$$HH(A) = \left| \begin{array}{c} \downarrow \cdots \downarrow \\ A \otimes A \otimes A \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ A \otimes A \\ \downarrow\uparrow\downarrow \\ A \end{array} \right| \longleftarrow T(A) = \left| \begin{array}{c} \downarrow \cdots \downarrow \\ HA \wedge HA \wedge HA \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ HA \wedge HA \\ \downarrow\uparrow\downarrow \\ HA \end{array} \right|$$

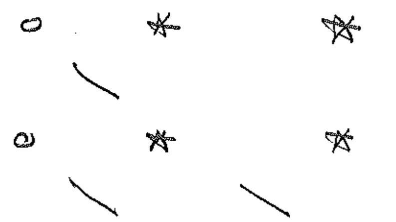
Connes: cyclic objects $\Rightarrow$ $\mathbb{T}$-spectra

$$TR^n_q(A;p) = [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(A)]_{\mathbb{T}}$$

maps in $\mathbb{T}$-stable cat

As $n \geq 1$ and $q \geq 0$ vary, these groups are related by maps

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
TR^3_0(A;p) & \xrightarrow{d} & TR^3_1(A;p) & \xrightarrow{d} & TR^3_2(A;p) & \to & \cdots \\
R\downarrow \; \downarrow F \; \uparrow V & & R\downarrow \; \downarrow F \; \uparrow V & & R\downarrow \; \downarrow F \; \uparrow V & & \\
TR^2_0(A;p) & \xrightarrow{d} & TR^2_1(A;p) & \xrightarrow{d} & TR^2_2(A;p) & \to & \cdots \\
R\downarrow \; \downarrow F \; \uparrow V & & R\downarrow \; \downarrow F \; \uparrow V & & R\downarrow \; \downarrow F \; \uparrow V & & \\
TR^1_0(A;p) & \xrightarrow{d} & TR^1_1(A;p) & \xrightarrow{d} & TR^1_2(A;p) & \to & \cdots
\end{array}$$



$$A \leftarrow A\otimes A \leftarrow A\otimes A\otimes A \leftarrow \cdots$$

cells : $D^q \times G/H$

htpy grps : maps from

$$\frac{D^q \times G/H}{\partial(D^q \times G/H)} = S^q \wedge (G/H)_+$$

$\mathbb{T}$ = circle group.

A comm. $\Rightarrow$ $TR^n_*(A;p)$ gr. ring and there is a mult. map

$$A \xrightarrow{[-]_n} TR_0^n(A;p).$$

Explain maps:

$$\begin{tikzcd}
TR_q^n(A;p) \arrow[r, shift left, "F"] \arrow[d, phantom, "\shortparallel"] & TR_q^{n-1}(A;p) \arrow[l, shift left, "V"] \arrow[d, phantom, "\shortparallel"] \\
{[S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(A)]} \arrow[r, "f^*"] \arrow[d, "\sim", "susp"'] & {[S^q \wedge (\mathbb{T}/C_{p^{n-2}})_+, T(A)]} \arrow[d, "\sim", "susp"'] \\
{[S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+ \wedge S^\lambda, T(A) \wedge S^\lambda]} & {[S^q \wedge (\mathbb{T}/C_{p^{n-2}})_+ \wedge S^\lambda, T(A) \wedge S^\lambda]} \arrow[l, "v^*"']
\end{tikzcd}$$

f = proj.

v = transfer: choose equivariant emb.

$$\mathbb{T}/C_{p^{n-2}} \xhookrightarrow{i} \lambda \quad \text{orthogonal } \mathbb{T}\text{-repr.}$$

The normal bdl. of the product emb.

$$\mathbb{T}/C_{p^{n-2}} \xhookrightarrow{(f,i)} \mathbb{T}/C_{p^{n-1}} \times \lambda$$

has a canonical trivialization, so the Pontryagin-Thom constr. gives

$$(\mathbb{T}/C_{p^{n-1}})_+ \wedge S^\lambda \xrightarrow{v} (\mathbb{T}/C_{p^{n-2}})_+ \wedge S^\lambda .$$

$$\begin{array}{ccc} TR^n_q(A;p) & \xrightarrow{d} & TR^n_{q+1}(A;p) \\ \| & & \| \\ [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(A)] & \xrightarrow{\delta^*} & [S^{q+1} \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(A)] \end{array}$$

$$\begin{array}{ccc} [S^{q+1} \wedge (\mathbb{T}/C_{p^{n-1}})_+, S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+]_{\mathbb{T}\text{-spectra}} & \ni & \delta \\ \uparrow \text{susp}_{\mathbb{T}}(-) & & \uparrow \\ [S^{q+1} \wedge (\mathbb{T}/C_{p^{n-1}})_+, S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+]_{\mathbb{T}\text{-spaces}} & \ni & \delta \\ \uparrow \sim \quad (q \geq 2) & & \uparrow \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & & (1,0) \end{array}$$

$$\begin{tikzcd}
TR^n_q(A;p) \arrow[rr, "R"] \arrow[d, phantom, "\cong" rotate=90] & & TR^{n-1}_q(A;p) \arrow[ddd, equal] \\
{[S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(A)]_{\mathbb{T}}} & & \\
{[S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(A)^{C_p}]_{\mathbb{T}/C_p}} \arrow[u, "\sim"'] & & \\
{[S^q \wedge (\mathbb{T}/C_{p^{n-2}})_+, \rho_p^* T(A)^{C_p}]_{\mathbb{T}}} \arrow[u, "\sim"'] \arrow[rr, "r_*"] & & {[S^q \wedge (\mathbb{T}/C_{p^{n-2}})_+, T(A)]_{\mathbb{T}}}
\end{tikzcd}$$

$$\mathbb{T} \xrightarrow[\sim]{\rho_p} \mathbb{T}/C_p \qquad z \longmapsto z^{1/p} C_p$$

$$\rho_p^* T(A)^{C_p} \xrightarrow{r} T(A) \qquad \text{"cyclotomic" str.}$$

Canonical map of $\mathbb{T}$-spaces

$$N^{cy}(A) = \begin{vmatrix} \downarrow \cdots \downarrow \\ A \wedge A \wedge A \\ \downarrow \uparrow \downarrow \uparrow \downarrow \\ A \wedge A \\ \downarrow \uparrow \downarrow \\ A \end{vmatrix} \xrightarrow{\gamma} \Omega^\infty T(A)$$

$$N^{cy}(A) \xrightarrow[\sim]{\Delta} \rho_{p^{n-1}}^* N^{cy}(A)^{C_{p^{n-1}}} \xrightarrow{\gamma} \rho_{p^{n-1}}^* \Omega^\infty T(A)^{C_{p^{n-1}}}$$

induced map of components

$$A \xrightarrow{[-]_n} TR_0^n(A;p).$$

These maps satisfy:

- $TR_*^n(A;p)$ is a differential graded ring.

- R is a ring-homomorphism and commutes with all other maps.

- F is a ring-homomorphism and

$$F d[a]_n = [a]_{n-1}^{p-1} d[a]_{n-1}$$

- V is additive and

$$x \cdot V(y) = V(F(x) \cdot y)$$

$$FV = p$$

$$FdV = d \qquad (p \text{ odd})$$

- $[-]_n$ is mult. and

$$W_n(A) \xrightarrow{\sim} TR_0^n(A;p)$$

$$(a_0, a_1, \dots, a_{n-1}) \mapsto \sum_{0 \le s < n} V^s([a_s]_{n-s})$$

is a ring-homomorphism (actually iso.)

Call this algebraic structure a Witt complex over A. Universal example: de Rham–Witt cx.

$$W_n\Omega^q_A \xrightarrow{\sim} TR^n_q(A;p).$$

Prop This is an iso. for $q \leq 1$.

Thm Let A be a regular $\mathbb{F}_p$-algebra. Then, as $n \geq 1$ varies, the map

$$W_n\Omega^q_A \xrightarrow{\sim} TR^n_q(A;p)$$

is an isom. of pro-ab. groups.

A = $\mathbb{Z}_{(p)}$-algebra, p odd prime.

Def A Witt complex over A is a quadruple

$$(E, \lambda, F, V)$$

where

(i) $E = \{E_n^\cdot\}_{n\in\mathbb{N}}$ is a pro-differential graded ring,

(ii) λ is a strict map of pro-rings

$$W_n(A) \xrightarrow{\lambda} E_n^0$$

(iii) F is a strict map of pro-graded rings

$$E_n^\cdot \xrightarrow{F} E_{n-1}^\cdot,$$

(iv) V is a strict map of pro-graded abelian groups

$$E_{n-1}^\cdot \xrightarrow{V} E_n^\cdot,$$

such that the following relations hold,

$$F\lambda = \lambda F$$

$$Fd\lambda[a]_n = \lambda[a]_{n-1}^{p-1} d\lambda[a]_{n-1}$$

$$V\lambda = \lambda V$$

$$x\,V(y) = V(F(x)\,y)$$

$$FV = p$$

$$F\,dV = d.$$

A map of Witt complexes is a strict map of pro-differential graded rings $E_n^{\cdot} \xrightarrow{f} E_n'^{\cdot}$ s.t. $\lambda' = f\lambda$, $F'f = fF$, and $V'f = fV$.

Lemma The following relations hold

$$dF = pFd$$

$$Vd = p\,dV.$$

Pf We have

$$dF(x) = FdVF(x) = Fd(V(1).x)$$

$$= F(dV(1).x + V(1).dx)$$

$$= FdV(1).F(x) + FV(1).Fdx$$

$$= d(1).F(x) + pFd(x)$$

$$= pFd(x)$$

and

$$Vd(x) = VFdV(x) = V(1)\cdot dV(x)$$

$$= d(V(1)\cdot V(x)) - dV(1)\cdot V(x)$$

$$= dV(FV(1)\cdot x) - V(FdV(1)\cdot x)$$

$$= p\,dV(x) - V(d(1)\cdot x)$$

$$= p\,dV(x). \qquad //$$

~~Rem The (serious) relations $FdV = d$ and $Fd\lambda[a]_n = \lambda[a]_{n-1}^{p-1}d\lambda[a]_{n-1}$ are~~

Lemma (Freyd) Let $\mathcal{C}$ be a category which has all small (inverse) limits and which satisfies the solution set condition: There is a set of objects k_i, $i \in I$, in $\mathcal{C}$ s.t. for every object c in $\mathcal{C}$, there exists a morphism $k_i \to c$, for some $i \in I$. Then $\mathcal{C}$ has an initial object.

Pf Every obj. c in $\mathcal{C}$ receives a morphism from the obj.

$$k = \prod_{i \in I} k_i,$$

but this need not be unique. However, let d be the equalizer

$$d \longrightarrow k \overset{\Delta}{\underset{\Delta'}{\rightrightarrows}} \prod_{\varphi \in \mathrm{End}_{\mathcal{C}}(k)} k$$

where $\Delta_\varphi = \mathrm{id}$ and $\Delta'_\varphi = \varphi$. Then d is an initial object. //

Prop The category of Witt complexes over A has an initial object called the de Rham-Witt complex and written $W_n\Omega^{\cdot}_A$. In addition, the canonical map

$$\Omega^{\cdot}_{W_n(A)} \xrightarrow{\lambda} W_n\Omega^{\cdot}_A$$

is surjective.

pf We show that if (E, λ, F, V) is a Witt complex, then the image of the canonical map

$$\Omega^{\cdot}_{W_n(A)} \xrightarrow{\lambda} E^{\cdot}_n$$

is a sub-Witt cx. of E. This image is canonically isom. to the quotient of $\Omega^{\cdot}_{W_n(A)}$ by the kernel of λ. Since there are (only) a set of such quotients, the solution set cond. is satisfied.

