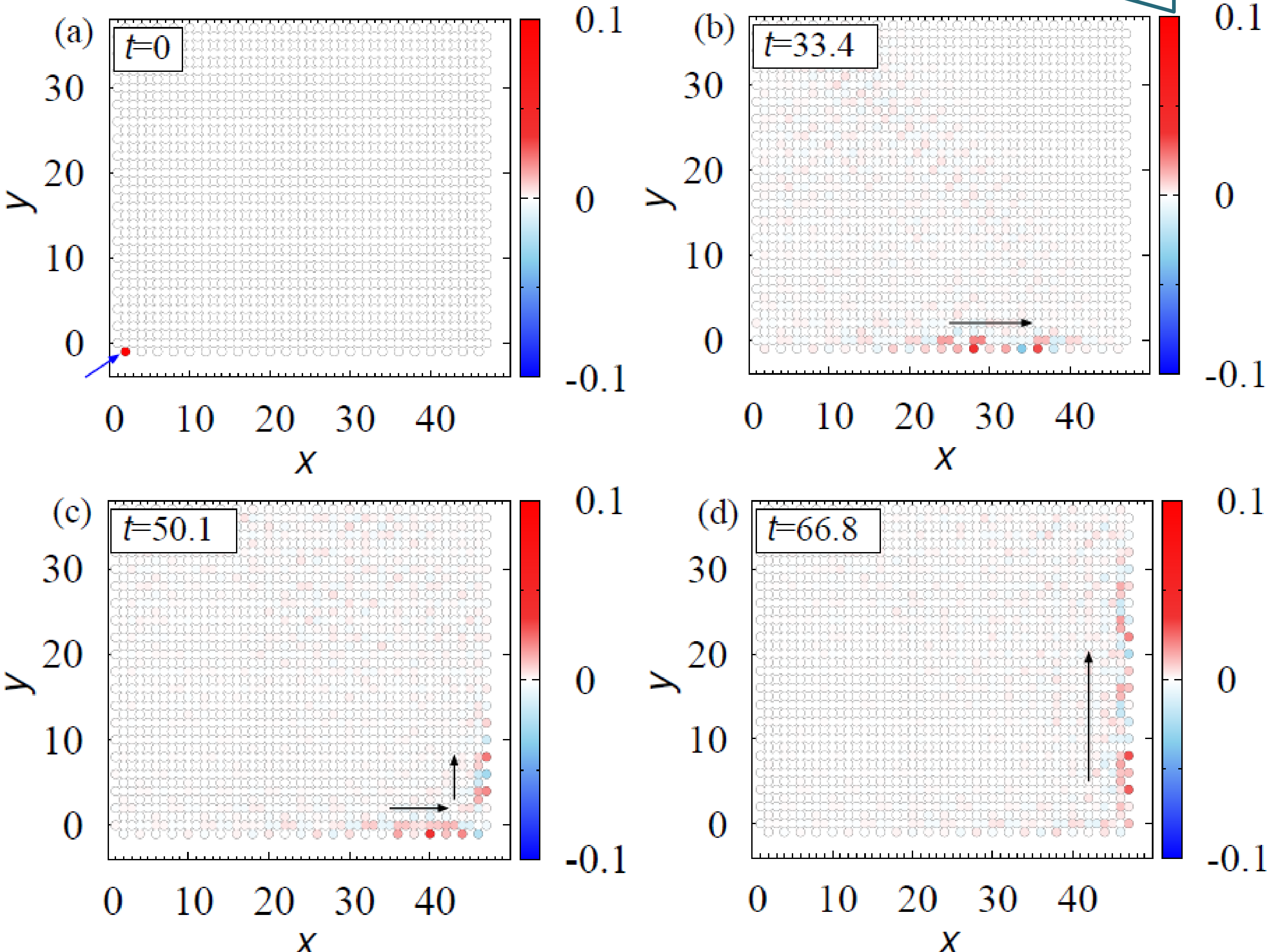
(cylinder geometry)



The payoff matrix *A* is topological !

