Vector Fields on Spheres

Lars Hesselholt

The unit sphere in Euclidean n-space is the subset $S^{n-1} \subset \mathbb{R}^n$ of all vectors x of norm 1. The *tangent space* to S^{n-1} at x is the hyperplane $T_x(S^{n-1}) \subset \mathbb{R}^n$ of all vectors $v \in \mathbb{R}^n$ that are perpendicular to x. A continuous *tangent vector field* on the sphere S^{n-1} is defined to be a continuous function

$$\mathfrak{X}\colon S^{n-1} \to \mathbb{R}^n$$

such that $\mathfrak{X}(x) \in T_x(S^{n-1})$, for all $x \in S^{n-1}$. The vector field problem asks for the maximal number $k(n)$ of continuous vector fields $\mathfrak{X}_1, \dots, \mathfrak{X}_k$ on S^{n-1} such that the vectors $\mathfrak{X}_1(x), \dots, \mathfrak{X}_k(x)$ are linearly independent, for all $x \in S^{n-1}$.

We note that it is equivalent to ask that the vectors $\mathfrak{X}_1(x), \dots, \mathfrak{X}_k(x)$ form an orthonormal frame, for all $x \in S^{n-1}$. To see this, we recall that the Gram-Schmidt process replaces the linearly independent vectors $\mathfrak{X}_1(x), \dots, \mathfrak{X}_k(x)$ by orthonormal vectors $\mathfrak{X}_1'(x), \dots, \mathfrak{X}_k'(x)$ that span the same subspace of $\mathbb{R}^n$. Moreover, this process is continuous, and therefore, the maps $\mathfrak{X}_1', \dots, \mathfrak{X}_k'\colon S^{n-1} \to \mathbb{R}^n$ defined in this way are again continuous vector fields on S^{n-1}.

One possible way to construct a vector field on S^{n-1} is as follows. Let A be an $n \times n$ matrix. Then the function $\mathfrak{X}\colon S^{n-1} \to \mathbb{R}^n$ defined by $\mathfrak{X}(x) = Ax$ is a tangent vector field if and only if the inner product $\langle x, Ax\rangle = 0$, for all $x \in S^{n-1}$. This, in turn, is equivalent to the requirement that A be skew symmetric, that is,

$$A + A^t = 0,$$

where A^t is the transpose of the matrix A. Indeed, suppose first that $\langle x, Ax\rangle = 0$, for all $x \in S^{n-1}$, or equivalently, for all $x \in \mathbb{R}^n$. Then

$$\begin{aligned}&\langle x, (A + A^t)y\rangle = \langle x, Ay\rangle + \langle Ax, y\rangle \\ &\quad = \langle x, Ax\rangle + \langle x, Ay\rangle + \langle Ax, y\rangle + \langle Ay, y\rangle = \langle x + y, A(x + y)\rangle = 0,\end{aligned}$$

for all $x, y \in \mathbb{R}^n$, and hence, $A + A^t = 0$. Conversely, if $A + A^t = 0$, then

$$\begin{aligned}\langle x, Ax\rangle &= \tfrac{1}{2}(\langle x, Ax\rangle + \langle Ax, x\rangle) \\ &= \tfrac{1}{2}(\langle x, Ax\rangle + \langle x, A^t x\rangle) = \tfrac{1}{2}\langle x, (A + A^t)x\rangle = 0.\end{aligned}$$

We will say that the vector field $\mathfrak{X}$ obtained in this way is a *linear* vector field.

Let $\mathfrak{X}_1, \dots, \mathfrak{X}_k\colon S^{n-1} \to \mathbb{R}^n$ be linear vectors fields corresponding to the skew symmetric $n \times n$ matrices $A_1, \dots, A_k$. Then the vectors $\mathfrak{X}_1(x), \dots, \mathfrak{X}_k(x)$ form an orthonormal frame, for all $x \in S^{n-1}$ if and only if $A_i^t A_j + A_j^t A_i = 0$, for all $1 \leqslant i < j \leqslant k$, and $A_i^t A_i = I$, for all $1 \leqslant i \leqslant k$. Since the matrices A_i

are skew symmetric, these requirements are equivalent to the requirements that $A_iA_j + A_jA_i = 0$, for all $1 \leqslant i < j \leqslant k$ and $A_i^2 = -I$, for all $1 \leqslant i \leqslant k$. Here are some examples: If $n = 2$, the skew symmetric matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

satisfies $A^2 = -I$ which shows that S^1 has one linear unit vector field. If $n = 4$, the three skew symmetric matrices

$$A_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

satisfy $A_1^2 = A_2^2 = A_3^2 = -I$ and $A_1A_2 + A_2A_1 = A_1A_3 + A_3A_1 = A_2A_3 + A_3A_2 = 0$ which shows that S^3 has three orthonormal linear vector fields. If $n = 6$, the matrix

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

satisfies $A^2 = -I$ which shows that S^5 has one linear unit vector field. The following result was proved independently by Hurwitz [**5**] and Radon [**7**] around 1923; see also Eckmann [**4**].

