MODULE CORRESPONDENCES IN ROUQUIER BLOCKS OF FINITE GENERAL LINEAR GROUPS

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1. Introduction

[1] Let ℓ be a prime number. Let $\mathcal{O}$ be a complete discrete valuation ring with fraction field K of characteristic 0 and residue field $\Bbbk = \mathcal{O}/J(\mathcal{O})$ of characteristic ℓ. We assume that K is big enough for all groups in this paper. Let $GL_n(q)$ be the general linear group over field of q elements, where q is a prime power. We assume that ℓ dose not divide q. Let e be the multiplicative order of q in $\Bbbk$. The unipotent blocks of $\mathcal{O}GL_n(q)$ are parametrized by e-weights (semisimple parts) and e-cores (unipotent parts) [FS82]. Let $B_{w,\rho}$ be the unipotent block of $\mathcal{O}GL_n(q)$ with e-weight w and e-core ρ.

Let ρ be an e-core satisfying the following property : ρ has an (e-runner) abacus representation such that $\Gamma_{i-1} + w - 1 \leq \Gamma_i$ where Γ_i is the number of beads on the i-th runner [Rou98],[CK02]. Let $m = ew + |\rho|$. Let $N_w = (GL_e(q) \wr \mathfrak{S}_w) \times GL_{|\rho|}(q)$. Let f be the block idempotent of $B_0(\mathcal{O}(GL_e(q) \wr \mathfrak{S}_w)) \otimes B_{0,\rho}$ where $B_0(\mathcal{O}(GL_e(q) \wr \mathfrak{S}_w))$ is the principal block of the wreath product $GL_e(q) \wr \mathfrak{S}_w$. Let D be a Sylow ℓ-subgroup of $GL_e(q) \wr \mathfrak{S}_w$. Then, as an $\mathcal{O}(GL_m(q) \times N_w)$-module, $B_{w,\rho}f$ has a unique indecomposable direct summand X_w with vertex $\Delta D = \{(d, d) \mid d \in D\}$. W.Turner [Tur02]proved the following which is analogoues to the result of Chuang and Kesser for symmetric groups [CK02].

Theorem 1.1. *If $w < \ell$, then a $(B_{w,\rho}, \mathcal{O}N_w f)$-bimodule X_w induces a Morita equivalence between $\mathcal{O}N_w f$ and $B_{w,\rho}$.*

This result was proved by the second author independently [Miy01].

Note that using this result, combining with [Mar96],[Rou95] and [Chu99], one can prove Broué's conjecture [Bro90], [Bro92] for weight two unipotent blocks of finite general linear groups very easily.

In this paper, we consider the correspondences of various modules under the equivalence above. In section 3, we consider simple modules. In section 4, we treat Young modules and Specht type modules. Throughout the paper, we assume $w < \ell$. The main results of this paper, namely, Corollary 3.3 and Theorem 4.9, enable us to calculate not only decomposition numbers of $B_{w,\rho}$ but also the radical series of Specht type modules lying in $B_{w,\rho}$. These graded decomposition numbers are calculated in [Miy01] explicitly in terms of the Littlewood-Richardson coefficients. (Moreover, we can also calculate the Loewy layers for Young modules lying in $B_{w,\rho}$.)

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[1]The original title of this paper at 7/18/2001 was " Module correspondences in some blocks of finite general linear groups". There were some overlaps between W. Turner's work [Tur02] and the original version of this paper. In this paper, we shall remove the overlaps from the original version of this paper.

The second author conjectured that under the condition "$w < \ell$" these graded decomposition numbers lying in $B_{w,\rho}$ are coincident with crystallized decomposition numbers introduced in [LT96] (see, also [LLT96]). This conjecture is proved by [LM02].