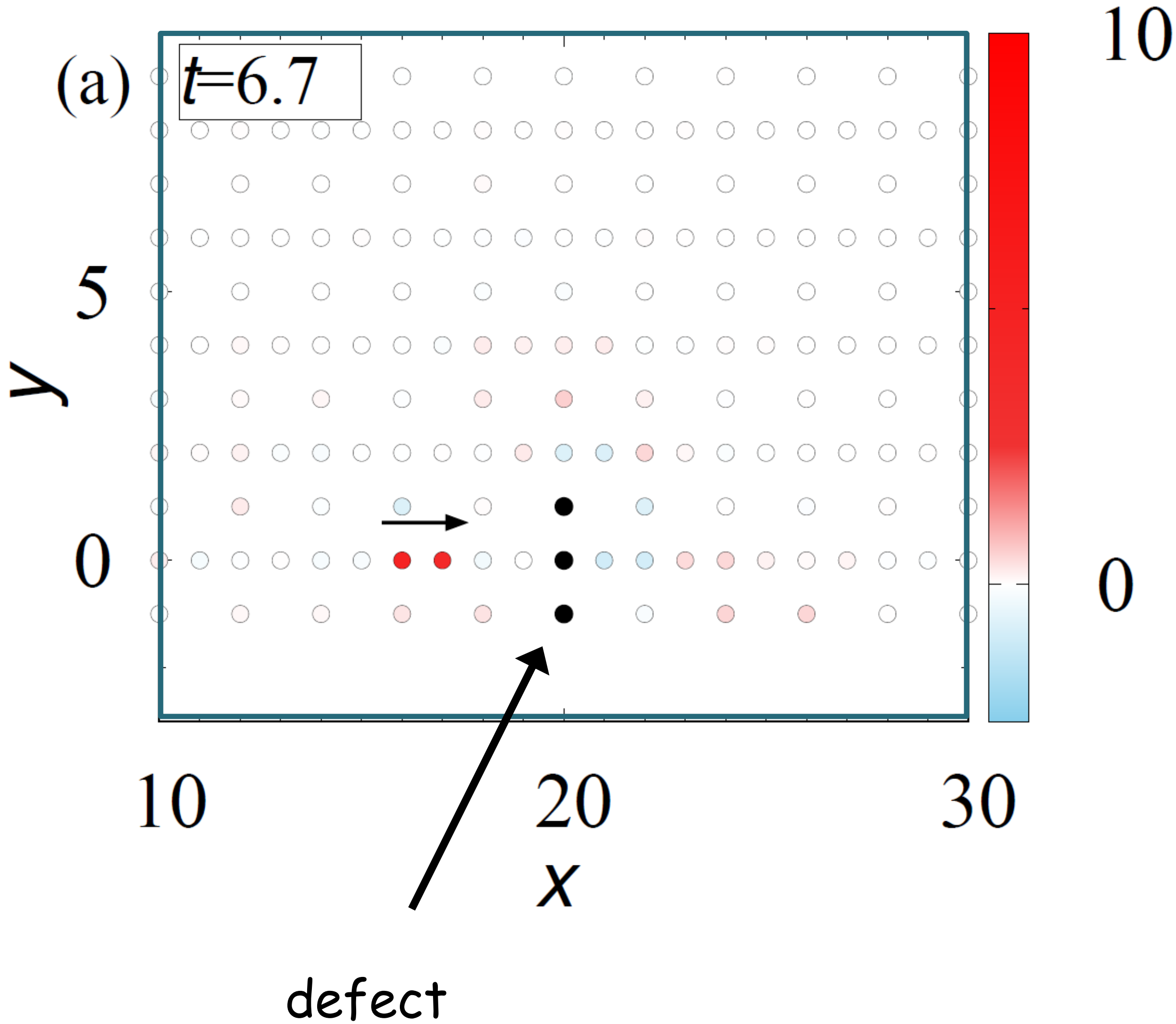
Unidirectional propagation

Deviation of population from a stationary state δx