To show that the image of λ is a sub-Witt cx. of $\underline{E}$ we must show that is preserved by F and V. This is easy for V:

$$V(\lambda(x_0)\, d\lambda(x_1) — d\lambda(x_q))$$

$$= V(\lambda(x_0))\, dV(\lambda(x_1)) — dV(\lambda(x_q))$$

$$= \lambda(V(x_0))\, d\lambda(V(x_1)) — d\lambda(V(x_q)).$$

Since F is mult., it is enough to show that for all $a \in W_n(A)$, $Fd\lambda(a)$ is in the image of λ. But every $a \in W_n(A)$ can be written uniquely as

$$a = [a_0]_n + V([a_1]_{n-1}) + \cdots + V^{n-1}([a_{n-1}]_1)$$

so

$$Fd\lambda(a) = Fd\lambda([a_0]_n) + Fd\lambda(V([a_1]_{n-1})) + \cdots$$
$$\cdots + Fd\lambda(V^{n-1}([a_{n-1}]_1))$$

$$= \lambda([a_0]_{n-1}^{p-1})\, d\lambda([a_0]_{n-1}) + d\lambda([a_1]_{n-1}) + \cdots$$
$$\cdots + d\lambda(V^{n-2}([a_{n-1}]_1))$$

which is in the image.

Finally, we show that

$$\Omega^{\bullet}_{W_n(A)} \xrightarrow{\lambda} W_n\Omega^{\bullet}_A$$

is surjective. We have just shown that the image $E^{\bullet}_n$ of λ forms a sub-Witt cx. of $W_n\Omega^{\bullet}_A$. Hence, there is a (unique) map of Witt complexes

$$W_n\Omega^{\bullet}_A \xrightarrow{f} E^{\bullet}_n .$$

The composite

$$W_n\Omega^{\bullet}_A \xrightarrow{f} E^{\bullet}_n \hookrightarrow W_n\Omega^{\bullet}_A$$

is an endomorphism of an initial obj. and hence the identity map. Thus, the inclusion

$$E^{\bullet}_n \hookrightarrow W_n\Omega^{\bullet}_A$$

is the identity map. $\square$

$$W_n(\mathbb{Z}_{(p)}) = \prod_{0 \le s < n} \mathbb{Z}_{(p)} \cdot V^s([1]_{n-s})$$

$$V^s([1]_{n-s}) \cdot dV^t([1]_{n-t})$$

$$= V^s([1]_{n-s}\, F^s dV^t([1]_{n-t}))$$

$$= \begin{cases} p^s\, dV^t([1]_{n-t}) & 0 \le s < t < n \\ 0 & 0 \le t \le s < n \end{cases}$$

$$p^s\, dV^s([1]_{n-s}) = V^s d([1]_{n-s}) = 0 .$$

so

$$\prod_{0 < s < n} \mathbb{Z}/p^s\mathbb{Z} \cdot dV^s([1]_{n-s})$$

$$\twoheadrightarrow W_n \Omega^1_{\mathbb{Z}_{(p)}} .$$

$$dV^s([1]_{n-s}) \cdot dV^t([1]_{n-t})$$

$$= d\,(V^s([1]_{n-s}) \cdot dV^t([1]_{n-t})) = 0$$

so

$$W_n \Omega^q_{\mathbb{Z}_{(p)}} = 0 , \qquad q \ge 2 .$$

$$F\,d\,[u^{-1}]_n = F\,d\,[u]_n^{-1}$$

$$= -F\left([u]_n^{-2}\,d[u]_n\right)$$

$$= -[u]_{n-1}^{-2p}\,[u]_{n-1}^{p-1}\,d[u]_{n-1}$$

$$= -[u]_{n-1}^{-p-1}\,d[u]_{n-1}$$

$$= [u]_{n-1}^{-(p-1)}\,d[u]_{n-1}^{-1}$$

$$[-1]_n = -[1]_n \qquad (p \text{ odd})$$

Symmetric spectrum E :

(i) (left) Σ_n-space E_n , $n \geq 0$

(ii) $\Sigma_m \times \Sigma_n$-equivariant maps

$$S^m \wedge E_n \xrightarrow{\sigma_{m,n}} E_{m+n}$$

s.t. $\sigma_{0,n}$ is the can. isom. and s.t.

$$\begin{tikzcd}
S^k \wedge S^m \wedge E_n \arrow[r, "id \wedge \sigma_{m,n}"] \arrow[d, "can"', "\sim"] & S^k \wedge E_{m+n} \arrow[d, "\sigma_{k,m+n}"] \\
S^{k+m} \wedge E_n \arrow[r, "\sigma_{k+m,n}"] & E_{k+m+n}
\end{tikzcd}$$

commutes, for all $k, m, n \geq 0$. //

Alternatively: (index) category $\mathcal{J}$ enriched in pointed topological spaces :

$$\text{ob}\, \mathcal{J} = \mathbb{N}_0 = \text{non-neg. integers}$$

$$\underline{\text{Hom}}_{\mathcal{J}}(n, m+n) = (\Sigma_{m+n})_+ \underset{\Sigma_m}{\wedge} S^m$$

~~Composition~~ Composition

$$\underline{\text{Hom}}_{\mathcal{J}}(m+n, k+m+n) \wedge \underline{\text{Hom}}_{\mathcal{J}}(m+n, n)$$

$$\xrightarrow{\circ} \underline{\text{Hom}}_{\mathcal{J}}(n, k+m+n)$$

$$(\Sigma_{k+m+n})_+ \underset{\Sigma_k}{\wedge} S^k \wedge (\Sigma_{m+n})_+ \underset{\Sigma_m}{\wedge} S^m$$

$$\longrightarrow (\Sigma_{k+m+n})_+ \underset{\Sigma_{k+m}}{\wedge} S^{k+m}$$

A symm. spectrum is a cts. functor

$$\mathcal{J} \xrightarrow{E} \mathrm{Top}.$$

$$n \longmapsto E_n$$

$$\underline{\mathrm{Hom}}_{\mathcal{J}}(n, m+n) \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{Top}}(E_n, E_{m+n}) \quad \text{cts.}$$

$$\Big\updownarrow$$

$$(\Sigma_{m+n})_+ \underset{\Sigma_m \times \Sigma_n}{\wedge} E_n \longrightarrow E_{m+n}.$$

Adjoint pair (left Kan extension)

$$\mathrm{Top} \underset{ev_n}{\overset{F_n}{\rightleftarrows}} \mathrm{Top}^{\mathcal{J}}$$

$$\underline{\mathrm{Hom}}_{\mathrm{Top}}(X, E_n) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathrm{Top}^{\mathcal{J}}}(F_n X, E)$$

$$(F_n X)_m = (\Sigma_m)_+ \underset{\Sigma_{m-n}}{\wedge} S^{m-n} \wedge X.$$

level model structure (or object-wise fibration structure):

weak equiv. = object-wise weak equiv.

fibrations = objectwise fibrations

cofibrations = LLP w.r.t. trivial fibrations.

Generators

$$FI : F_m(S^{n-1}_+) \rightarrowtail F_m(D^n_+)$$

$$FJ : F_m(D^n_+) \overset{\sim}{\rightarrowtail} F_m((D^n \times [0,1])_+) . \quad //$$

The stable model structure on $Top^{\mathcal{J}}$ is obtained by localizing w.r.t. the following set of maps

$$A = \{ F_{n+1} S^1 \xrightarrow{\lambda_n} F_n S^0 \mid n \geq 0 \}$$

$$\begin{array}{ccc} (F_{n+1} S^1)_m & \xrightarrow{\lambda_n} & (F_n S^0)_m \\ \| & & \| \\ (\Sigma_m)_+ \underset{\Sigma_{m-n-1}}{\wedge} S^{m-n-1} \wedge S^1 & \xrightarrow{proj.} & (\Sigma_m)_+ \underset{\Sigma_{m-n}}{\wedge} S^{m-n} \end{array}$$

A spectrum E is A-<u>local</u> if (it is fibrant and) the induced maps

$$[F_n S^0, E]_{level} \xrightarrow{\lambda_n^*} [F_{n+1} S^1, E]_{level}$$

are bijections, or equivalently, if the adjoint structure maps

$$E_n \longrightarrow \Omega E_{n+1}$$

are w.e. of pointed spaces.

A map $X \xrightarrow{f} X'$ of symm. spectra is an <u>A-local</u> <u>weak</u> <u>equivalence</u> if for all A-local E,

$$[X', E]_{level} \xrightarrow{f^*} [X, E]_{level}$$

is a bijection.

Stable model structure:

weak equiv. = A-local weak equiv.

cofibrations = level cofibrations

fibrations = RLP w.r.t. trivial cof.

Generators:

FI: $F_m(S^{n-1}_+) \rightarrowtail F_m(D^n_+)$

K : $F_m(D^n_+) \xrightarrow{\sim} F_m((D^m \times [0,1])_+)$

and the following maps (i_m, k_n):

$$\begin{tikzcd}
F_{n+1}S^1 \arrow[rr, "\lambda_n"] \arrow[d, shift right, "i_0"'] & & F_n S^0 \arrow[d, shift right] \\
{[0,1]_+ \wedge F_{n+1}S^1} \arrow[rr] \arrow[u, shift right, "r"'] & & \mathrm{Cyl}(\lambda_n) \arrow[u, shift right, "r_n"'] \\
F_{n+1}S^1 \arrow[u, "i_1"] \arrow[urr, "k_n"'] & &
\end{tikzcd}$$

id

r_n = level equiv.

k_n = cofibration,

$$\begin{tikzcd}
S^{m-1}_+ \wedge F_{n+1}S^1 \arrow[r, "\mathrm{id}\wedge k_n"] \arrow[d, "i_m \wedge \mathrm{id}"] \arrow[dr, phantom, "\text{p.o.}"] & S^{m-1}_+ \wedge \mathrm{Cyl}(\lambda_n) \arrow[d] \arrow[ddr, bend left, "i_m \wedge \mathrm{id}"] & \\
D^m_+ \wedge F_{n+1}S^1 \arrow[r] \arrow[drr, bend right, "\mathrm{id}\wedge k_n"'] & D^m_+ \wedge F_{n+1}S^1 \cup S^{m-1}_+ \wedge \mathrm{Cyl}(\lambda_n) \arrow[dr, "{(i_m, k_n)}"] & \\
& & D^m_+ \wedge \mathrm{Cyl}(\lambda_n)
\end{tikzcd}$$

"

Homotopy groups

$$\pi_q(X) = \operatorname{colim}_n \pi_{q+n}(X_n)$$

Let $X \xrightarrow{f} X'$ be a map of symm. spectra

$\pi_*(f)$ isom. $\Rightarrow$ f stable equiv. ✓

f stable equiv. $\Rightarrow$ $\pi_*(f)$ isom. ?

There is a canonical isom.

$$\pi_q(X) \xrightarrow{\sim} [S^q, X] .$$

This is true if and only if $\pi_*(\lambda_n)$ are isom., where

$$F_{n+1}S^1 \xrightarrow{\lambda_n} F_n S^0 .$$

And this is true if and only if $\pi_*(\lambda_n \wedge S^n)$ is an isom. Consider

$$\begin{array}{ccc}
((F_{n+1}S^1)\wedge S^n)_m & \xrightarrow{\lambda_n \wedge \mathrm{id}} & ((F_n S^0)\wedge S^n)_m \\
\| & & \| \\
(\Sigma_m)_+ \wedge_{\Sigma_{m-n-1}} S^{m-n-1}\wedge S^1 \wedge S^n & \xrightarrow{\mathrm{proj.}} & (\Sigma_m)_+ \wedge_{\Sigma_{m-n}} S^{m-n}\wedge S^n \\
\uparrow \sim & & \uparrow \sim \\
(\Sigma_m/\Sigma_{m-n-1})_+ \wedge S^m & \xrightarrow{\mathrm{proj.} \wedge \mathrm{id}} & (\Sigma_m/\Sigma_{m-n})_+ \wedge S^m
\end{array}$$

so $\pi_0(\lambda_n \wedge S^n)$ is not an isom.