2. Modules for the wreath product

In this paper, *modules* always mean finitely generated right modules, unless stated otherwise. Let $\lambda \vdash n$. In [Jam86], James defined a $K[GL_n(q)]$-module (resp. $\Bbbk GL_n(q)$-module) $S_K(\lambda)$ (resp. $S_{\Bbbk}(\lambda)$). $S_K(\lambda)$ is simple and affords an irreducible unipotent character. On the other hand, $S_{\Bbbk}(\lambda)$ has a unique simple quotient $D(\lambda)$. Moreover, $\{D(\lambda) \mid \lambda \vdash m,\ e\text{-core of } \lambda \text{ is } \rho\}$ is a complete set of representatives of isomorphic classes of simple $\Bbbk \otimes B_{w,\rho}$-modules [DJ86]. Let $X_{\mathcal{O}}(\lambda)$ be the Young $\mathcal{O}GL_n(q)$-module corresponding to λ [DJ89]. Then $X_{\mathcal{O}}(\lambda)$ is an indecomposable direct summand of a permutation $\mathcal{O}GL_n(q)$-module induced from a parabolic subgroup. For $R \in \{K, \Bbbk\}$ let $X_R(\lambda)$ be $R \otimes_{\mathcal{O}} X_{\mathcal{O}}(\lambda)$. Let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_{e-1})$ be a multipartition of w. Let $R \in \{K, \mathcal{O}, \Bbbk\}$. For each $\boldsymbol{\lambda}_i$, we write $S_R^{\boldsymbol{\lambda}_i}$ for the Specht $R\mathfrak{S}_{|\boldsymbol{\lambda}_i|}$-module corresponding to $\boldsymbol{\lambda}_i$. Note that, since $w < \ell$, $S_{\Bbbk}^{\boldsymbol{\lambda}_i}$ is simple projective. Hence there is a unique projective $\mathcal{O}\mathfrak{S}_{|\boldsymbol{\lambda}_i|}$-module $S_{\mathcal{O}}^{\boldsymbol{\lambda}_i}$ which is a lift of $S_{\Bbbk}^{\boldsymbol{\lambda}_i}$. Moreover, $S_K^{\boldsymbol{\lambda}_i} = K \otimes S_{\mathcal{O}}^{\boldsymbol{\lambda}_i}$. Let $\nu_i = (i+1, 1^{e-1-i}) \vdash e$ for $0 \le i \le e-1$. Let T_R be one of X_K, $X_{\Bbbk}$, S_K, $S_{\Bbbk}$ and D. Then $T_R(\nu_i)^{\otimes|\boldsymbol{\lambda}_i|} \otimes S_R^{\boldsymbol{\lambda}_i}$ is an $R[GL_e(q) \wr \mathfrak{S}_{|\boldsymbol{\lambda}_i|}]$-module [JK81]. We set,

$$\tilde{T}_R(\boldsymbol{\lambda}) = \mathrm{Ind}_{GL_e(q)\wr\mathfrak{S}_{\boldsymbol{\lambda}}}^{GL_e(q)\wr\mathfrak{S}_w}\Big(\bigotimes_{i=0}^{e-1}(T_R(\nu_i)^{\otimes|\boldsymbol{\lambda}_i|} \otimes S_R^{\boldsymbol{\lambda}_i}\Big) \otimes S_R(\rho)$$

where $S_{\mathcal{O}}(\rho)$ is a projective indecomposable $B_{0,\rho}$-module (note that $B_{0,\rho}$ has defect zero), $S_R(\rho) = R \otimes S_{\mathcal{O}}(\rho)$ is a simple $R[N_w]f$-module for $R = K, \Bbbk$. Then

$$\{\tilde{D}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} : e\text{-tuple partition of } w\}$$

is a complete set of isomorphism classes of simple $\Bbbk N_w f$-modules.

Let $\mathcal{P}(\rho, w)$ be the set of all partitions of m with e-core ρ. Let $\lambda \in \mathcal{P}(\rho, w)$. Using the abacus representation of ρ mentioned in Introduction, we can consider the e-quotient of λ [JK81]. We denote the e-quotient of λ by

$$\bar{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(e-1)}).$$

The correspondence

$$\lambda \leftrightarrow \bar{\lambda}$$

gives a bijection between $\mathcal{P}(\rho, w)$ and the set of all e-tuples partitions of w. Hence,

$$\{\tilde{D}(\bar{\lambda}) \mid \lambda \in \mathcal{P}(\rho, w)\}$$

is a complete set of representatives of isomorphism classes of simple $\Bbbk N_w f$-modules.