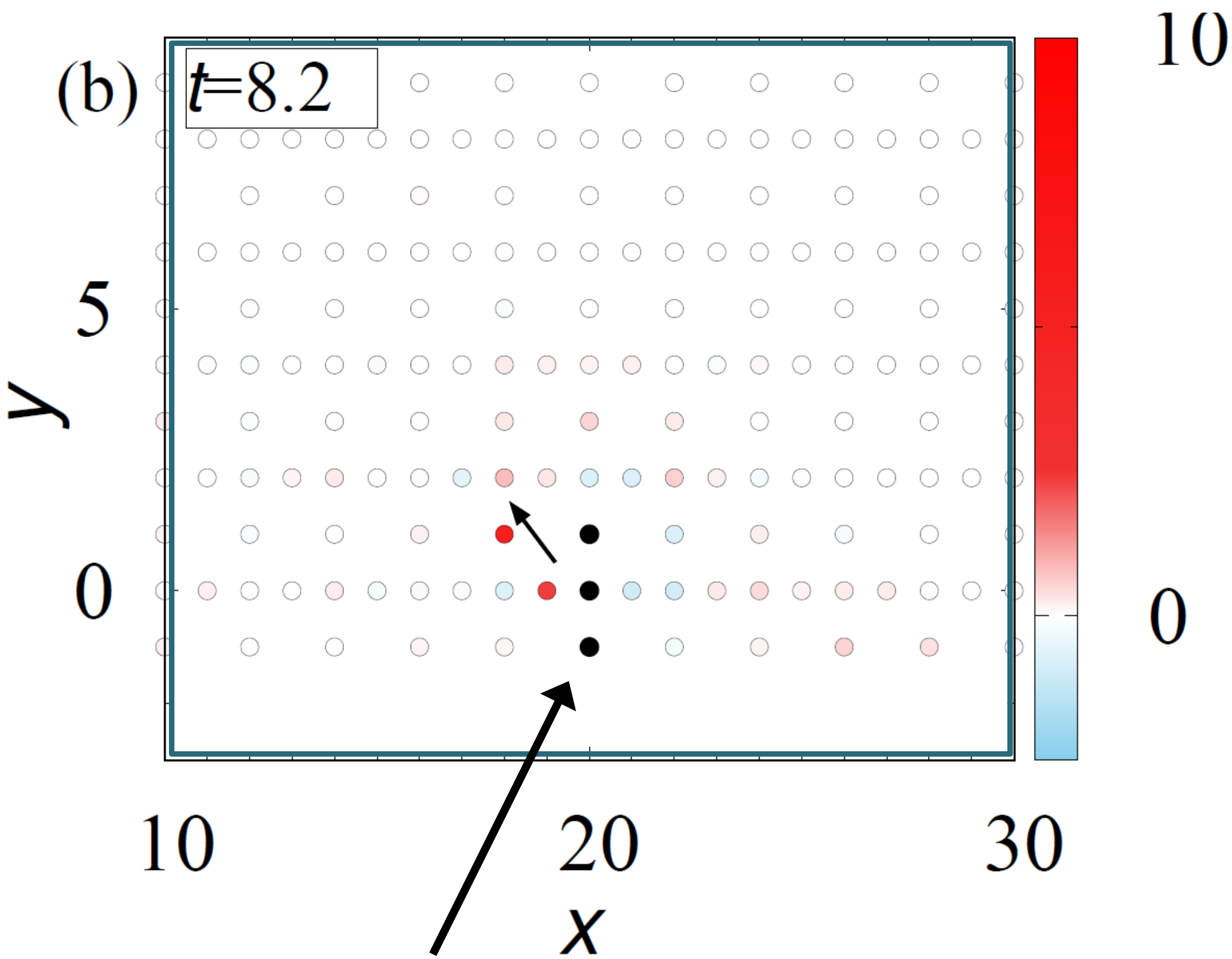


Players show unidirectional propagation

Unidirectional propagation with a defect