However, the corresponding projection

$$(O(m)/O(m-n-1))_+ \wedge S^n \longrightarrow (O(m)/O(m-n))_+ \wedge S^n$$

is $(2m-n-1)$-conn. So if we replace Σ_n by $O(n)$ everywhere we get a (stable model) category of orthogonal spectra, where stable equivalences are the π_*-isom. Moreover, there is a

Quillen equivalence

$$\mathrm{Top}^{\mathcal{J}_\Sigma} \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathrm{Top}^{\mathcal{J}_0}$$

where ι_* is the forgetful functor and ι^* the left Kan extension. The total derived functors

$$\mathrm{Ho\,Top}^{\mathcal{J}_\Sigma} \underset{\mathbb{R}\iota_*}{\overset{\mathbb{L}\iota^*}{\rightleftarrows}} \mathrm{Ho\,Top}^{\mathcal{J}_0}$$

is an adjoint equivalence of categories.

Smash product of symm. spectra:

$$\begin{array}{c}
\bigvee_{k+\ell+m=n} (\Sigma_{k+\ell+m})_+ \wedge_{\Sigma_k\times\Sigma_\ell\times\Sigma_m} (E_k \wedge S^\ell \wedge E'_m) \\
f \downarrow \quad \downarrow g \\
\bigvee_{p+q=n} (\Sigma_{p+q})_+ \wedge_{\Sigma_p\times\Sigma_q} (E_p \wedge E'_q) \\
\downarrow \\
(E \wedge E')_n
\end{array}$$

$$f(\alpha, e, x, e') = (\alpha \circ \rho_{k,\ell}^{-1}, \sigma_{\ell,k}(x,e), e')$$

$$g(\alpha, e, x, e') = (\alpha, e, \sigma'_{\ell,m}(x,e')).$$

A symmetric ring spectrum is a monoid in the symmetric monoidal category of symm. spectra and smash product, or equivalently, a sequence of (left) Σ_n-spaces E_n together with unit and multiplication maps

$$S^n \xrightarrow{\eta_n} E_n$$

$$E_m \wedge E_n \xrightarrow{\mu_{m,n}} E_{m+n}$$

that are Σ_n and $\Sigma_m \times \Sigma_n$-equiv. and such that the following unity, assoc., and centrality of unit diagrams commute

$$\begin{tikzcd} S^m \wedge S^n \arrow[r, "\eta_m \wedge \eta_n"] \arrow[d, "\sim"', "\mathrm{can}"] & E_m \wedge E_n \arrow[d, "\mu_{m,n}"] \\ S^{m+n} \arrow[r, "\eta_{m+n}"] & E_{m+n} \end{tikzcd}$$

$$\begin{tikzcd} E_k \wedge E_m \wedge E_n \arrow[r, "\mu_{k,m} \wedge \mathrm{id}"] \arrow[d, "\mathrm{id} \wedge \mu_{m,n}"] & E_{k+m} \wedge E_n \arrow[d, "\mu_{k+m,n}"] \\ E_k \wedge E_{m+n} \arrow[r, "\mu_{k,m+n}"] & E_{k+m+n} \end{tikzcd}$$

$$\begin{tikzcd} S^m \wedge E_n \arrow[r, "\eta_m \wedge \mathrm{id}"] \arrow[d, "tw"] & E_m \wedge E_n \arrow[r, "\mu_{m,n}"] & E_{m+n} \arrow[d, "\rho_{m,n}"] \\ E_n \wedge S^m \arrow[r, "\mathrm{id} \wedge \eta_m"] & E_n \wedge E_m \arrow[r, "\mu_{n,m}"] & E_{n+m} \end{tikzcd}$$

and such that

$$\begin{tikzcd} S^0 \wedge E_n \arrow[r, "\eta_0 \wedge \mathrm{id}"] \arrow[dr, "can", "\sim"'] & E_0 \wedge E_n \arrow[d, "\mu_{0,n}"] \\ & E_n \end{tikzcd} .$$

A symm. ring spectrum E is commutative if and only if the following diagr. comm.

$$\begin{tikzcd} E_m \wedge E_n \arrow[r, "\mu_{m,n}"] \arrow[d, "tw"] & E_{m+n} \arrow[d, "\rho_{m,n}"] \\ E_n \wedge E_m \arrow[r, "\mu_{n,m}"] & E_{n+m} \end{tikzcd} .$$

Ex 1) A monoid Π in the symm. monoidal category of pointed spaces and smash product determines a symm. ring spectrum with n'th space

$$S^n \wedge \Pi$$

and with structure maps

$$S^n \xrightarrow{\text{can}} S^n \wedge S^0 \xrightarrow{\text{id}\wedge\eta} S^n \wedge \Pi$$

$$(S^m \wedge \Pi) \wedge (S^n \wedge \Pi) \xrightarrow[\sim]{\text{can}} S^{m+n} \wedge \Pi \wedge \Pi$$

$$\xrightarrow{\text{id}\wedge\mu} S^{m+n} \wedge \Pi .$$

It is commutative if Π is.

2) A ring A determines a symm. ring spectrum $\tilde{A}$. Let

$$S^n_\bullet = \underbrace{\Delta[1]/\partial\Delta[1] \wedge \cdots \wedge \Delta[1]/\partial\Delta[1]}_{n}$$

Then

$$\tilde{A}_n = \left| [k] \mapsto A\{S^n_k\}/A\{s_{0,k}\} \right|$$

with structure maps

$$S^n \xrightarrow[\sim]{\text{can}} \left| [k] \mapsto S^n_k \right|$$

$$\xrightarrow{\eta} \left| [k] \mapsto A\{S^n_k\}/A\{s_{0,k}\} \right| = \tilde{A}_n$$

$$\tilde{A}_m \wedge \tilde{A}_n = \left| [k] \mapsto A\{S^m_k\}/A\{s_{0,k}\} \right| \wedge \left| [k] \mapsto A\{S^n_k\}/A\{s_{0,k}\} \right|$$

$$\xleftarrow[\sim]{pr_1 \wedge pr_2} \left| [k] \mapsto A\{S^m_k\}/A\{s_{0,k}\} \wedge A\{S^n_k\}/A\{s_{0,k}\} \right|$$

$$\xrightarrow{\mu} \left| [k] \mapsto A\{S^m_k \wedge S^n_k\}/A\{s_{0,k}\} \right| = \tilde{A}_{m+n} .$$

It is commutative if A is. //

Recall

$$\mathrm{Top}^{I} \underset{\Delta}{\overset{\mathrm{colim}_I}{\rightleftarrows}} \mathrm{Top}$$

$$(i \mapsto X) \longleftarrow\!\shortmid \quad X$$

$$\mathrm{Top}^{I} \underset{}{\overset{\mathrm{hocolim}_I}{\rightleftarrows}} \mathrm{Top}$$

$$(i \mapsto F(|i/I|_+, X)) \longleftarrow\!\shortmid \quad X$$

so canonical homeomorphism

$$\mathrm{hocolim}_{I}\,(X_i \wedge Y) \xrightarrow{\sim} (\mathrm{hocolim}_{I}\, X_i) \wedge Y$$

and

$$I \xrightarrow{g} I' \xrightarrow{X} \mathrm{Top} \implies$$

$$\mathrm{hocolim}_{I}\,(X \circ g) \xrightarrow{\mathrm{can}} \mathrm{hocolim}_{I'}\, X .$$

Let E be a symm. ring spectrum, and let X be a pointed space. We define a cyclic pointed space with k-simpl.

$$\mathrm{THH}(E;X)[k] = \mathrm{hocolim}_{I^{k+1}}\, G_k(E;X)$$

The (index) category I has

ob I : $\underline{i} = \{1, 2, \ldots, i\}$, $i \geq 1$, $\underline{0} = \emptyset$.

mor I : <u>all</u> injective maps.

Every morphism in $\mathcal{I}$ is (non-uniquely) the composite of the standard incl.

$$\begin{array}{ccc} \underline{i} & \xrightarrow{\iota} & \underline{i}' \\ \\ s & \longmapsto & s \end{array}$$

and an automorphism of $\underline{i}'$.

The functor

$$\mathcal{I}^{k+1} \xrightarrow{G_k(E;X)} Top$$

is given on objects by

$$G_k(\underline{i_0}, \dots, \underline{i_k}) = F(S^{i_0} \wedge \dots \wedge S^{i_k}, E_{i_0} \wedge \dots \wedge E_{i_k} \wedge X).$$

To define G_k on morphism, write $i_r' = i_r + j_r$ and let $(\underline{i_0}, \dots, \iota, \dots, \underline{i_k})$ be the morphism in $\mathcal{I}^{k+1}$ given by the standard incl. $\underline{i_r} \xrightarrow{\iota} \underline{i_r'}$. Then the induced map

$$G_k(\underline{i_0}, \dots, \underline{i_r}, \dots, \underline{i_k}) \longrightarrow G_k(\underline{i_0}, \dots, \underline{i_r'}, \dots, \underline{i_k})$$

takes the map

$$S^{i_0} \wedge \dots \wedge S^{i_r} \wedge \dots \wedge S^{i_k} \xrightarrow{f} E_{i_0} \wedge \dots \wedge E_{i_r} \wedge \dots \wedge E_{i_k} \wedge X$$

to the following composite

$$S^{i_0} \wedge \dots \wedge S^{i_r'} \wedge \dots \wedge S^{i_k} \underset{\sim}{\xrightarrow{can}} S^{i_0} \wedge \dots \wedge S^{i_r} \wedge \dots \wedge S^{i_k} \wedge S^{j_r}$$

$$\xrightarrow{f \wedge \eta} E_{i_0} \wedge \dots \wedge E_{i_r} \wedge \dots \wedge E_{i_k} \wedge X \wedge E_{j_r}$$

$$\xrightarrow[\sim]{\text{can}} E_{i_0} \wedge \dots \wedge E_{i_r} \wedge E_{j_r} \wedge \dots \wedge E_{i_k} \wedge X$$

$$\xrightarrow{\mu} E_{i_0} \wedge \dots \wedge E_{i_r'} \wedge \dots \wedge E_{i_r} \wedge X .$$

The symmetric group $\Sigma_{i_r'}$ acts on $S^{i_r'}$ and $E_{i_r'}$ and on $G_k(\underline{i_0}, \dots, \underline{i_r'}, \dots, \underline{i_k})$ by conjugation. This defines $G_k(E;X)$ on morphisms. It is well-defined since μ_{i_r, j_r} is $\Sigma_{i_r} \times \Sigma_{j_r}$ - equivariant.