For $\alpha \in P(\rho, w-1)$, let

$$\Gamma(\alpha, i) = \{\lambda \in \mathcal{P}(\rho, w) \mid \alpha^{(i)} \subseteq \lambda^{(i)},\ \lambda^{(j)} = \alpha^{(j)}\ (i \neq j)\}.$$

On the other hand, let $\mathcal{P}(w)$ be the set of all partitions of w and let $\mathcal{P}_\alpha(w) = \{\lambda \in \mathcal{P}(w) \mid \alpha \subset \lambda\}$ for $\alpha \vdash w-1$. We need the following combinatorial lemma.

Lemma 2.1. *(1) Let σ and τ be bijections from $\mathcal{P}(w)$ to $\mathcal{P}(w)$. Suppose $w > 2$. If $\sigma(\mathcal{P}_\alpha(w)) = \tau(\mathcal{P}_\alpha(w))$ for any $\alpha \vdash w-1$, then $\sigma = \tau$.*
(2) Let σ and τ be bijections from $\mathcal{P}(\rho, w)$ to $\mathcal{P}(\rho, w)$. Suppose $w > 2$. If $\sigma(\Gamma(\alpha, i)) = \tau(\Gamma(\alpha, i))$ for any α, i, then $\sigma = \tau$.

Proof. (1) Let $\lambda = (\lambda_1, \lambda_2, \cdots) \vdash w$. Suppose that $\lambda_i > \lambda_{i+1} \geq \cdots \geq \lambda_j > 0$ for some $i \neq j$. We define two partitions α, β as follows:

$$\alpha = (\alpha_1, \alpha_2, \cdots),\ \alpha_i = \lambda_i - 1,\ \alpha_k = \lambda_k\ (k \neq i)$$

$$\beta = (\beta_1, \beta_2, \cdots),\ \beta_j = \lambda_j - 1,\ \beta_k = \lambda_k\ (k \neq j).$$

Then $\mathcal{P}_\alpha(w) \cap \mathcal{P}_\beta(w) = \{\lambda\}$, and $\sigma(\lambda) = \tau(\lambda)$.

So we may assume that $\lambda = (\lambda_1^m)$. Let $\gamma = (\lambda_1^{m-1}, \lambda_1 - 1)$. Then $\sigma(\mu) = \tau(\mu)$ for $\mu \in \mathcal{P}_\gamma(w)$ other than λ by the above argument. Hence we have $\sigma(\lambda) = \tau(\lambda)$.
(2) Let $\lambda \in \mathcal{P}(\rho, w)$. If there exists i such that $\lambda^{(j)} = \emptyset$ for any $j \neq i$, then $\sigma(\lambda) = \tau(\lambda)$ by (1). So we may assume that $\lambda^{(i)} \neq \emptyset$, $\lambda^{(j)} \neq \emptyset$ for some $i \neq j$. Take $\alpha, \beta \in \mathcal{P}(\rho, w-1)$ satisfying:

$$\alpha^{(i)} \subset \lambda^{(i)},\ \beta^{(j)} \subset \lambda^{(j)}.$$

Then $\Gamma(\alpha, i) \cap \Gamma(\beta, j) = \{\lambda\}$, and $\sigma(\lambda) = \tau(\lambda)$. □

3. Simple modules

Definition

Groups: Let G be $GL_n(q)$ and let G' be a Levi subgroup of G corresponding to a composition $(e, n-e)$, which is isomorphic to $GL_e(q) \times GL_{n-e}(q)$. Let $N = N_w$(resp. N') be the normalizer of a Levi subgroup corresponding to a composition $(e^w, |\rho|)$ in G(resp. G'), which is isomorphic to $GL_e(q) \wr \mathfrak{S}_w \times GL_{|\rho|}(q)$ (resp. $GL_e(q) \times N_{w-1}$).

Blocks: Let B_0 be the principal block of $\mathcal{O}GL_e(q)$. Let b be the block idempotent of $B_{w,\rho}$. Let f (resp. f') be the the block idempotent of $B_0(\mathcal{O}[GL_e(q) \wr \mathfrak{S}_w]) \otimes B_{0,\rho}$ (resp. $B_0 \otimes B_0(\mathcal{O}[GL_e(q) \wr \mathfrak{S}_{w-1}]) \otimes B_{0,\rho}$). For a block ideal B of $\mathcal{O}H$ for some subgroup H of G we denote $\Bbbk \otimes B$ by $\bar{B}$.