THEOREM A. *Let n be a positive integer and write $n = 2^{4\alpha+\beta}u$, where u is odd, $\alpha \geqslant 0$, and $0 \leqslant \beta < 4$. Then the maximal number of orthonormal linear vector fields on S^{n-1} is equal to $l(n) = 8\alpha + 2^\beta - 1$.*

The theorem of Hurwitz and Radon determines the maximal number $l(n)$ of orthogonal *linear* vector fields on S^{n-1}. However, the maximal number $k(n)$ of orthogonal *continuous* vector fields on S^{n-1} could possibly be larger. It was proved by Adams in 1962 that, in fact, $k(n) = l(n)$. To explain how one may prove such a thing, we first reformulate the problem.

Let $p \leqslant n$ be positive integers. The *Stiefel manifold* $V_{n,p}$ is defined to be the set of all p-tuples $(x_1, \dots, x_p)$ of orthonormal vectors in $\mathbb{R}^n$. Let $x_{i,s}$ be the sth coordinate of the vector x_i. Then $V_{n,p} \subset (\mathbb{R}^n)^p = \mathbb{R}^{np}$ is equal to the set of solutions to the $p(p+1)/2$ equations

$$\sum_{s=1}^{n} x_{i,s}x_{j,s} = \delta_{i,j} \qquad (1 \leqslant i \leqslant j \leqslant p).$$

The implicit function theorem shows that, locally, we can express $p(p+1)/2$ of the np coordinates $x_{i,s}$, $1 \leqslant i \leqslant p$, $1 \leqslant s \leqslant n$, as smooth functions of the remaining coordinates. This shows that $V_{n,p}$ is a smooth manifold of dimension $np - p(p+1)/2$. Hence, locally, $V_{n,p}$ is diffeomorphic to Euclidean $np - p(p+1)/2$ space. However, globally, $V_{n,p}$ has a rich topology. For example, $V_{n,1}$ is the unit sphere S^{n-1}, $V_{n,n} = O(n)$ is the Lie group of orthogonal $n \times n$ matrices, and $V_{n,n-1} = SO(n)$ is the closed subgroup of orthogonal $n \times n$ matrices whose determinant is equal to 1. Now, there is a continuous projection map

$$\pi \colon V_{n,p} \to V_{n,1} = S^{n-1}$$

that takes the p-frame $x_1, \dots, x_p$ to the last vector x_p. Suppose that $\mathfrak{X}_1, \dots, \mathfrak{X}_{p-1}$ are orthonormal continuous vector fields on S^{n-1}. Then the map

$$\sigma \colon S^{n-1} \to V_{n,p}$$

defined by $\sigma(x) = (\mathfrak{X}_1(x), \dots, \mathfrak{X}_{p-1}(x), x)$ is continuous and the composite map

$$S^{n-1} \xrightarrow{\sigma} V_{n,p} \xrightarrow{\pi} S^{n-1}$$

is equal to the identity map $\mathrm{id}_{S^{n-1}}$. Conversely, if $\sigma \colon S^{n-1} \to V_{n,p}$ is continuous and $\pi \circ \sigma = \mathrm{id}_{S^{n-1}}$, then the maps $\mathfrak{X}_1, \dots, \mathfrak{X}_{p-1} \colon S^{n-1} \to \mathbb{R}^n$ defined by the formula $\sigma(x) = (\mathfrak{X}_1(x), \dots, \mathfrak{X}_{p-1}(x), x)$ are continuous orthonormal vector fields on S^{n-1}. Hence, we wish to prove that if $p \geqslant l(n) + 2$, then there does not exists a continuous map $\sigma \colon S^{n-1} \to V_{n,p}$ such that $\pi \circ \sigma = \mathrm{id}_{S^{n-1}}$.