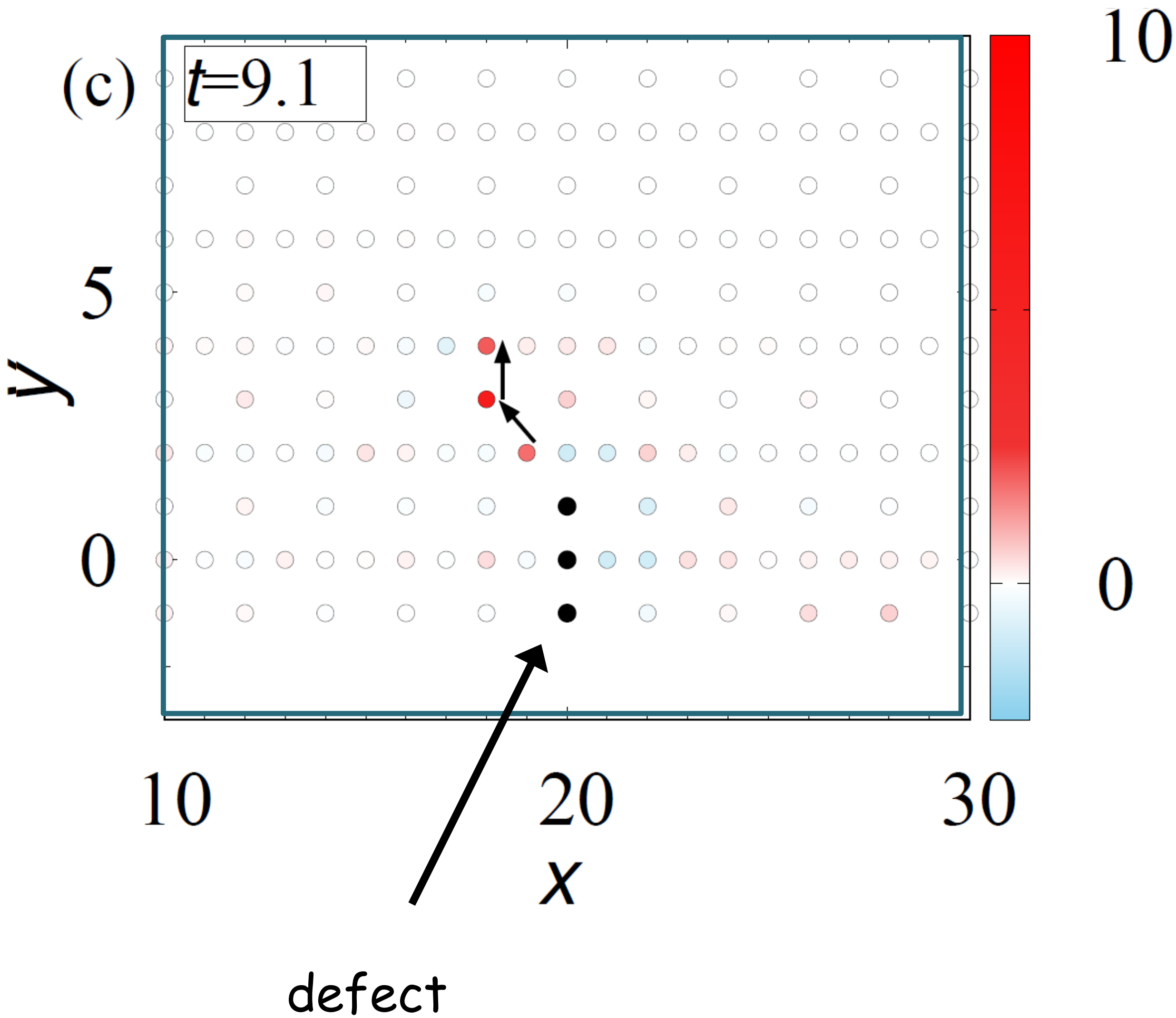


Unidirectional propagation with a defect