Functors: For a finite dimensional algebra A, we denote by mod-A the category of finitely generated right A-modules. For any Levi subgroup L of G, we denote by $\mathcal{R}_L^G$ (resp. ${}^*\mathcal{R}_L^G$) the Harish-Chandra induction (resp. restriction) functor for G and L. Let $X_w^\vee$ be $\mathrm{Hom}_{B_{w,\rho}}(X_w, B_{w,\rho})$. Define functors $F_w, (F_w)_K$ and $\bar{F}_w$ as follows:

$$F_w = - \otimes_{\mathcal{O}Nf}\ X_w^\vee :\ \text{mod-}\mathcal{O}Nf \longrightarrow \text{mod-}B_{w,\rho}$$

$$(F_w)_K = - \otimes_{K[N]f} (K \otimes X_w^\vee)\ :\ \text{mod-}K[N]f \longrightarrow \text{mod-}K \otimes B_{w,\rho}$$

$$\bar{F}_w = - \otimes_{\Bbbk N\bar{f}} \bar{X}_w^\vee\ :\ \text{mod-}\Bbbk N\bar{f} \longrightarrow \text{mod-}\bar{B}_{w,\rho}$$

For $F \in \{F_w, (F_w)_K, \bar{F}_w\}$, we denote the inverse of F by F^*.

Lemma 3.1. $\mathrm{Res}^{G\times N}_{G'\times N'}(X_w) \cong b\mathcal{R}^{G\times N}_{G'\times N'}(B_0 \otimes X_{w-1})$.

Proof. First,

$$\mathrm{Ind}^{G\times N'}_{N'\times N'}(\mathcal{O}N'f') \cong \mathcal{R}^{G\times N'}_{G'\times N'}(B_0 \otimes X_{w-1}) \oplus V$$

where V is a direct sum of indecomposable modules with vertex not conjugate of ΔD. On the other hand, $\mathrm{Res}^{G\times N}_{G\times N'}(X_w)$ is a direct summand of $\mathrm{Ind}^{G\times N'}_{N'\times N'}(\mathcal{O}N'f')$.

Since $\mathrm{Res}^{G\times N}_{G\times N'}(X_w)$ is indecomposable by [Tur02], $\mathrm{Res}^{G\times N}_{G\times N'}(X_w)$ is a direct summand of $\mathcal{R}^{G\times N'}_{G'\times N'}(B_0\otimes X_{w-1})$. Since

$$(F_w)_K(\mathrm{Ind}^N_{N'}(K[N']f')) \cong b(\mathcal{R}^G_{G'})(\mathrm{id}\otimes (F_{w-1})_K)(K[N']f')$$

by [Tur02],

$$K\otimes X_w \cong b(\mathcal{R}^{G\times N'}_{G'\times N'}(K\otimes (B_0\otimes X_{w-1}))$$

as $K[G]$-modules. Hence

$$\mathrm{Res}^{G\times N}_{G'\times N'}(X_w) \cong b\mathcal{R}^{G\times N}_{G'\times N'}(B_0\otimes X_{w-1}).$$

□

In order to compare the labels of some $\bar{B}_{w,\rho}$-modules with that of some $\Bbbk Nf$-modules, the following definition will be important.

Definition Let $\check{\lambda}\in\mathcal{P}(\rho,w)$ be the partition such that

$$\check{\lambda}^{(i)} = \begin{cases} \lambda^{(i)} & \text{if } e+i\ :\ \text{odd},\\ \lambda^{(i)\prime} & \text{if } e+i\ :\ \text{even}.\end{cases}$$

Theorem 3.2. $(F_w)_K(\tilde{S}_K(\bar{\lambda}))\cong S_K(\check{\lambda})$.