The method of algebraic topology is to construct an “image” in algebra of our problem in topology. Here is one such “image.” Let M be a smooth manifold such as $V_{n,p}$. Then we have the notion of a differential q-form ω on M. The differential $d\omega$ of a differential q-form on M is a differential $(q+1)$-form on M. We say that ω is a *closed* differential q-form, if $d\omega = 0$, and we say that ω is an *exact* differential q-form, if $\omega = d\eta$, for some differential $(q-1)$-form η. The set of all closed differential q-forms on M forms a real vector space, and the set of all exact differential q-forms on M forms a real subspace of this vector space. These vector spaces are both infinite dimensional. But the quotient vector space

$$H^q_{\mathrm{dR}}(M) = \frac{\{\text{closed differential } q\text{-forms on } M\}}{\{\text{exact differential } q\text{-forms on } M\}}$$

is often a finite dimensional vector space. This is the case, for instance, if M is a compact smooth manifold such as $V_{n,p}$. The vector space $H^q_{\mathrm{dR}}(M)$ is called the qth *de Rham cohomology group* of M. Suppose that $f \colon N \to M$ is a smooth map from a smooth manifold N to the smooth manifold M. Then a differential q-form ω on M gives rise to a differential q-form $f^*\omega$ on N called the pull-back of ω by f. The pull-back $f^*\omega$ is closed, if ω is closed, and exact, if ω is exact, and therefore, we have a well-defined map $f^* \colon H^q_{\mathrm{dR}}(N) \to H^q_{\mathrm{dR}}(M)$ that takes the class of ω to the class of $f^*\omega$. This map is a linear map from the real vector space $H^q_{\mathrm{dR}}(N)$ to the real vector space $H^q_{\mathrm{dR}}(M)$. In fact, one can use the Weierstrauss approximation theorem to associate a linear map $f^* \colon H^q_{\mathrm{dR}}(M) \to H^q_{\mathrm{dR}}(N)$ to every continuous map $f \colon N \to M$. This association has the following properties:

(i) $(\mathrm{id}_M)^* = \mathrm{id}_{H^q_{\mathrm{dR}}(M)}$.

(ii) $(f \circ g)^* = g^* \circ f^*$.

We say that $H^q_{\mathrm{dR}}(-)$ is a *functor*

$$\left\{ \begin{matrix} \text{smooth manifolds} \\ \text{continuous maps} \end{matrix} \right\} \xrightarrow{H^q_{\mathrm{dR}}(-)} \left\{ \begin{matrix} \text{real vector spaces} \\ \text{linear maps} \end{matrix} \right\}$$

from the category of smooth manifolds and continuous maps to the category of vector spaces and linear maps. We refer to Madsen and Tornehave’s book [**6**] for a detailed introduction to differential forms and de Rham cohomology.

Let n be an odd number. Then $l(n) = 0$ and we wish to prove that there does not exist a continuous map $\sigma\colon S^{n-1} \to V_{n,2}$ such that the composition

$$S^{n-1} \xrightarrow{\sigma} V_{n,2} \xrightarrow{\pi} S^{n-1}$$

is the identity map $\mathrm{id}_{S^{n-1}}$. So we assume that such a map σ exists and proceed to derive a contradiction. The maps σ and π give rise to linear maps

$$H^q_{\mathrm{dR}}(S^{n-1}) \xleftarrow{\sigma^*} H^q_{\mathrm{dR}}(V_{n,2}) \xleftarrow{\pi^*} H^q_{\mathrm{dR}}(S^{n-1}),$$

and since $H^q_{\mathrm{dR}}(-)$ is a functor, the composition of these two maps is the identity map of the real vector space $H^q_{\mathrm{dR}}(S^{n-1})$. Now, one calculates

$$\dim_{\mathbb{R}} H^q_{\mathrm{dR}}(S^{n-1}) = \begin{cases} 1 & (q = 0 \text{ or } q = n-1) \\ 0 & (\text{otherwise}) \end{cases}$$

and, if n is odd,

$$\dim_{\mathbb{R}} H^q_{\mathrm{dR}}(V_{n,2}) = \begin{cases} 1 & (q = 0 \text{ or } q = 2n-3) \\ 0 & (\text{otherwise}). \end{cases}$$

Hence, for $q = n-1$, the composite map

$$H^{n-1}_{\mathrm{dR}}(S^{n-1}) \xleftarrow{\sigma^*} H^{n-1}_{\mathrm{dR}}(V_{n,2}) \xleftarrow{\pi^*} H^{n-1}_{\mathrm{dR}}(S^{n-1})$$

is the zero map, because the real vector space in the middle is zero. But then this map is not the identity map of the 1-dimensional real vector space $H^{n-1}_{\mathrm{dR}}(S^{n-1})$ which is a contradiction. We can therefore conclude that there are no continuous unit vector fields on S^{n-1} if n is odd.