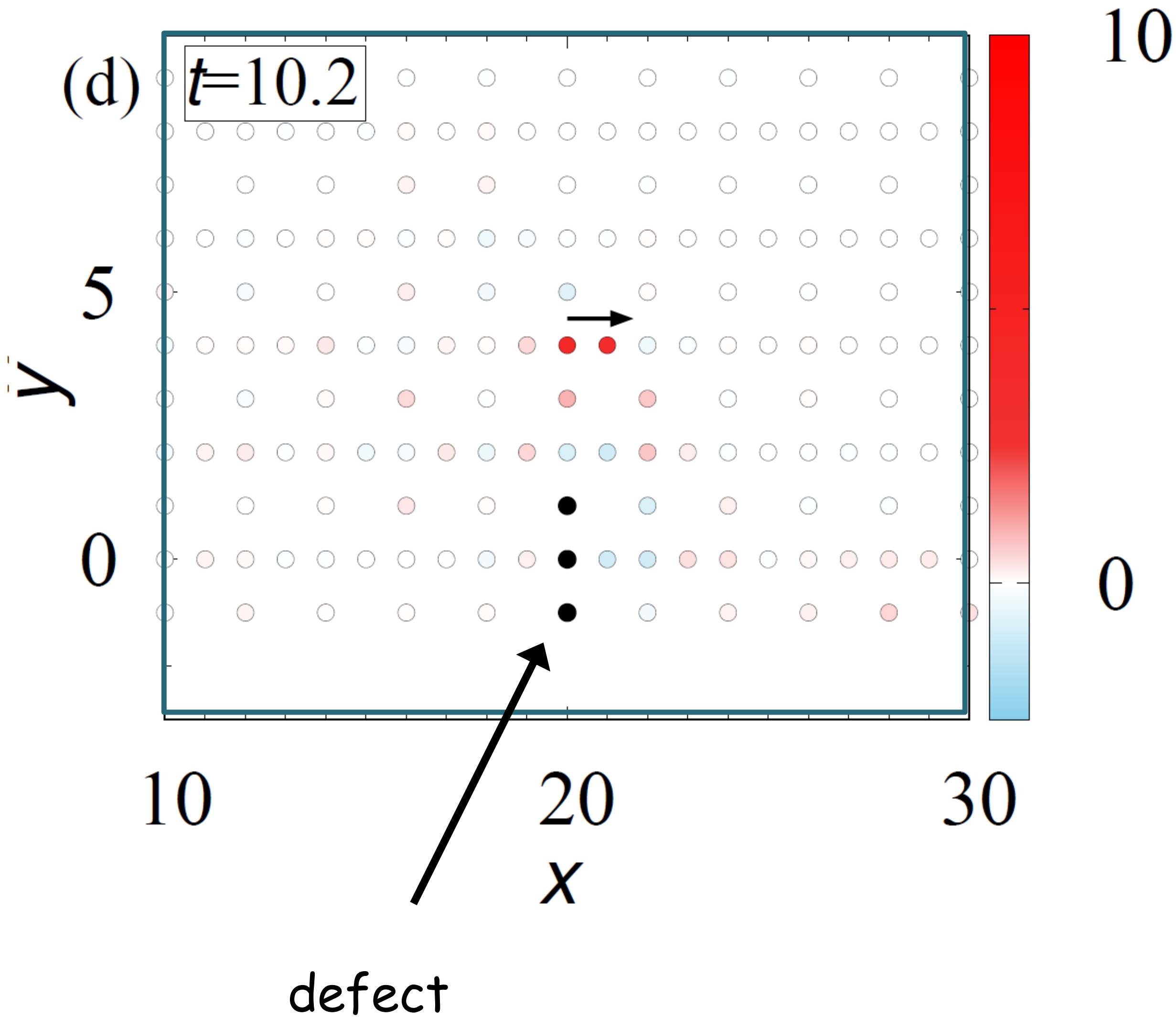


defect

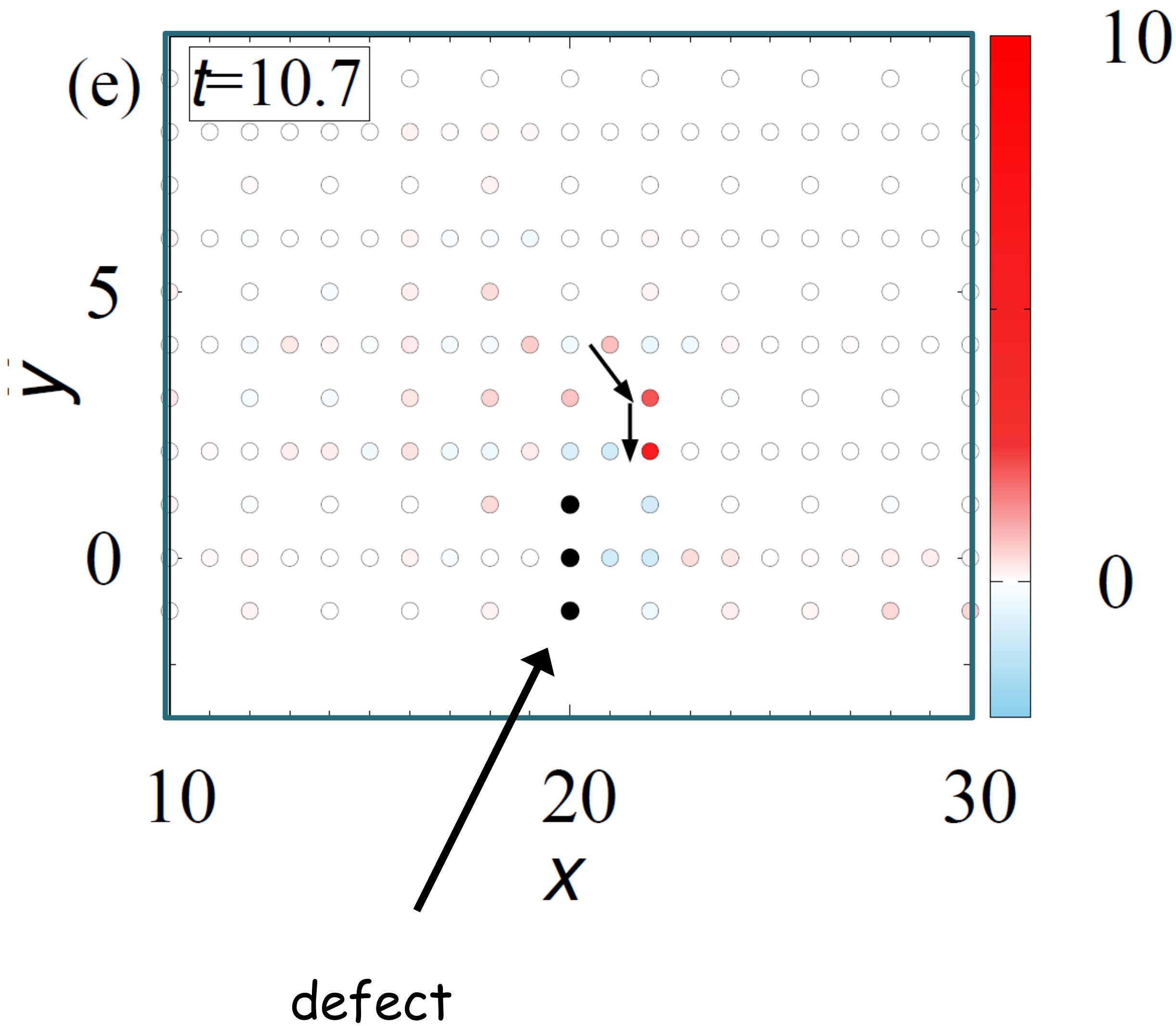
Unidirectional propagation with a defect