Proof. We proceed by induction on w. The case $w=2$ is proved in [Tur02]. So we assume $w>2$. Define a bijection $\sigma:\mathcal{P}(\rho,w)\longrightarrow\mathcal{P}(\rho,w)$ by

$$(F_w)_K(\tilde{S}_K(\bar{\lambda}))\cong S_K(\sigma(\lambda)).$$

It suffices to show that $\sigma(\lambda)=\check{\lambda}$. By the Littlewood-Richardson rule,

$$(F_w)_K(\mathrm{Ind}^N_{N'}S_K(\nu_i)\otimes\tilde{S}_K(\bar{\alpha})) = (F_w)_K(\bigoplus_{\mu\in\Gamma(\alpha,i)}\tilde{S}_K(\bar{\mu})) = \bigoplus_{\mu\in\Gamma(\alpha,i)} S_K(\sigma(\mu))$$

for $\alpha\in\mathcal{P}(\rho,w-1)$ and $0\le i\le e-1$. On the other hand,

$$\begin{aligned} b\mathcal{R}^G_{G'}(\mathrm{id}\otimes (F_{w-1})_K)(S_K(\nu_i)\otimes\tilde{S}_K(\bar{\alpha})) &= b\mathcal{R}^G_{G'}(S_K(\nu_i)\otimes S_K(\check{\alpha}))\\ &= \bigoplus_{\mu\in\Gamma(\check{\alpha},i)} S_K(\mu) = \bigoplus_{\mu\in\Gamma(\alpha,i)} S_K(\check{\mu})\end{aligned}$$

by induction and [Tur02]. Since these two modules are isomorphic by Lemma 3.1, we have $\sigma(\Gamma(\alpha,i))=(\Gamma(\alpha,i))^{\vee}$. Hence $\sigma(\lambda)=\check{\lambda}$ by Lemma 2.1. □

Corollary 3.3. *$\bar{F}_w(\tilde{D}(\bar{\lambda}))\cong D(\check{\lambda})$ for $\lambda\in\mathcal{P}(\rho,w)$.*

Proof. Let $\lambda\in\mathcal{P}(\rho,w)$. We will show that $\bar{F}_w(\tilde{D}(\bar{\check{\lambda}}))\cong D(\lambda)$ by induction on λ. Suppose that $\bar{F}_w(\tilde{D}(\bar{\check{\mu}}))\cong D(\mu)$ for any $\mu\in\mathcal{P}(\rho,w)$, $\mu>\lambda$. Suppose that $\bar{F}_w(\tilde{D}(\bar{\nu}))\cong D(\lambda)$ for $\nu\in\mathcal{P}(\rho,w)$. We write $V\leftrightarrow W$ if modules V and W have the same composition factors. Then

$$S_{\Bbbk}(\lambda)\leftrightarrow(\bigoplus_{\mu>\lambda} d_{\lambda\mu}D(\mu))\oplus D(\lambda)$$

for some nonnegative integers $d_{\lambda\mu}$. By Theorem 3.2,

$$\tilde{S}_{\Bbbk}(\bar{\check{\lambda}})\leftrightarrow(\bigoplus_{\mu>\lambda} d_{\lambda\mu}\tilde{D}(\bar{\check{\mu}}))\oplus\tilde{D}(\bar{\nu}).$$

Since $\tilde{S}_{\Bbbk}(\bar{\check{\lambda}})$ has $\tilde{D}(\bar{\check{\lambda}})$ as a composition factor by definition, we have $\nu=\check{\lambda}$. □

4. Young modules

Recall that

$$\tilde{X}_R(\boldsymbol{\lambda}) = \mathrm{Ind}_{GL_e(q)\wr \mathfrak{S}_{\boldsymbol{\lambda}}}^{GL_e(q)\wr \mathfrak{S}_w}(\bigotimes_{i=0}^{e-1} X_R(\nu_i)^{\otimes|\boldsymbol{\lambda}_i|} \otimes S_R^{\boldsymbol{\lambda}_i}) \otimes S_R(\rho)$$

and $\tilde{S}_R(\boldsymbol{\lambda})$ is a submodule of $\tilde{X}_R(\boldsymbol{\lambda})$ by definition.

Lemma 4.1. $[\tilde{X}_{\Bbbk}(\boldsymbol{\lambda}) : \tilde{D}(\boldsymbol{\lambda})] = 1$.