Let us also consider the case $n = 6$. We have $l(6) = 1$ and wish to show that also $k(6) = 1$. Again, we assume that there exists a smooth map $\sigma\colon S^5 \to V_{6,3}$ such that $\pi \circ \sigma$ is the identity map of S^5. However, in this case, one calculates

$$\dim_{\mathbb{R}} H^q_{\mathrm{dR}}(V_{6,3}) = \begin{cases} 1 & (q = 0, 5, 7, \text{ or } 12) \\ 0 & (\text{otherwise}), \end{cases}$$

so we cannot rule out that the linear maps

$$H^q_{\mathrm{dR}}(S^5) \xleftarrow{\sigma^*} H^q_{\mathrm{dR}}(V_{6,3}) \xleftarrow{\pi^*} H^q_{\mathrm{dR}}(S^5)$$

exist. Therefore, we need an invariant that more fully captures the topology of the manifold $V_{n,p}$ than does de Rham cohomology. The more suttle invariant that turns out to give the solution to the problem is called *topological K-theory* and was introduced by Atiyah and Hirzebruch [**3**] based on ideas of Grothendieck. It assigns to the topological space X, a λ-ring $KO(X)$, and to the continuous map $f\colon X \to Y$, a λ-ring homomorphism $f^*\colon KO(Y) \to KO(X)$ such that $(\mathrm{id}_X)^* = \mathrm{id}_{KO(X)}$ and $(f \circ g)^* = g^* \circ f^*$. Hence, $KO(-)$ is a functor

$$\left\{ \begin{matrix} \text{topological spaces} \\ \text{continuous maps} \end{matrix} \right\} \xrightarrow{KO(-)} \left\{ \begin{matrix} \lambda\text{-rings} \\ \lambda\text{-ring homomorphisms} \end{matrix} \right\}$$

from the category of topological spaces and continuous maps to the category of λ-rings and λ-ring homomorhisms. We will not give the definition of $KO(-)$ here but refer to Atiyah's book [**2**].

Now, let $p = l(n) + 2$ and assume there exists a continuous map $\sigma\colon S^{n-1} \to V_{n,p}$ such that the composition

$$S^{n-1} \xrightarrow{\sigma} V_{n,p} \xrightarrow{\pi} S^{n-1}$$

is the identity map of S^{n-1}. Then the composition

$$KO(S^{n-1}) \xleftarrow{\sigma^*} KO(V_{n,p}) \xleftarrow{\pi^*} KO(S^{n-1})$$

is also the identity map, because $KO(-)$ is a functor. It is now possible as before to derive a contradiction and conclude that the map σ cannot exist. This was achieved by Adams [**1**] in 1962 who proved the following result.

THEOREM B. *Let n be a positive integer and write $n = 2^{4\alpha+\beta}u$ where u is an odd integer, and α and β integers with $\alpha \geqslant 0$ and $0 \leqslant \beta < 4$. Then there are at most $k(n) = 8\alpha + 2^\beta - 1$ linearly independent continuous vector fields on S^{n-1}.*

Together the theorems of Hurwicz-Radon and Adams show that there exists exactly $k(n) = l(n)$ linearly independent continuous vector fields on the unit sphere S^{n-1} in Euclidean n-space.

References

[1] J. F. Adams, *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–632.

[2] M. F. Atiyah, *K-theory. Notes by D. W. Anderson. Second edition*, Advanced Book Classics, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989.

[3] M. F. Atiyah and F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. **65** (1959), 276–281.

[4] B. Eckmann, *Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon über die Komposition quadratischer Formen*, Comment. Math. Helv. **15** (1943), 358–366.

[5] A. Hurwitz, *Über die Komposition der quadratischen Formen*, Math. Ann. **88** (1923), 1–25.

[6] I. Madsen and J. Tornehave, *From calculus to cohomology. De Rham cohomology and characteristic classes*, Cambridge University Press, Cambridge, 1997.

[7] J. Radon, *Lineare Scharen orthogonaler Matrizen*, Abh. Sem. Hamburg **I** (1923), 1–14.