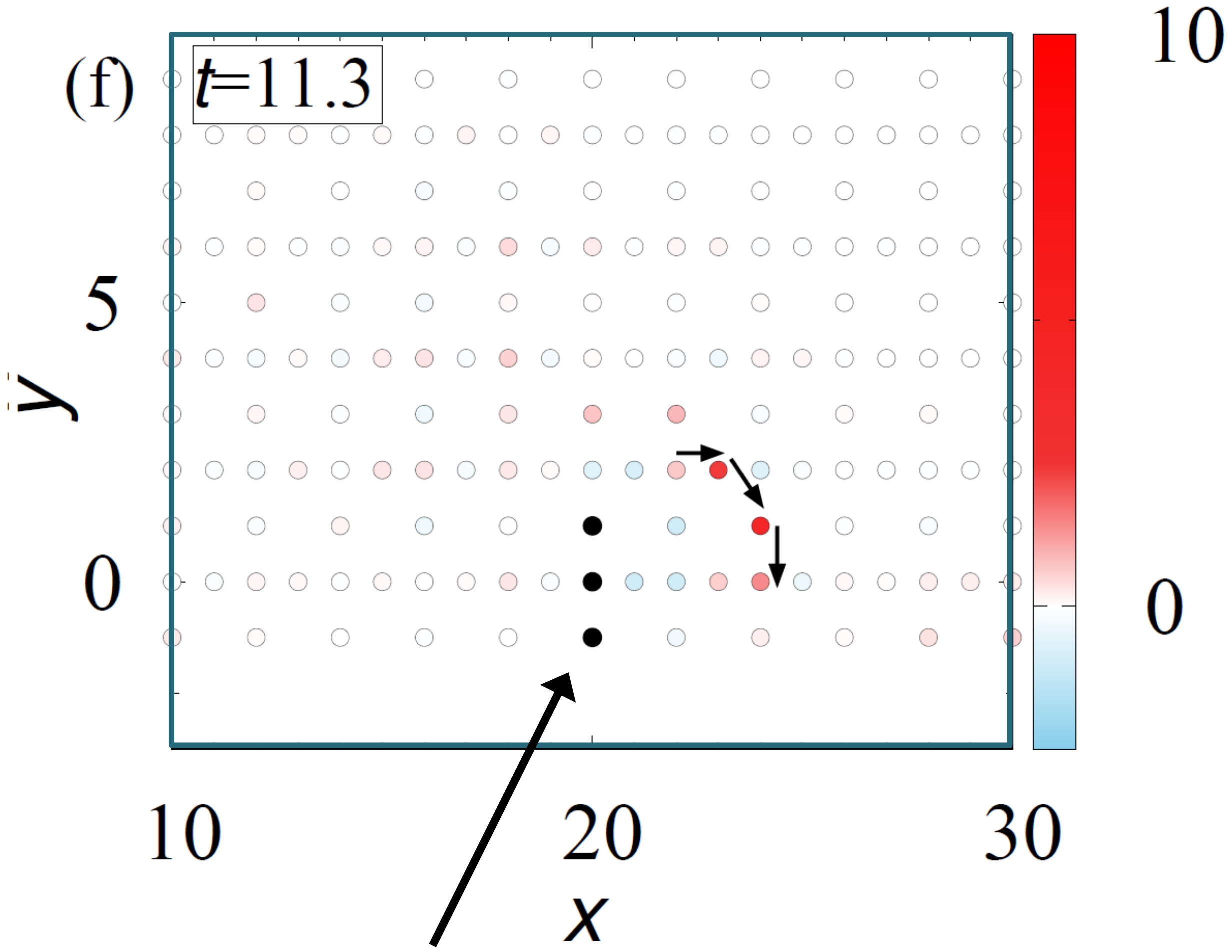
Unidirectional propagation with a defect