Proof. By Mackey's decomposition theorem, we know

$$\mathrm{Res}_{GL_e(q)^{\times w}}^{GL_e(q)\wr \mathfrak{S}_w} \mathrm{Ind}_{GL_e(q)\wr \mathfrak{S}_{\boldsymbol{\lambda}}}^{GL_e(q)\wr \mathfrak{S}_w}(V) \cong \bigoplus_{g\in[\mathfrak{S}_{\boldsymbol{\lambda}}\backslash \mathfrak{S}_w]} (\mathrm{Res}_{GL_e(q)^{\times w}}^{GL_e(q)\wr \mathfrak{S}_{\boldsymbol{\lambda}}}(V))^g$$

for any $R[GL_e(q)\wr \mathfrak{S}_{\boldsymbol{\lambda}}]$-module V. Here, $[\mathfrak{S}_{\boldsymbol{\lambda}}\backslash \mathfrak{S}_w]$ is a set of the right coset representatives of $\mathfrak{S}_{\boldsymbol{\lambda}}\backslash \mathfrak{S}_w$.

In particular,

$$\begin{aligned} &\mathrm{Res}_{GL_e(q)^{\times w}}^{GL_e(q)\wr \mathfrak{S}_w}(D(\boldsymbol{\lambda})) \\ (1)\qquad &\cong \left(\prod_{i=0}^{e-1} \dim_R S_{\Bbbk}^{\boldsymbol{\lambda}_i}\right) \cdot \bigoplus_{g\in[\mathfrak{S}_{\boldsymbol{\lambda}}\backslash \mathfrak{S}_w]} \left(\bigotimes_{i=0}^{e-1}(D(\nu_i))^{\otimes|\boldsymbol{\lambda}_i|}\right)^g \otimes S_{\Bbbk}(\rho).\end{aligned}$$

Comparing the composition factors of $\mathrm{Res}_{GL_e(q)^{\times w}}^{GL_e(q)\wr \mathfrak{S}_{\boldsymbol{\lambda}}} \bigotimes_{i=0}^{e-1} X_{\Bbbk}(\nu_i)^{\otimes|\boldsymbol{\lambda}_i|} \otimes S_{\Bbbk}^{\boldsymbol{\lambda}_i}$ with $\bigotimes_{i=0}^{e-1} D(\nu_i)^{\otimes|\boldsymbol{\lambda}_i|}$, we get

$$[\mathrm{Res}_{GL_e(q)^{\times w}}^{GL_e(q)\wr \mathfrak{S}_{\boldsymbol{\lambda}}} \bigotimes_{i=0}^{e-1} X_{\Bbbk}(\nu_i)^{\otimes|\boldsymbol{\lambda}_i|} \otimes S_{\Bbbk}^{\boldsymbol{\lambda}_i} : \bigotimes_{i=0}^{e-1} D(\nu_i)^{\otimes|\boldsymbol{\lambda}_i|}] = \prod_{i=0}^{e-1} \dim_{\Bbbk} S_{\Bbbk}^{\boldsymbol{\lambda}_i}.$$

Therefore, by (1) we have $[\tilde{X}_{\Bbbk}(\boldsymbol{\lambda}) : \tilde{D}(\boldsymbol{\lambda})] = 1$. □

Let $\eta_{\boldsymbol{\lambda}}$ (resp. $\chi_{\boldsymbol{\lambda}}$) be the character afforded by $\tilde{X}_{\mathcal{O}}(\boldsymbol{\lambda})$ (resp. $\tilde{S}_K(\boldsymbol{\lambda})$).

For e-tuple multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ we define a total order $\boldsymbol{\lambda} \succ \boldsymbol{\mu}$ by

$\boldsymbol{\lambda} \succ \boldsymbol{\mu}$ if there exists m such that $\boldsymbol{\lambda}_m > \boldsymbol{\mu}_m$ and $\boldsymbol{\lambda}_j = \boldsymbol{\mu}_j$ for $j > m$.