We define the cyclic structure maps

$$THH(E;X)[k] \xrightarrow{d_r} THH(E;X)[k-1] \qquad 0 \le r \le k$$

$$THH(E;X)[k] \xrightarrow{s_r} THH(E;X)[k+1] \qquad 0 \le r \le k$$

$$THH(E;X)[k] \xrightarrow{t_k} THH(E;X)[k] .$$

The category I has a strict moncidal structure given by concatenation :

$$\begin{array}{ccc} I \times I & \xrightarrow{\mu} & I \\ (\underline{i}, \underline{i}') & \longmapsto & \underline{i+i'} \\ \downarrow (f, f') & & \downarrow f \mu f' \\ (\underline{j}, \underline{j}') & \longmapsto & \underline{j+j'} \end{array}$$

$$(f \mu f')(s) = \begin{cases} f(s) & 1 \le s \le i \\ f'(s-i) + j & i+1 \le s \le i+i' \end{cases}$$

Define functors of index categories

$$I^{k+1} \xrightarrow{\partial_r} I^{k} \qquad 0 \le r \le k$$

$$I^{k+1} \xrightarrow{s_r} I^{k+2} \qquad 0 \le r \le k$$

$$I^{k+1} \xrightarrow{t_k} I^{k+1}$$

by

$$\partial_r(\underline{i_0}, \dots, \underline{i_k}) = (\underline{i_0}, \dots, \underline{i_r} \amalg \underline{i_{r+1}}, \dots, \underline{i_k}) \qquad 0 \le r < k$$

$$= (\underline{i_k} \amalg \underline{i_0}, \underline{i_1}, \dots, \underline{i_{k-1}}) \qquad r = k$$

$$s_r(\underline{i_0}, \dots, \underline{i_k}) = (\underline{i_0}, \dots, \underline{i_r}, \underline{0}, \underline{i_{r+1}}, \dots, \underline{i_k}) \quad 0 \le r \le k$$

$$t_k(\underline{i_0}, \dots, \underline{i_k}) = (\underline{i_k}, \underline{i_0}, \underline{i_1}, \dots, \underline{i_{k-1}})$$

and natural transformations

$$G_k(E;X) \xrightarrow{\delta_r} G_{k-1}(E;X) \circ \partial_r \qquad 0 \le r \le k$$

$$G_k(E;X) \xrightarrow{\sigma_r} G_{k+1}(E;X) \circ s_r \qquad 0 \le r \le k$$

$$G_k(E;X) \xrightarrow{\tau_k} G_k(E;X) \circ t_k$$

where δ_r takes the map

$$S^{i_0} \wedge \dots \wedge S^{i_k} \xrightarrow{f} E_{i_0} \wedge \dots \wedge E_{i_k} \wedge X$$

to the composite

$$S^{i_0} \wedge \dots \wedge S^{i_r + i_{r+1}} \wedge \dots \wedge S^{i_k} \underset{\sim}{\xrightarrow{\text{can}}} S^{i_0} \wedge \dots \wedge S^{i_r} \wedge S^{i_{r+1}} \wedge \dots \wedge S^{i_k}$$

$$\xrightarrow{f} E_{i_0} \wedge \cdots \wedge E_{i_r} \wedge E_{i_{r+1}} \wedge \cdots \wedge E_{i_k} \wedge X$$

$$\xrightarrow{\mu} E_{i_0} \wedge \cdots \wedge E_{i_r + i_{r+1}} \wedge \cdots \wedge E_{i_k} \wedge X ,$$

if $0 \le r < k$, and to the composite

$$S^{i_k + i_0} \wedge S^{i_1} \wedge \cdots \wedge S^{i_{k-1}} \xrightarrow[\sim]{\text{can}} S^{i_0} \wedge \cdots \wedge S^{i_{k-1}} \wedge S^{i_k}$$

$$\xrightarrow{f} E_{i_0} \wedge \cdots \wedge E_{i_{k-1}} \wedge E_{i_k} \wedge X$$

$$\xrightarrow[\sim]{\text{can}} E_{i_k} \wedge E_{i_0} \wedge \cdots \wedge E_{i_{k-1}} \wedge X$$

$$\xrightarrow{\mu} E_{i_k + i_0} \wedge E_{i_1} \wedge \cdots \wedge E_{i_{k-1}} \wedge X ,$$

if $r = k$, and where σ_r and τ_k are defined similarly. Then the cyclic structure maps are the composites

$$d_r : \operatorname*{hocolim}_{I^{k+1}} G_k \xrightarrow{\delta_r} \operatorname*{hocolim}_{I^{k+1}} G_{k-1} \circ \partial_r \xrightarrow{\text{can}} \operatorname*{hocolim}_{I^{k-1}} G_{k-1}$$

$$s_r : \operatorname*{hocolim}_{I^{k+1}} G_k \xrightarrow{\sigma_r} \operatorname*{hocolim}_{I^{k+1}} G_{k+1} \circ \delta_r \xrightarrow{\text{can}} \operatorname*{hocolim}_{I^{k+2}} G_{k+1}$$

$$t_k : \operatorname*{hocolim}_{I^{k+1}} G_k \xrightarrow{\tau_k} \operatorname*{hocolim}_{I^{k+1}} G_k \circ t_k \xrightarrow{\text{can}} \operatorname*{hocolim}_{I^{k+1}} G_{k+1} .$$

To show that δ_r, σ_r, and τ_k are actually natural transformations and that d_r, s_r, and t_k satisfy the cyclic relations, one uses all the axioms of a symm. ring spectrum.

Define the pointed $\mathbb{T}$ - space

$$THH(E;X) = |[k] \mapsto THH(E;X)[k]| .$$

We need a slight modification. Let (n) be the finite ordered set

$$(n) = \{1, 2, \ldots, n\}, \qquad (0) = \emptyset,$$

and let $\underline{I}^{(n)}$ be the product category. It is strict monoidal under componentwise concatenation $\sqcup^{(n)}$. The functor

$$\underline{I}^{(n)} \xrightarrow{\sqcup_n} \underline{I}$$

$$(\underline{i}_1, \ldots, \underline{i}_n) \longmapsto \underline{i}_1 \sqcup \cdots \sqcup \underline{i}_n$$

does not preserve the monoidal structure (unless $n = 0$ or $n = 1$). Let

$$G_k^{(n)}(E;X) = G_k(E;X) \circ (\sqcup_n)^{k+1}$$

and define

$$THH^{(n)}(E;X)[k] = \operatorname*{hocolim}_{(\underline{I}^{(n)})^{k+1}} G_k^{(n)}(E;X) .$$

This gives a cyclic pointed space as before, and hence, a pointed $\mathbb{T}$ - space

$$THH^{(n)}(E;X) = |[k] \mapsto THH^{(n)}(E;X)[k]| .$$

We define a symmetric spectrum $T(E)$ with n'th space

$$T(E)_n = THH^{(n)}(E; S^n).$$

There are two commuting Σ_n-actions on this space. One is induced from the permutation action on $I^{(n)}$; the other from the action on S^n. We give $T(E)_n$ the diagonal Σ_n-action. Let

$$I^{(n)} \xrightarrow{\iota_{m,n}} I^{(m+n)}$$

be the inclusion as the last n factors. Then the canonical maps define a natural transformation

$$S^m \wedge G_k^{(n)}(E; S^n) \xrightarrow{\alpha_{m,n}} G_k^{(m+n)}(E; S^{m+n}) \circ \iota_{m,n}^{k+1}$$

and hence we have a map

$$S^m \wedge \operatorname*{hocolim}_{(I^{(n)})^{k+1}} G_k^{(n)}(E; S^n)$$

$$\xleftarrow[\sim]{can} \operatorname*{hocolim}_{(I^{(n)})^{k+1}} S^m \wedge G_k^{(n)}(E; S^n)$$

$$\xrightarrow{\alpha_{m,n}} \operatorname*{hocolim}_{(I^{(n)})^{k+1}} G_k^{(m+n)}(E; S^{m+n}) \circ \iota_{m,n}^{k+1}$$

$$\xrightarrow{can} \operatorname*{hocolim}_{(I^{(m+n)})^{k+1}} G_k^{(m+n)}(E; S^{m+n}).$$

or

$$S^m \wedge THH^{(n)}(E; S^n)[k] \longrightarrow THH^{(m+n)}(E; S^{m+n}).$$

This is a map of cyclic pointed spaces, so we get

$$S^m \wedge T(E)_n \xrightarrow{\sigma_{m,n}} T(E)_{m+n}.$$

This makes $T(E)$ a symm. spectrum.

There is an isom. of categories

$$(I^{(m)})^{k+1} \times (I^{(n)})^{k+1} \xrightarrow[\sim]{\tau_{m,n}} (I^{(m+n)})^{k+1}$$

$$(((i_{0,1}, .., i_{0,m}), .., (i_{k,1}, .., i_{k,m})), (i'_{0,1}, .., i'_{0,n}), .., (i'_{k,1}, .., i'_{k,n})))$$

$$\mapsto ((i_{0,1}, .., i_{0,m}, i'_{0,1}, .., i'_{0,n}), .., (i_{k,0}, .., i_{k,m}, i'_{k,0}, .., i'_{k,n}))$$

and the canonical maps and the mult. in E gives a natural transf.

$$G_k^{(m)}(E; S^m) \wedge G_k^{(n)}(E; S^n) \longrightarrow G_k^{(m+n)}(E; S^{m+n}) \circ \tau_{m,n}$$

and hence a map

$$\operatorname*{hocolim}_{(I^{(m)})^{k+1}} G_k^{(m)}(E; S^m) \wedge \operatorname*{hocolim}_{(I^{(n)})^{k+1}} G_k^{(n)}(E; S^n)$$

$$\xleftarrow[\sim]{can} \operatorname*{hocolim}_{(I^{(m)})^{k+1} \times (I^{(n)})^{k+1}} G_k^{(m)}(E; S^m) \wedge G_k^{(n)}(E; S^n)$$

$$\longrightarrow \operatorname*{hocolim}_{(I^{(m)})^{k+1} \times (I^{(n)})^{k+1}} G_k^{(m+n)}(E; S^{m+n}) \circ \tau_{m,n}$$

$$\xrightarrow{\text{can}} \underset{(I^{(m+n)})^{k+1}}{\mathrm{hocolim}}\, G_k^{(m+n)}(E; S^{m+n})$$

or

$$THH^{(m)}(E; S^m)[k] \wedge THH^{(n)}(E; S^n)[k]$$
$$\longrightarrow THH^{(m+n)}(E; S^{m+n})[k].$$

If E is commutative, these maps form a map of cyclic pointed spaces so we get

$$T(E)_m \wedge T(E)_n \xrightarrow{\mu_{m,n}} T(E)_{m+n}.$$

There is a map

$$S^n \xrightarrow{\eta_n} T(E)_n$$

defined similarly to $\sigma_{m,n}$. These maps make $T(E)$ a symmetric ring spectrum (with a $\mathbb{T}$-action).

G = cpt. Lie group.

$\mathcal{J}_G$ = (index) category enriched in pointed G-spaces

obj.: all f.d. orthogonal G-repr.

morph.: let $O(\lambda, \lambda')$ be the G-space of linear isometries from λ to λ' with G acting by conjugation; let

$$\begin{array}{ccc} E(\lambda,\lambda') & \hookrightarrow & O(\lambda,\lambda') \times \lambda' \\ \downarrow & & \downarrow \mathrm{pr}_1 \\ O(\lambda,\lambda') & = & O(\lambda,\lambda') \end{array}$$

be the subbundle of pairs (f, x) with $x \in \lambda' \setminus f(\lambda)$; it is a sub-$G$-bundle and by definition

$$\underline{\mathrm{Hom}}_{\mathcal{J}_G}(\lambda, \lambda') = \mathrm{Th}(E(\lambda,\lambda')).$$

comp.: is the G-map

$$\underline{\mathrm{Hom}}_{\mathcal{J}_G}(\lambda', \lambda'') \wedge \underline{\mathrm{Hom}}_{\mathcal{J}_G}(\lambda, \lambda') \xrightarrow{\circ} \underline{\mathrm{Hom}}_{\mathcal{J}_G}(\lambda, \lambda'')$$

$$((g,y),(f,x)) \longmapsto (g \circ f, g(x)+y).$$

An <u>orthogonal</u> G-<u>spectrum</u> is a cts.

G-equivariant functor

$$\mathcal{J}_G \xrightarrow{E} \mathrm{Top}_G$$

where Top_G is the category of pointed G-spaces and (pointed G-spaces of) non-equivariant maps.