Unidirectional propagation with a defect

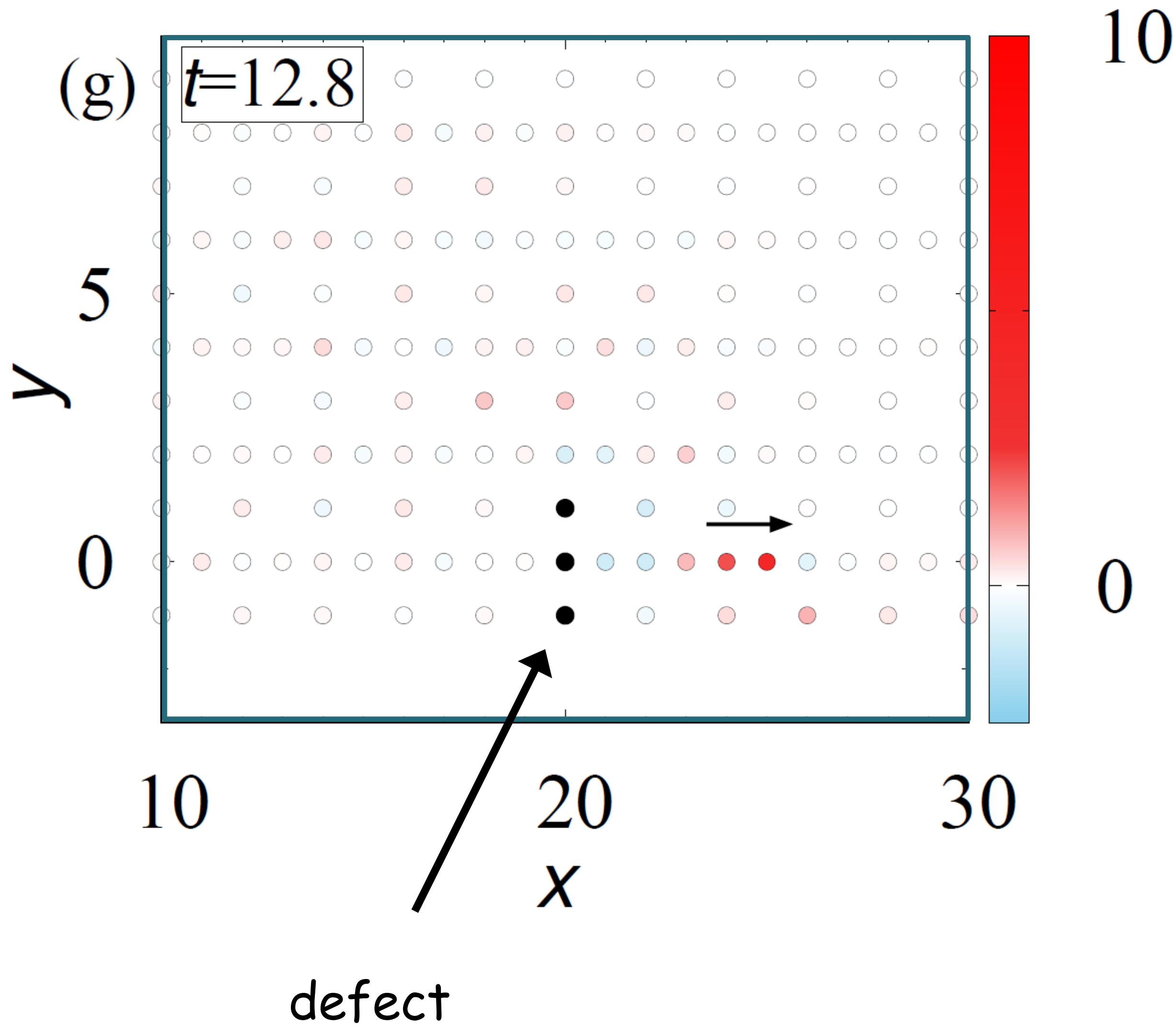


Unidirectional propagation with a defect

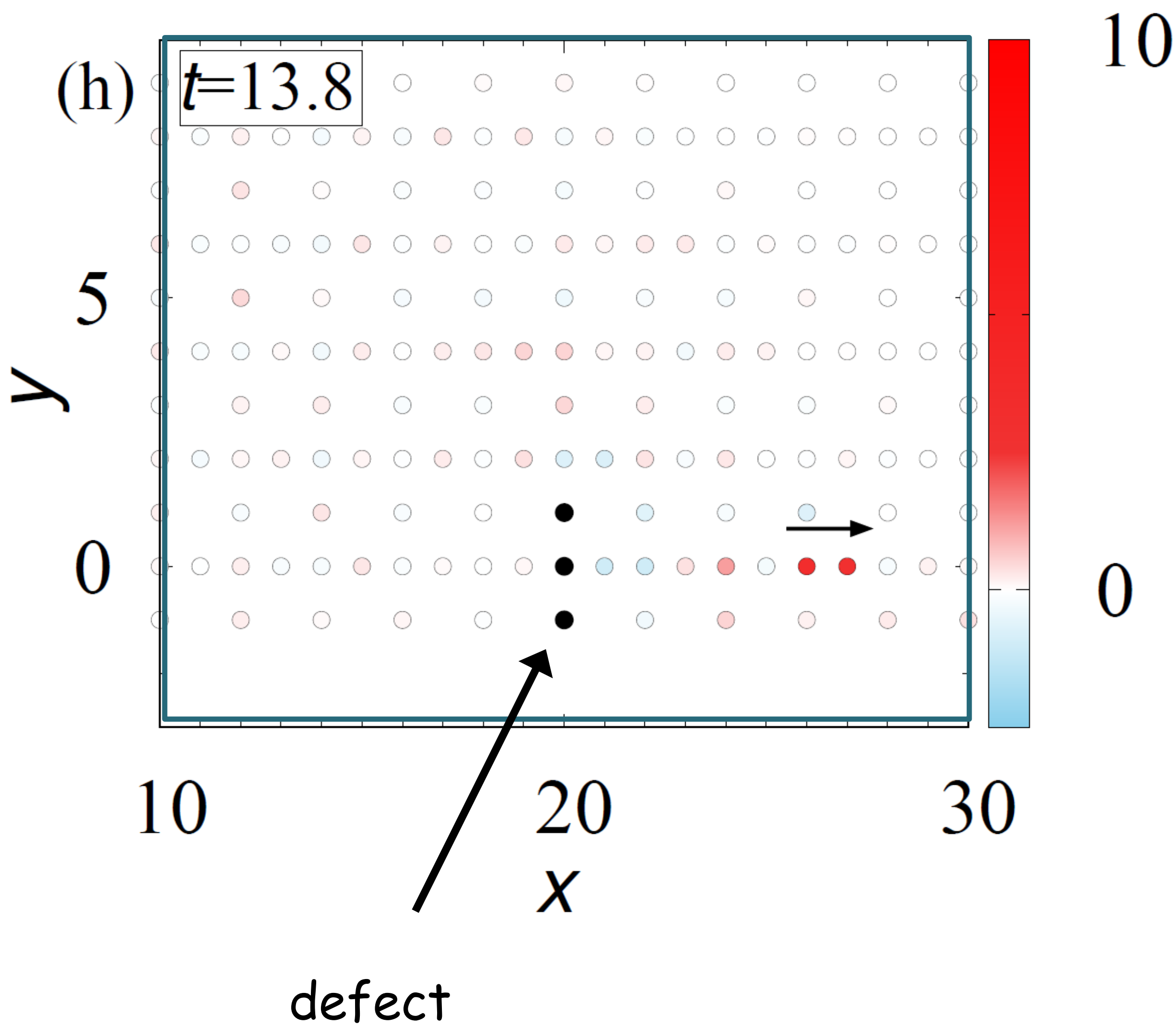


defect

Unidirectional propagation with a defect



Unidirectional propagation with a defect

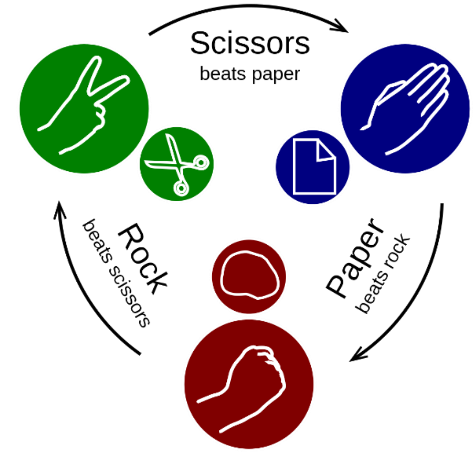


The chiral edge mode is robust!

Summary of part II