Then, by definition clearly the following holds:

Proposition 4.2. *$\langle \eta_{\boldsymbol{\lambda}}, \chi_{\boldsymbol{\lambda}}\rangle = 1$ and $\langle \eta_{\boldsymbol{\lambda}}, \chi_{\boldsymbol{\mu}}\rangle = 0$ if $\boldsymbol{\lambda} \succ \boldsymbol{\mu}$.*

In particular, we know the following:

Corollary 4.3. *If $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$, then $\tilde{X}_{\Bbbk}(\boldsymbol{\lambda}) \ncong \tilde{X}_{\Bbbk}(\boldsymbol{\mu})$.*

Lemma 4.4. $\mathrm{Soc}(\tilde{X}_{\Bbbk}(\boldsymbol{\lambda})) = \mathrm{Soc}(\tilde{S}_{\Bbbk}(\boldsymbol{\lambda}))$

Proof. Since the functor $\mathrm{Ind}_{GL_e(q)\wr \mathfrak{S}_{\boldsymbol{\lambda}}}^{GL_e(q)\wr \mathfrak{S}_w}$ preserves the Loewy layers of modules,

$$\mathrm{Soc}(\tilde{X}_{\Bbbk}(\boldsymbol{\lambda})) = \mathrm{Ind}_{GL_e(q)\wr \mathfrak{S}_{\boldsymbol{\lambda}}}^{GL_e(q)\wr \mathfrak{S}_w}(\bigotimes_{i=0}^{e-1}(\mathrm{Soc}(X_{\Bbbk}(\nu_i)))^{\otimes|\boldsymbol{\lambda}_i|} \otimes S_{\Bbbk}^{\boldsymbol{\lambda}_i}) \otimes S_{\Bbbk}(\rho).$$

Similarly, we have

$$\mathrm{Soc}(\tilde{S}_{\Bbbk}(\boldsymbol{\lambda})) = \mathrm{Ind}_{GL_e(q)\wr \mathfrak{S}_{\boldsymbol{\lambda}}}^{GL_e(q)\wr \mathfrak{S}_w}(\bigotimes_{i=0}^{e-1}(\mathrm{Soc}(S_{\Bbbk}(\nu_i)))^{\otimes|\boldsymbol{\lambda}_i|} \otimes S_{\Bbbk}^{\boldsymbol{\lambda}_i}) \otimes S_{\Bbbk}(\rho).$$

However, $\mathrm{Soc}(X_{\Bbbk}(\nu_i)) = \mathrm{Soc}(S_{\Bbbk}(\nu_i))$ for $i = 0, \ldots, e-1$. Hence, we are done. $\square$

Since $\tilde{S}_{\Bbbk}(\boldsymbol{\lambda})$ is indecomposable, by Lemma 4.1 we immediately know

Corollary 4.5. *$\tilde{X}_{\Bbbk}(\boldsymbol{\lambda})$ is indecomposable.*

Lemma 4.6. *$\bar{F}_w^*(X_{\Bbbk}(\lambda)) \cong \tilde{X}_{\Bbbk}(\bar{\nu})$ for some $\nu \in \mathcal{P}(\rho, w)$.*

Proof. Let $L := L_{(e^w,\rho)}$ be the Levi subgroup of G corresponding to the composition $(e^w, \rho) \models m$. $\mathrm{Res}_L^N(\bar{F}_w^*(V))$ is a direct summand of $b_w \cdot {}^*\mathcal{R}_L^G(V)$ for any $\bar{B}_{w,\rho}$-module V. Any indecomposable direct summand of $\mathrm{Res}_L^N(\bar{F}_w^*(X_{\Bbbk}(\lambda)))$ has the following shape :

$$\bigotimes_{i=1}^{w} X(\nu_{m(i)}) \otimes S_{\Bbbk}(\rho).$$

Here, $m(i)$ is an element of $\{0, 1, 2, \ldots, e-1\}$ for any $i \in \{1, 2, \ldots, w\}$. Hence, any indecomposable direct summand of $\bar{F}_w^*(X_{\Bbbk}(\lambda))$ is a direct summand of

$$\left(\mathrm{Ind}_L^N(\bigotimes_{i=0}^{e-1} X(\nu_i)^{\otimes n_i}) \right) \otimes S_{\Bbbk}(\rho)$$

for some $(n_0, n_1, \ldots, n_{e-1}) \models w$. Since $\bar{F}_w^*$ is an equivalence, we get

$$\bar{F}_w^*(X_{\Bbbk}(\lambda)) \cong \tilde{X}_{\Bbbk}(\nu) \text{ for some } \nu.$$

$\square$

Theorem 4.7. *$\bar{F}_w(\tilde{X}_{\Bbbk}(\bar{\lambda})) \cong X_{\Bbbk}(\check{\lambda})$ for any $\lambda \in \mathcal{P}(\rho, w)$.*