Rem 1) If $\dim(\lambda) = \dim(\lambda')$, then

$$\underline{\mathrm{Hom}}_{\mathcal{J}_G}(\lambda, \lambda') = O(\lambda, \lambda')_+$$

and

$$O(\lambda, \lambda')_+ \underset{O(\lambda)}{\wedge} E_\lambda \xrightarrow{\sim} E_{\lambda'} .$$

2) We have

$$\underline{\mathrm{Hom}}_{\mathcal{J}_G}(\lambda, \lambda' \oplus \lambda) = O(\lambda' \oplus \lambda)_+ \underset{O(\lambda')}{\wedge} S^{\lambda'}$$

and $G \times O(\lambda') \times O(\lambda)$-equivariant structure maps

$$S^{\lambda'} \wedge E_\lambda \longrightarrow E_{\lambda' \oplus \lambda} . \quad "$$

Adjoint pair of cts. G-functors

$$\mathrm{Top}_G \underset{ev_\lambda}{\overset{F_\lambda}{\rightleftarrows}} (\mathrm{Top}_G)^{\mathcal{J}_G}$$

(pointed G-spaces of maps)

$$(F_\lambda A)_{\lambda' \oplus \lambda} = O(\lambda' \oplus \lambda)_+ \wedge_{O(\lambda')} S^{\lambda'} \wedge A .$$

Level model structure :

weak equiv. = object-wise weak equiv. of G - spaces

fibrations = object-wise Serre fibr. of G - spaces.

cofibrations = LLP w.r.t. trivial fibr.

Generators :

$$FI : F_\lambda((G/H \times S^{n-1})_+) \rightarrowtail F_\lambda((G/H \times D^n)_+)$$

$$FJ : F_\lambda((G/H \times D^n)_+) \xrightarrow{\sim} F_\lambda((G/H \times D^n \times [0,1])_+)$$

where $n \geq 0$, λ runs through a set of repr. of isom. classes of all f.d. ortho. G-repr., and $H \subset G$ runs through all closed subgroups. //

Obtain stable model structure by localizing w.r.t. the following set of maps

$$A = \{ F_{\lambda' \oplus \lambda} S^{\lambda'} \xrightarrow{l_{\lambda,\lambda'}} F_\lambda S^0 \}$$

where $l_{\lambda,\lambda'}$ is the adjoint of

$$S^{\lambda'} \longrightarrow (F_\lambda S^0)_{\lambda\oplus\lambda'} = O(\lambda'\oplus\lambda)_+ \wedge_{O(\lambda')} S^{\lambda'}$$

$$x' \longmapsto (e, x')$$

An orthogonal G-spectrum E is A-local if for all λ, λ',

$$[F_\lambda S^0, E]_{\text{level}} \xrightarrow{l^*_{\lambda,\lambda'}} [F_{\lambda'\oplus\lambda} S^{\lambda'}, E]_{\text{level}},$$

is a bijection, or equivalently, if

$$E_\lambda \longrightarrow \Omega^{\lambda'}(E_{\lambda'\oplus\lambda})$$

is a weak equivalence of pointed G-spaces.

A map $X \xrightarrow{f} X'$ of orthogonal G-spectra is an A-local weak equiv. if for all A-local E,

$$[X', E]_{\text{level}} \xrightarrow{f^*} [X, E]_{\text{level}}$$

is a bijection.

Stable model structure:

weak equiv. = A-local weak equiv.
cofibrations = level cofibrations
fibrations = RLP w.r.t. trivial cof.

Generators :

$$FI : F_\lambda((G/H \times S^{n-1})_+) \rightarrowtail F_\lambda((G/H \times D^n)_+)$$

$$K : F_\lambda((G/H \times D^n)_+) \xrightarrow{\sim} F_\lambda((G/H \times D^n \times [0,1])_+)$$

and the maps $(i_{\lambda'',m}, k_{\lambda,\lambda'})$ where

$$F_{\lambda'\oplus\lambda} S^{\lambda'} \xrightarrow{k_{\lambda,\lambda'}} \mathrm{Cyl}(\ell_{\lambda,\lambda'}) . \qquad //$$

Define homotopy groups. Choose a sequence of f.d. ortho. G-repr.

$$\lambda_1 \subset \lambda_2 \subset \lambda_3 \subset \cdots$$

s.t. every f.d. ortho. G-repr. is isom. to a subrepr. of some λ_i. Then

$$\pi_q^H(X) := \operatorname*{colim}_i \pi_q^H(\Omega^{\lambda_i} X_{\lambda_i}) .$$

<u>Prop</u> A map $X \xrightarrow{f} X'$ of orthogonal G-spectra is a stable equivalence if and only if $\pi_q^H(f)$ is an isom. for all integers q and all closed subgroups $H \subset G$. //

If E is a symmetric ring spectrum, we define an ortho. $\mathbb{T}$-spectrum $T(E)$ with λ'th space

$$T(E)_\lambda = THH(E; S^\lambda).$$

There are two $\mathbb{T}$-actions on this space: One comes from the cyclic structure of $THH(E; S^\lambda)[-]$; the other is induced by the $\mathbb{T}$-action on S^λ. We give $T(E)$ the diagonal $\mathbb{T}$-action. The $O(\lambda)$-action on $T(E)$ is induced from the $O(\lambda)$-action on S^λ.

Actually, it is ~~better~~ to define $T(E)$ to be the symmetric orthogonal $\mathbb{T}$-spectrum with (n, λ)'th space

$$T(E)_{n,\lambda} := THH^{(n)}(E; S^n \wedge S^\lambda)$$

with diagonal $\mathbb{T}$- and Σ_n-actions. The advantage is that for E a commutative symmetric ring spectrum, there is a product map

$$T(E) \wedge T(E) \longrightarrow T(E)$$

that makes $T(E)$ a comm. symm. orthogonal $\mathbb{T}$-spectrum. Hence, the homotopy groups

$$TR^n_q(E; p) = [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(E)]$$

form a graded-commutative ring (for fixed n and varying q).

Let G be a cpt. Lie grp. A pointed G-CW-complex is a pointed (left) G-space X together with a sequence of pointed sub-G-spaces

$$* = sk_{-1}X \subset sk_0 X \subset sk_1 X \subset \cdots \subset X$$

and for all $n \geq 0$, a push-out square of un-pointed G-spaces

$$\begin{tikzcd}
\coprod_\alpha G/H_\alpha \times S^{n-1} \arrow[r, "\varphi_n"] \arrow[d] & sk_{n-1}X \arrow[d] \\
\coprod_\alpha G/H_\alpha \times D^n \arrow[r, "\varphi_n"] & sk_n X
\end{tikzcd}$$

s.t. the can. pointed G-map

$$\operatorname{colim}_n sk_n X \longrightarrow X$$

is a homeomorphism.

<u>Lemma</u> Let X be a pointed G-CW-complex, and let $d(H)$ be the maximum of the dimension of the cells of orbit-type G/H. Let Y be a pointed G-space such that Y^H is $n(H)$-connected. Then the equivariant mapping space

$$F_G(X, Y) = F(X, Y)^G$$

is m-connected, where

$$m = \min\{ n(H) - d(H) \mid H \in \mathcal{O}_G(X) \}.$$

Here $\mathcal{O}_G(X)$ is the set of subgroups $H \subset G$ s.t. X has a cell of orbit-type G/H.

pf Show by induction that $F(sk_n X, Y)^G$ is m-connected, the case $n = -1$ being trivial. The map $sk_{n-1} X \rightarrowtail sk_n X$ has the G-htpy.-extension-property, so

$$F(sk_n X, Y)^G \longrightarrow F(sk_{n-1} X, Y)^G$$

has the homotopy-lifting-property. The fiber over the base-pt. is can. homeo. to

$$\begin{aligned} & F(sk_n X / sk_{n-1} X, Y)^G \\ & \xrightarrow[\sim]{\varphi_n} F(\bigvee_\alpha (G/H_\alpha)_+ \wedge S^n, Y)^G \\ & \xrightarrow{\sim} \prod_\alpha F((G/H_\alpha)_+ \wedge S^n, Y)^G \\ & \xleftarrow{\sim} \prod_\alpha F(S^n, Y^{H_\alpha}). \end{aligned}$$

Since $F(S^n, Y^{H_\alpha})$ is $(n(H_\alpha) - n)$-conn. and since $n(H_\alpha) - n \geq m$, the induction step follows. Finally,

$$F(X, Y)^G \xrightarrow{\sim} \operatorname{holim}_n F(sk_n X, Y)^G. \qquad //$$

Cor Let $X \xrightarrow{f} Y$ be a map of ortho. G-spectra, and let $H \subset G$ be a closed subgroup. Suppose that for all f.d. ortho. G-repr. λ, the map

$$X_\lambda^H \xrightarrow{f_\lambda^H} Y_\lambda^H$$

is $(\dim(\lambda) + \varepsilon_H(\lambda))$-connected, where $\varepsilon_H(\lambda)$ tends to infinity with λ. Then $\pi_*^H(f)$ is an isomorphism. //

We define a natural map of pointed $\mathbb{T}$-spaces

$$\rho_p^*(T(A)_{n,\lambda})^{C_p} \xrightarrow{r_p} T(A)_{n,\rho_p^*\lambda^{C_p}}$$

and show that the induced map of $C_{p^{s-1}}$-fixed sets

$$(T(A)_{n,\lambda})^{C_{p^s}} \longrightarrow (T(A)_{n,\rho_p^*\lambda^{C_p}})^{C_{p^{s-1}}}$$

is $(n + \dim(\lambda) - 1)$-conn. We have

$$T(A)_{n,\lambda} := THH^{(n)}(A; S^{n+\lambda})$$

$$= |\, THH^{(n)}(A; S^{n+\lambda})[-]\,|$$

$$\underset{\sim}{\xleftarrow{D}} |\, sd_{p^s} THH^{(n)}(A; S^{n+\lambda})[-]\,|$$

where

$$(sd_r X)[k] = X[r(k+1)-1]$$

is the r-fold edge-wise subdivision. Here the action by (the generator $e^{2\pi i/r}$) of $C_r \subset \mathbb{T}$ is given by the (cyclic) operator $t_{r(k+1)-1}^{k+1}$, so

$$(T(A)_{n,\lambda})^{C_r} \xleftarrow{\sim} |\, (sd_r THH^{(n)}(A; S^{n+\lambda})[-])^{C_r} \,|.$$

There is a canonical homeomorphism

$$(sd_r THH(A; S^{n+\lambda})[k])^{C_r} \xleftarrow{\sim}$$

$$\operatorname*{hocolim}_{(I^{(n)})^{k+1}} F((S^{j_0}\wedge\dots\wedge S^{j_k})^{\wedge r}, (\tilde{A}_{j_0}\wedge\dots\wedge\tilde{A}_{j_k})^{\wedge r}\wedge S^{n+\lambda})^{C_r}$$

where C_r acts on $(--)^{\wedge r}$ by cyclically permuting the smash factors, on $S^{n+\lambda}$ by the action induced from the C_r-action on λ, and on the mapping space by conjugation. The map r_p is then induced from the map

$$\rho_p^* F((S^{j_0}\wedge\dots\wedge S^{j_k})^{\wedge r}, (\tilde{A}_{j_0}\wedge\dots\wedge\tilde{A}_{j_k})^{\wedge r}\wedge S^{n+\lambda})^{C_p}$$
$$\to \rho_p^* F(((S^{j_0}\wedge\dots\wedge S^{j_k})^{\wedge r})^{C_p}, ((\tilde{A}_{j_0}\wedge\dots\wedge\tilde{A}_{j_k})^{\wedge r}\wedge S^{n+\lambda})^{C_p})$$
$$= F((S^{j_0}\wedge\dots\wedge S^{j_k})^{\wedge \bar{r}}, (\tilde{A}_{j_0}\wedge\dots\wedge\tilde{A}_{j_k})^{\wedge\bar{r}}\wedge S^{n+\rho_p^*\lambda^{C_p}})$$

induced by the inclusion

$$((S^{j_0}\wedge\dots\wedge S^{j_k})^{\wedge r})^{C_p} \hookrightarrow (S^{j_0}\wedge\dots\wedge S^{j_k})^{C_p}.$$