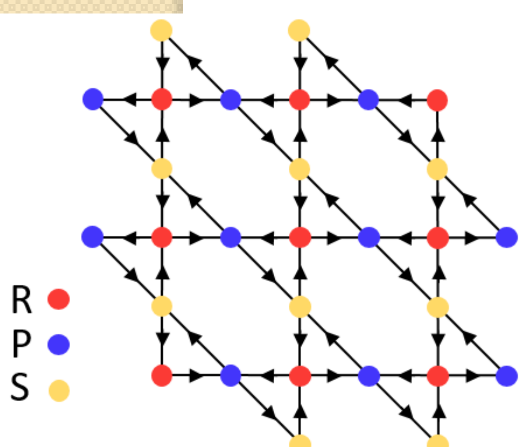
*bacterium (biology)
*human behaviors
(social science)

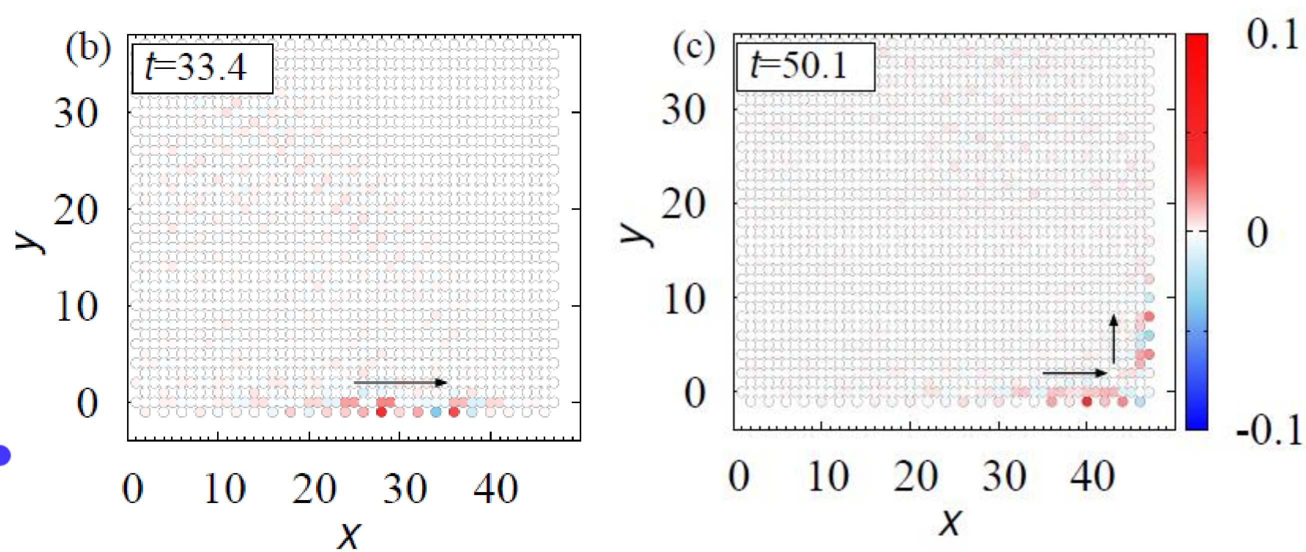
(from Wikipedia)



Chiral edge modes in evolutionary game theory:

When players play a type of rock-paper-scissors game, a chiral edge mode is observed





TY-Mizoguchi-Hatsugai, PRE (2021)

Thank you !