Proof. It suffices to show that

$$\bar{F}_w^*(X_{\Bbbk}(\lambda)) \cong \tilde{X}_{\Bbbk}(\bar{\bar{\lambda}}) \tag{2}$$

for any $\lambda \in \mathcal{P}(\rho, w)$. We proceed by induction on $(>, \mathcal{P}(\rho, w))$. By Corollary 3.3, we have already shown (2) for the maximal element of $\mathcal{P}(\rho, w)$. Suppose that the claim (2) holds for any $\mu \gneq \lambda$. By Lemma 4.6 there exists $\nu \in \mathcal{P}(\rho, w)$ such that

$$\bar{F}_w^*(X_{\Bbbk}(\lambda)) \cong \tilde{X}_{\Bbbk}(\bar{\nu}).$$

In particular, $\tilde{X}_{\Bbbk}(\bar{\nu})$ must have $\tilde{D}(\bar{\nu})$ as a composition factor. So, we deduce by Corollary 3.3 that $X_{\Bbbk}(\lambda) \cong \bar{F}_w(\tilde{X}_{\Bbbk}(\bar{\nu}))$ must have $D(\nu) \cong \bar{F}_w(\tilde{D}(\bar{\nu}))$ as a composition factor. Hence, by [Jam84, 16.3] we deduce $\nu \trianglerighteq \lambda$. Suppose that $\nu \gneq \lambda$. By the assumption of induction, we have

$$\bar{F}_w^*(X_{\Bbbk}(\nu)) \cong \tilde{X}_{\Bbbk}(\bar{\nu}).$$

In other words, we have

$$\bar{F}_w^*(X_{\Bbbk}(\nu)) \cong \bar{F}_w^*(X_{\Bbbk}(\lambda)).$$

Therefore, we have $\nu = \lambda$ and get a contradiction. $\square$

Let $P(\lambda)$ (resp. $\tilde{P}(\bar{\lambda})$) be the projective indecomposable module corresponding to $D(\lambda)$ (resp. $\tilde{D}(\bar{\lambda})$). Then, Specht type modules $S_{\Bbbk}(\lambda), \tilde{S}_{\Bbbk}(\bar{\lambda})$ and Young modules $X_{\Bbbk}(\lambda), \tilde{X}_{\Bbbk}(\bar{\lambda})$ enjoy the following properties:

Lemma 4.8. (1) *If $\tau \in \mathrm{Hom}_{\Bbbk G}(P(\lambda), X_{\Bbbk}(\lambda))$ satisfies $\mathrm{Top}(\tau(P(\lambda))) \cong D(\lambda)$, then $\tau(P(\lambda)) \cong S_{\Bbbk}(\lambda)$.*

(2) *If* $\psi \in \mathrm{Hom}_{\Bbbk N}(\tilde{P}(\bar{\lambda}), \tilde{X}_{\Bbbk}(\bar{\lambda}))$ *satisfies* $\mathrm{Top}(\psi(\tilde{P}(\bar{\lambda}))) \cong \tilde{D}(\bar{\lambda})$, *then* $\psi(\tilde{P}(\bar{\lambda})) \cong \tilde{S}_{\Bbbk}(\bar{\lambda})$.

Proof. (1) is clear by [DJ89]. (2): By Lemma 4.1 we have already known $[\tilde{X}_{\Bbbk}(\bar{\lambda}) : \tilde{D}(\bar{\lambda})] = 1$. Moreover, $\tilde{S}_{\Bbbk}(\bar{\lambda})$ is a submodule of $\tilde{X}_{\Bbbk}(\bar{\lambda})$ and have its unique top $\tilde{D}(\bar{\lambda})$. So, $\psi(\tilde{P}(\bar{\lambda}))$ must be isomorphic to $\tilde{S}_{\Bbbk}(\bar{\lambda})$. □

By Corollary 3.3, Theorem 4.7 and Lemma 4.8, we have

Theorem 4.9. $\bar{F}_w(\tilde{S}_{\Bbbk}(\bar{\lambda})) \cong S_{\Bbbk}(\check{\lambda})$ *for any* $\lambda \in \mathcal{P}(\rho, w)$.

Ackowledgments

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