Here $\bar{r} = r/p$. Similarly, the map $(r_p)^{C_{p^{s-1}}}$ is given by the maps

$$F((S^{j_0}\wedge\dots\wedge S^{j_k})^{\wedge r}, (\tilde{A}_{j_0}\wedge\dots\wedge\tilde{A}_{j_k})^{\wedge r}\wedge S^{n+\lambda})^{C_{p^s}}$$
$$\to F((S^{j_0}\wedge\dots\wedge S^{j_k})^{\wedge\bar{r}}, (\tilde{A}_{j_0}\wedge\dots\wedge\tilde{A}_{j_k})^{\wedge\bar{r}}\wedge S^{n+\rho_p^*\lambda^{C_p}})^{C_{p^{s-1}}}$$

induced from the above inclusion. The fiber of the latter map is the equiv. mapping space

$$F\left((S^{\lambda_{j_0}}\wedge\cdots\wedge S^{\lambda_{j_k}})^{\wedge r}/(S^{\lambda_{j_0}}\wedge\cdots\wedge S^{\lambda_{j_k}})^{\wedge\bar r},\ (\bar A_{j_0}\wedge\cdots\wedge\bar A_{j_k})^{\wedge r}\wedge S^{n+\lambda}\right)^{C_{p^s}}.$$

Now for every C_{p^s}-CW-cx. X, the quotient X/X^{C_p} is a free C_{p^s}-CW-cx., i.e. all cells have orbit-type C_{p^s}. Hence, the equiv. mapping space above is m-connected with

$$\begin{aligned} m &= n(\{1\}) - d(\{1\}) \\ &= jr + n + \dim(\lambda) - 1 - jr \\ &= n + \dim(\lambda) - 1. \end{aligned}$$

We use the map

$$\rho_p^*(T(A)_{n,\lambda})^{C_p} \xrightarrow{r_p} T(A)_{n,\rho_p^*\lambda}{}^{C_p}$$

to understand the groups

$$\begin{aligned} TR_q^n(A;p) &= \pi_q^{C_{p^{n-1}}}(T(A)) \\ &= \operatorname*{colim}_{m,i} \pi_q\left(F(S^{m+\lambda_i}, T(A)_{m,\lambda_i})^{C_{p^n}}\right). \end{aligned}$$

In general, if G is a cpt. Lie group, we let $E = EG$ be a free G-CW-complex that non-equivariantly is contractible, and define $\tilde E = \tilde EG$ as the mapping cone

$$E_+ \to S^0 \to \tilde{E} \to E[-1].$$

Then for closed subgrps. $H \subset G$

$$(EG_+)^H = \begin{cases} \{*\} & H \neq \{1\} \\ \simeq S^0 & H = \{1\} \end{cases}$$

$$(\tilde{E}G)^H = \begin{cases} S^0 & H \neq \{1\} \\ \simeq pt & H = \{1\} \end{cases}$$

If T is a G-spectrum, we get an induced cofibration sequence of G-spectra

$$E_+ \wedge T \to T \to \tilde{E} \wedge T \to E_+ \wedge T[-1]$$

which induces a l.e.s. of htpy. grps.

$$\cdots \to \pi_q^H(E_+ \wedge T) \to \pi_q^H(T) \to \pi_q^H(\tilde{E} \wedge T) \to \cdots$$

For $H \subset G$ finite, we write

$$\mathbb{H}_q(H, T) := \pi_q^H(E_+ \wedge T).$$

The skeleton filtr. of E_+ as a pointed H-CW-complex gives

$$E^2_{s,t} = H_s(H, \pi_t(T)) \Rightarrow \mathbb{H}_{s+t}(H, T)$$

so this term is accessible to calculations. The term $\pi_q^H(\tilde{E}\wedge T)$ is in general difficult to understand. But for $T = T(A)$ we have the following result.

Prop There is a canonical isom.

$$\pi_q^{G_{p^{n-1}}}(\tilde{E}\wedge T(A)) \xrightarrow{\sim} \pi_q^{G_{p^{n-2}}}(T(A)).$$

pf The isom. is given by the following composite.

$$\begin{aligned}
&\pi_q^{G_{p^{n-1}}}(\tilde{E}\wedge T(A)) \\
&= \operatorname*{colim}_{m,i} \pi_q(F(S^{m+\lambda_i}, \tilde{E}\wedge T(A)_{m,\lambda_i})^{G_{p^{n-1}}}) \\
&\xrightarrow{f_*} \operatorname*{colim}_{m,i} \pi_q(F(S^{m+\lambda_i^{G_p}}, (\tilde{E}\wedge T(A)_{m,\lambda_i})^{G_p})^{G_{p^{n-1}}/G_p}) \\
&= \operatorname*{colim}_{m,i} \pi_q(F(S^{m+\rho_p^*\lambda_i^{G_p}}, \rho_p^*(T(A)_{m,\lambda_i})^{G_p})^{G_{p^{n-2}}}) \\
&\xrightarrow{r_{p*}} \operatorname*{colim}_{m,i} \pi_q(F(S^{m+\rho_p^*\lambda_i^{G_p}}, T(A)_{m,\rho_p^*\lambda_i^{G_p}})^{G_{p^{n-2}}}) \\
&= \pi_q^{G_{p^{n-2}}}(T(A)).
\end{aligned}$$

We show that the map f_* is an isom. This is induced from the inclusion

$$S^{m+\lambda_i^{G_p}} \hookrightarrow S^{m+\lambda_i}$$

so the domain and range of f_* sit

in a l.e.s. where the remaining terms are the htpy. grps. of the mapping space

$$F(S^{m+\lambda_i}/S^{m+\lambda_i^{C_p}}, \tilde{E}\wedge T(A)_{m,\lambda_i})^{C_{p^{n-1}}}.$$

But these are all zero. Indeed, the space $S^{m+\lambda_i}/S^{m+\lambda_i^{C_p}}$ has the structure of a free pointed $C_{p^{n-1}}$-CW-complex, and $\tilde{E}$, and hence $\tilde{E}\wedge T(A)_{m,\lambda_i}$, is non-equivariantly contractible.

We show that the map r_{p*} is an isom. By the lemma on the connectivity of an equivariant mapping space, the map induced by r_p

$$F(S^{m+\rho_p^*\lambda_i^{C_p}}, \rho_p^*(T(A)_{m,\lambda_i})^{C_p})^{C_{p^{n-2}}}$$
$$\to F(S^{m+\rho_p^*\lambda_i^{C_p}}, T(A)_{m,\rho_p^*\lambda_i^{C_p}})^{C_{p^{n-2}}}$$

is r-connected, where

$$r = \min\{n(C_{p^s}) - d(C_{p^s}) \mid 0 \le s \le n-2\}.$$

We showed earlier that

$$n(C_{p^s}) = m + \dim(\lambda_i) - 1,$$

and

$$d(G_{p^s}) = m + \dim(\lambda_i^{G_{p^{s+1}}}) .$$

Hence,

$$r = \dim(\lambda_i) - \dim(\lambda_i^{G_{p^{1}}}) - 1$$

which tends to infinity with i. So r_{p*} is an isom.

Finally, if

$$\lambda_1 \subset \lambda_2 \subset \lambda_3 \subset \cdots$$

is a sequence of f.-d. ortho. π-repr. s.t. every f.-d. ortho. π-repr. is isom. to a sub-repr. of some λ_i, then the same is true for the sequence

$$\rho_p^* \lambda_1^{G_p} \subset \rho_p^* \lambda_2^{G_p} \subset \rho_p^* \lambda_3^{G_p} \subset \cdots .$$

This justifies the last equality. //

We obtain the following fundamental long-exact sequence

$$\cdots \to H_q(G_{p^{n-1}}, T(A)) \xrightarrow{N} TR_q^n(A;p) \xrightarrow{R} TR_q^{n-1}(A;p) \to \cdots$$

Let G be a cpt. Lie group, let $E = EG$ be a free G-CW-complex that is non-equivariantly contractible, and recall the cofibration sequence of G-CW-complexes

$$E_+ \to S^0 \to \tilde{E} \to E_+[-1].$$

Let T be a G-spectrum, and let

$$T \xrightarrow{\gamma} F(E_+, T)$$

be the map induced from the can. projection $E_+ \to S^0$. The cofibration sequence above and the map γ gives rise to a diagram of T-spectra

$$\begin{array}{ccccccc}
E_+ \wedge T & \to & T & \to & \tilde{E} \wedge T & \to & E_+ \wedge T[-1] \\
\downarrow & & \downarrow \Gamma & & \downarrow \hat{\Gamma} & & \downarrow \\
E_+ \wedge F(E_+, T) & \to & F(E_+, T) & \to & \tilde{E} \wedge F(E_+, T) & \to & E_+ \wedge F(E_+, T)[-1]
\end{array}$$

where the rows are cofibration sequences of T-spectra. We shall see shortly that the left-hand (and right-hand) map is an equivalence of G-spectra. If $H < G$ is a finite subgroup, we write the induced diagram on $\pi_*^H(-)$

in the following way.

$$
\begin{array}{ccccccccc}
\cdots \to & \mathbb{H}_q(H,T) & \to & \pi_q^H(T) & \to & \pi_q^H(\tilde{E}\wedge T) & \xrightarrow{\partial} & \mathbb{H}_{q-1}(H,T) & \to \cdots \\
& \| & & \downarrow \Gamma & & \downarrow \hat{\Gamma} & & \| & \\
\cdots \to & \mathbb{H}_q(H,T) & \to & \mathbb{H}^{-q}(H,T) & \to & \hat{\mathbb{H}}^{-q}(H,T) & \xrightarrow{\partial} & \mathbb{H}_{q-1}(H,T) & \to \cdots
\end{array}
$$

The skeleton filtrations of E_+ and $\tilde{E}$ give rise to spectral sequences for the terms in the lower sequence. We now describe these spectral sequences.

Let k be a commutative ring, and let kH be the group algebra. It is a co-commutative Hopf-algebra, and the category of chain complexes of (left) kH-modules and chain homotopy classes of chain maps is a triangulated category and closed symmetric monoidal category and the two structures are compatible. For example, if X and Y are two chain-complexes of kH-modules, then the tensor-product $X \otimes Y$ and the function-complex $\underline{\mathrm{Hom}}(X,Y)$ are defined by

$$(X\otimes Y)_n = \bigoplus_{s+t=n} X_s \otimes_k Y_t$$

$$d(x\otimes y) = dx\otimes y + (-1)^{|x|} x\otimes dy$$

$$\underline{\mathrm{Hom}}(X,Y)_n = \prod_{t-s=n} \mathrm{Hom}_k(X_s, Y_t)$$

$$d(f(x)) = (df)(x) + (-1)^{|f|} f(dx).$$

We also recall the norm-element

$$N_H = \sum_{h \in H} h \in kH.$$

If M is a left kH-module, then multiplication by N_H defines a map of k-modules

$$M_H \xrightarrow{N_H} M^H$$

from the co-invariants

$$M_H := k \otimes_{kH} M$$

to the invariant

$$M^H := \mathrm{Hom}_{kH}(k, M).$$

The map is an isomorphism if $M = P$ is a projective kH-module. We also recall the canonical isomorphisms

$$(M \otimes N)_H \approx c^*M \otimes_{kH} N$$

$$\underline{\mathrm{Hom}}(M,N)^H \approx \mathrm{Hom}_{kH}(M,N)$$

where c^*M denotes M viewed as

a right kH-module with multiplication $m \cdot h = h^{-1}m$.

Let $P \xrightarrow{\varepsilon} k$ be a resolution of the trivial kH-module k by projective kH-modules, and let $\tilde{P}$ be the mapping-cone of ε s.t. we have an exact triangle

$$P \xrightarrow{\varepsilon} k \longrightarrow \tilde{P} \longrightarrow P[-1].$$

We define the Tate cohomology of H with coeff. in a left kH-module M to be

$$\hat{H}^i(H,M) = H_{-i}((\tilde{P} \otimes \underline{\mathrm{Hom}}(P,M))^H).$$

To see that this agrees with the usual definition, we need.

<u>Lemma</u> The following maps are quasi-isomorphisms

$$(P \otimes M)_H \underset{\sim}{\xrightarrow{N}} (P \otimes M)^H \longrightarrow (P \otimes \underline{\mathrm{Hom}}(P,M))^H.$$

<u>pf</u> The norm-map on the left is an isomorphism, since P is projective. For the right-hand map, we filter $(P \otimes \underline{\mathrm{Hom}}(P,M))^H$ after the first tensor

factor. This gives a strongly convergent fourth quadrant spectral sequence

$$E^1_{s,t} = H_t((P_s \otimes \underline{\mathrm{Hom}}(P,M))^H)$$
$$\Rightarrow H_{s+t}((P \otimes \underline{\mathrm{Hom}}(P,M))^H)$$

so it suffices to show that

$$(P_s \otimes M)^H \xrightarrow{\varepsilon^*} (P_s \otimes \underline{\mathrm{Hom}}(P,M))^H$$

is a quasi-isomorphism. Since both sides commute with filtered colimits, we can assume that P_s is a f.g. proj. kH-module. Then also the dual

$$DP_s = \underline{\mathrm{Hom}}(P_s, k)$$

is a f.g. proj. kH-module. Consider the diagr.

$$\begin{tikzcd}
(P_s \otimes M)^H \arrow[r, "\mathrm{id} \otimes \varepsilon^*"] \arrow[d, "\sim"] & (P_s \otimes \underline{\mathrm{Hom}}(P,M))^H \arrow[d, "\sim"] \\
\underline{\mathrm{Hom}}(DP_s, M)^H \arrow[r, "(\varepsilon \otimes \mathrm{id})^*"] & \underline{\mathrm{Hom}}(P \otimes DP_s, M)^H
\end{tikzcd}$$

The vertical maps are isom. and the map

$$P \otimes DP_s \xrightarrow{\varepsilon \otimes \mathrm{id}} DP_s$$

is a quasi-isom. between bounded below complexes of projective kH-modules. But then it is a chain-homotopy equivalence, and hence so is the top map. //

It follows that there are canonical isomorphisms

$$\hat{H}^i(H,M) \approx \begin{cases} H^i(H,M) & , i \geq 1 \\ H_{-i-1}(H,M) & , i \leq -1 \end{cases}$$

and a canonical exact sequence

$$0 \to \hat{H}^{-1}(H,M) \to H_0(H,M) \xrightarrow{N} H^0(H,M) \to \hat{H}^0(H,M) \to 0 .$$

So our definition agrees with the usual one. The advantage of the approach taken here is that it is easy to describe products. We choose chain maps

$$P \longrightarrow P \otimes P$$

$$\tilde{P} \otimes \tilde{P} \longrightarrow \tilde{P}$$

compatible with the canonical isom. $k \to k \otimes k$ and $k \otimes k \to k$. Such maps exist and are unique up to chain-homotopy. Then the cup product

$$\hat{H}^i(H, M) \otimes \hat{H}^j(H, M) \xrightarrow{\smile} \hat{H}^{i+j}(H, M \otimes N)$$

is defined to be the map on homology induced by the composite

$$(\tilde{P} \otimes \underline{\mathrm{Hom}}(P, M))^H \otimes (\tilde{P} \otimes \underline{\mathrm{Hom}}(P, N))^H$$

$$\xrightarrow{\text{can}} (\tilde{P} \otimes \tilde{P} \otimes \underline{\mathrm{Hom}}(P \otimes P, M \otimes N))^H$$

$$\longrightarrow (\tilde{P} \otimes \underline{\mathrm{Hom}}(P, M \otimes N))^H$$

where the last map is induced from the chosen chain-maps. This makes $\hat{H}^*(H, k)$ a graded ring and $\hat{H}^*(H, M)$ a graded $\hat{H}^*(H, k)$-module.

<u>Lemma</u> Suppose either that p is odd or that $n \geq 2$. Then, as graded $\mathbb{F}_p$-algebras,

$$\hat{H}^*(C_{p^n}, \mathbb{F}_p) = \Lambda\{u_n\} \otimes S\{t^{\pm 1}\}$$

where $\deg u_n = 1$ and $\deg t = 2$. //

<u>pf</u> Let $g \in C_{p^n}$ be a generator, and let $\epsilon : P \to k$ be the standard resolution with $P_s = \mathbb{F}_p C_{p^n} \cdot x_s$, with diff.

$$d(x_s) = \begin{cases} N \cdot x_{s-1} & s \text{ even} \\ (g-1) \cdot x_{s-1} & s \text{ odd}. \end{cases}$$

and with augmentation $\epsilon(x_0) = 1$. Then

$\tilde{P}_0 = \mathbb{F}_p \cdot y_0$, $\tilde{P}_s = \mathbb{F}_p G y_s$, $s \geq 1$, with

$$d(y_s) = \begin{cases} -(g-1)\cdot y_{s-1} & s \text{ even} \\ -N \cdot y_{s-1} & s > 1 \text{ odd} \\ -y_0 & s = 1 \end{cases}$$

and the generators u_n and t are repr. by the cycles $y_0 \otimes N x_1^*$ and $y_0 \otimes N x_2^*$. //

We return to the spectral sequence

$$\hat{E}^r_{s,t}(H,T) \Rightarrow \hat{\mathbb{H}}^{-s-t}(H,T)$$

obtained from the skeleton filtrations of E_+ and $\tilde{E}$. Taking reduced cellular chains with coeff. in k, the cofibration sequence of H-CW-complexes

$$E_+ \to S^0 \to \tilde{E} \to E_+[-1]$$

gives an exact triangle of complexes of kH-modules

$$P \to k \to \tilde{P} \to P[-1].$$

Since E is a free H-CW-complex, P is a complex of free kH-modules, and since E is contractible, $P \to k$ is a resolution. Then there is a canonical isomorphism of chain complexes of abelian groups

$$\hat{E}^1_{*,t}(H,T) = (\tilde{P} \otimes \underline{\mathrm{Hom}}(P, \pi_t T))^H .$$

Hence, the sp. seq. takes the form

$$\hat{E}^2_{s,t}(H,T) = \hat{H}^{-s}(H, \pi_t T) \Rightarrow \hat{\mathbb{H}}^{-s-t}(H,T).$$

If T is a comm. G-ring spectrum, then $\hat{\mathbb{H}}^*(H,T)$ is a graded ring and the spectral sequence is multiplicative.

Similarly, there are spectral sequences

$$E^2_{s,t}(H,T) = H^{-s}(H, \pi_t T) \Rightarrow \mathbb{H}^{-s-t}(H,T)$$

and

$$E^2_{s,t}(H,T) = H_s(H, \pi_t T) \Rightarrow \mathbb{H}_{s+t}(H,T).$$

The spectral sequence converge conditionally in the sence of Boardman.

Lemma Let A be a commutative ring. Then the following sequence is exact.

$$0 \to \mathrm{TR}^1_0(A;p) \xrightarrow{V^{n-1}} \mathrm{TR}^n_0(A;p) \xrightarrow{R} \mathrm{TR}^{n-1}_0(A;p) \to 0.$$

pf The map V^{n-1} factors as follows.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \hat{\mathbb{H}}_0(C_{p^{n-1}}, T(A)) \arrow[r, "N"] & \mathrm{TR}^n_0(A;p) \arrow[r, "R"] & \mathrm{TR}^{n-1}_0(A;p) \arrow[r] & 0 \\
& H_0(C_{p^{n-1}}, \pi_0 T(A)) \arrow[u, "\sim", "\text{edge}"'] \arrow[d, equal] & & & \\
& \mathrm{TR}^1_0(A;p) \arrow[uur, bend right, "V^{n-1}"'] & & &
\end{tikzcd}$$

so the lemma amounts to showing that the boundary map

$$\mathrm{TR}^{n-1}_1(A;p) \xrightarrow{\partial} \hat{\mathbb{H}}_0(C_{p^{n-1}}, T(A))$$

is equal to zero. The composite

$$\mathrm{TR}^1_0(A;p) \xrightarrow{V^{n-1}} \mathrm{TR}^n_0(A;p) \xrightarrow{F^{n-1}} \mathrm{TR}^1_0(A;p)$$

is multiplication by p^{n-1}, so the statement holds if p is a non-zero-divisor in A. (Recall that $\mathrm{TR}^1_0(A;p) = A$.) In general we choose a surjection

$$A' \twoheadrightarrow A$$

s.t. p is a non-zero-divisor in A'. We will show by induction on $n \geq 1$ that

$$\mathrm{TR}^n_q(A';p) \to \mathrm{TR}^n_q(A;p)$$

is surjective for $q \leq 1$. The lemma then follows from the diagram

$$\begin{array}{ccc}
\mathrm{TR}^{n-1}_1(A';p) & \xrightarrow{\partial = 0} & \mathbb{H}_0(C_{p^{n-1}}, T(A')) \\
\downarrow\wr & & \downarrow \\
\mathrm{TR}^{n-1}_1(A;p) & \xrightarrow{\partial} & \mathbb{H}_0(C_{p^{n-1}}, T(A))
\end{array}.$$

To prove the induction step, we use the following diagram.

$$\begin{array}{ccccccc}
\mathbb{H}_q(C_{p^{n-1}}, T(A')) & \to & \mathrm{TR}^n_q(A';p) & \to & \mathrm{TR}^{n-1}_q(A';p) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{H}_q(C_{p^{n-1}}, T(A)) & \to & \mathrm{TR}^n_q(A;p) & \to & \mathrm{TR}^{n-1}_q(A;p) & &
\end{array}.$$

The rows are exact for $q = 0$ and $q = 1$ (by what was already proved), and the right-hand vertical map is surjective by induction. Hence, it suffices to shows that the left-hand vertical map is surjective for $q = 0$ and $q = 1$. We use the spectral sequence

$$E^2_{s,t} = H_s(C_{p^{n-1}}, \pi_t T(A)) \Rightarrow \mathbb{H}_{s+t}(C_{p^{n-1}}, T(A)),$$

Since the canonical map

$$\Omega^q_A \longrightarrow \pi_q T(A)$$

is an isom., for $q \le 1$, and since $\pi_q T(A)$ is a trivial $C_{p^{n-1}}$-module, the sp. seq. takes the form

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \\ \Omega^1_A & \Omega^1_A/p^{n-1}\Omega^1_A & \Omega^1_A[p^{n-1}] & \cdots \\ A & A/p^{n-1}A & A[p^{n-1}] & \cdots \end{array}$$

so we have

$$A \xrightarrow[\sim]{\text{edge}} \mathbb{H}_0(C_{p^{n-1}}, T(A))$$

$$\Omega^1_A \xrightarrow{\text{edge}} \mathbb{H}_1(C_{p^{n-1}}, T(A)) \longrightarrow A/p^{n-1}A \to 0.$$

The lemma follows. //

Cor The following map is ring-isom.

$$W_n(A) \xrightarrow{I_n} TR^n_0(A;p)$$

$$(a_0, \dots, a_{n-1}) \longmapsto \sum_{0 \le s < n} V^s([a_s]_{n-s}).$$

pf The lemma implies that the following sequence is exact.

$$0 \to TR^{n-1}_0(A;p) \xrightarrow{V} TR^n_0(A;p) \xrightarrow{R^{n-1}} TR^1_0(A;p) \to 0.$$

Moreover, the following diagr. commutes.

$$\begin{tikzcd} TR^n_0(A;p) \arrow[r, "R^{n-1}"] & TR^1_0(A;p) \\ A \arrow[u, "{[-]_n}"] \arrow[r, equal] & A \arrow[u, "{\sim}", "{[-]_1}"'] \end{tikzcd}$$

It follows that the map I_n is a bijection. To see that I_n is a ring-homom. we consider the composite

$$W_n(A) \xrightarrow{I_n} TR^n_0(A;p) \longrightarrow \prod_{0\le s<n} TR^1_0(A;p),$$

where the s'th component of the right-hand map is the ring-homom. $F^s R^{n-1-s}$. We have

$$\begin{aligned} & F^s R^{n-1-s}(I_n(a_0, a_1, \dots, a_{n-1})) \\ &= F^s R^{n-1-s}\Big(\sum_{0\le t<n} V^t([a_t]_{n-t})\Big) \\ &= F^s\Big(\sum_{0\le t\le s} V^t([a_t]_{s+1-t})\Big) \\ &= \sum_{0\le t\le s} p^t F^{s-t}([a_t]_{s+1-t}) \\ &= \sum_{0\le t\le s} p^t [a_t]_1^{p^{s-t}} \\ &= I_1(a_0^{p^t} + p a_1^{p^{t-1}} + \dots + p^t a_t), \end{aligned}$$

so this composite is equal to the composite of the ghost map and the isom. I_1 both of which are ring-homom. If p is a non-zero-divisor in A, the composite is injective, and hence I_n is a ring-homom. In general, we pick choose a surjection $A' \to A$ with p a non-zero-divisor in A'. Then the diagram

$$\begin{array}{ccc} W_n(A') & \xrightarrow{I_n} & TR_0^n(A';p) \\ \downarrow & & \downarrow \\ W_n(A) & \xrightarrow{I_n} & TR_0^n(A;p) \end{array}$$

shows that I_n is a ring-homom. //

We next evaluate $TR_*^n(\mathbb{F}_p;p)$, but first we briefly discuss homotopy groups with $\mathbb{Z}/m\mathbb{Z}$-coeff. Let M_m be the Moore-G-spectrum defined as the mapping cone

$$S^0 \xrightarrow{m} S^0 \xrightarrow{i} M_m \xrightarrow{\beta} S^1.$$

Then

$$\pi_q^H(T, \mathbb{Z}/m\mathbb{Z}) := \pi_q^H(M_m \wedge T).$$

We have a l.e.s.

$$\cdots \to \pi_q^H(T) \xrightarrow{m} \pi_q^H(T) \xrightarrow{\iota} \pi_q^H(T, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\beta} \pi_{q-1}^H(T) \to \cdots$$

which gives rise to a s.e.s.

$$0 \to \pi_q^H(T)/m \xrightarrow{\iota} \pi_q^H(T; \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\beta} \pi_{q-1}^H(T)[m] \to 0.$$

The Moore-G-spectrum M_{p^v} is a comm. ring-G-spectrum if $p \geq 5$, if $p = 3$ and $v \geq 2$, or if $p = 2$ and $v \geq 3$. So if this is the case and if T is a comm. ring-G-spectrum, then

$$\pi_*^H(T, \mathbb{Z}/m\mathbb{Z})$$

is a graded-comm. graded $\mathbb{Z}/m\mathbb{Z}$-alg. It is always a graded module over the graded-comm. graded ring $\pi_*^H(T)$.

We recall the structure of $\pi_* T(\mathbb{F}_p)$; see C. Ausoni "Topological Hochschild homology of topological K-theory" for a good exposition. First,

$$\pi_*(T(\mathbb{F}_p), \mathbb{Z}/p\mathbb{Z}) = \Lambda\{\varepsilon\} \otimes S\{\sigma\}$$

where

$$\pi_1(T(\mathbb{F}_p), \mathbb{Z}/p\mathbb{Z}) \xrightarrow[\sim]{\beta} \pi_0 T(\mathbb{F}_p)$$

$$\varepsilon \longmapsto 1$$

and Connes' operator

$$\pi_1(T(\mathbb{F}_p), \mathbb{Z}/p\mathbb{Z}) \xrightarrow[\sim]{d} \pi_2(T(\mathbb{F}_p), \mathbb{Z}/p\mathbb{Z}).$$

$$\varepsilon \longmapsto \bar{\sigma}$$

Then, as a graded ring,

$$\pi_* T(\mathbb{F}_p) = S\{\sigma\}$$

where

$$\pi_2 T(\mathbb{F}_p) \xrightarrow[\sim]{\iota} \pi_2(T(\mathbb{F}_p), \mathbb{Z}/p\mathbb{Z})$$

$$\sigma \longmapsto \bar{\sigma}$$

We consider the spectral sequence

$$\hat{E}^2_{s,t}(C_{p^{n-1}}, T(\mathbb{F}_p)) = \hat{H}^{-s}(C_{p^{n-1}}, \pi_t T(\mathbb{F}_p))$$
$$\Rightarrow \hat{H}^{-s-t}(C_{p^{n-1}}, T(\mathbb{F}_p)).$$

As a bi-graded ring,

$$\hat{E}^2(C_{p^{n-1}}, T(\mathbb{F}_p)) = \Lambda\{u_{n-1}\} \otimes S\{t^{\pm 1}, \sigma\}$$

where $\deg u_{n-1} = (-1, 0)$, $\deg t = (-2, 0)$, and $\deg \sigma = (0, 2)$.

$$\begin{array}{ccc}\hat{E}^r(\mathbb{T}, T(\mathbb{F}_p)) & \to & \hat{E}^r(\mathbb{T}, M_p \wedge T(\mathbb{F}_p)) \\ \downarrow & & \downarrow \\ \hat{E}^r(C_{p^{n-1}}, T(\mathbb{F}_p)) & \to & \hat{E}^r(C_{p^{n-1}}, M_p \wedge T(\mathbb{F}_p))\end{array}$$

The $\hat{E}^2$-terms are identified as

$$\begin{array}{ccc}S\{t^{\pm 1}, \sigma\} & \to & \Lambda\{\varepsilon\} \otimes S\{t^{\pm 1}, \bar{\sigma}\} \\ \downarrow & & \downarrow \\ \Lambda\{u_{n-1}\} \otimes S\{t^{\pm 1}, \sigma\} & \to & \Lambda\{u_{n-1}, \varepsilon\} \otimes S\{t^{\pm 1}, \bar{\sigma}\}\end{array}.$$

(The right-hand sides are only rings for $p \geq 5$, but they are always modules over the corresponding left-hand sides.)

In the spectral sequence $\hat{E}^r(\mathbb{T}, T(\mathbb{F}_p))$ all classes are in even total degree, so all differentials must be zero. It follows that t and σ are permanent cycles in all spectral sequences. It is a general fact that the d^2-differential is given by Connes' operator in the following way

$$d^2(x) = t \cdot dx .$$

Hence, in the right-hand sp. seq.

$$d^2(\varepsilon) = t\bar{\sigma}.$$

In the lower left-hand sp. seq u_{n-1} is a d^2-cycle for degree reasons. Hence, u_{n-1} is also a d^2-cycle in the lower right-hand sp. seq. It follows that

$$\hat{E}^3(G_p^{n-1}, M_p \wedge T(\mathbb{E}_p))$$

$$= \Lambda\{u_{n-1}\} \otimes S\{t^{\pm 1}\}$$

and for degree reason, all further differentials are zero.

$$\hat{E}^2 = \Lambda\{u_n\} \otimes S\{t^{\pm 1}, \sigma\}$$

$$\Rightarrow \pi_* \hat{\mathbb{H}}(C_{p^n}, T(\mathbb{F}_p))$$

t, σ are permanent cycles

$$\hat{E}^2 = \Lambda\{u_n, \varepsilon\} \otimes S\{t^{\pm 1}, \sigma\} \Rightarrow \pi_*(\hat{\mathbb{H}}(C_{p^n}, T(\mathbb{F}_p)), \mathbb{Z}_p)$$

t, σ are permanent cycles

u_n is a d^2-cycle.

$$d^2 \varepsilon = t\sigma$$

$$\hat{E}^3 = \Lambda\{u_n\} \otimes S\{t^{\pm 1}\}$$

$$\begin{array}{ccc} T(\mathbb{F}_p) & \xrightarrow{\hat{\Gamma}} & \hat{\mathbb{H}}(C_p, T(\mathbb{F}_p)) \\ 1 & \longmapsto & 1 \\ \varepsilon & \longmapsto & u_1 t^{-1} \\ \sigma & \longmapsto & t^{-1} \end{array}$$

By Tsalidis,

$$TR^n_q(\mathbb{F}_p; p) \underset{\sim}{\xrightarrow{\hat{\Gamma}_n}} \hat{\pi}_q \hat{\mathbb{H}}(C_{p^n}, T(\mathbb{F}_p))$$

for $q \geq 0$ and $n \geq 1$.

Since

$$W_n(\mathbb{F}_p) \xrightarrow{\sim} TR_0^n(\mathbb{F}_p; p),$$

there must be a differential

$$d^{2n+1}(t^{-1}u_n) = (t\sigma)^n$$

and then

$$\hat{E}^\infty = \hat{E}^{2n+2} = S\{t^{\pm 1}, \sigma\}/(\sigma^n).$$

Thm Let A be a $\mathbb{Z}_{(p)}$-algebra with p odd. Then every elem. $\omega^{(n)} \in TR^n_q(A[x];p)$ can be written uniquely as a sum

$$\omega^{(n)} = \sum_{j\in\mathbb{N}_0} a^{(n)}_{0,j}[x]^j_n + \sum_{j\in\mathbb{N}} b^{(n)}_{0,j}[x]^{j-1}_n d[x]_n$$
$$+ \sum_{\substack{1\le s<n \\ j\in I_p}} \left(V^s(a^{(n-s)}_{s,j}[x]^j_{n-s}) + dV^s(b^{(n-s)}_{s,j}[x]^j_{n-s})\right)$$

with $a^{(m)}_{s,j} \in TR^m_q(A;p)$ and $b^{(m)}_{s,j} \in TR^m_{q-1}(A;p)$. Here I_p is the set of positive integers prime to p.

Alternatively, the thm. gives an isom.

$$\bigoplus_{j\in\mathbb{N}_0} TR^n_q(A;p) \oplus \bigoplus_{j\in\mathbb{N}} TR^n_{q-1}(A;p)$$
$$\oplus \bigoplus_{\substack{1\le s<n \\ j\in I_p}} \left(TR^{n-s}_q(A;p) \oplus TR^{n-s}_{q-1}(A;p)\right)$$
$$\xrightarrow{\sim} TR^n_q(A[x];p)$$

with the map given by the formula of the thm., e.g. on the first summand (s,j):

$$TR^{n-s}_q(A;p) \longrightarrow TR^{n-s}_q(A[x];p)$$
$$\xrightarrow{\cdot [x]^j_{n-s}} TR^{n-s}_q(A[x];p) \xrightarrow{V^s} TR^n_q(A[x];p).$$

$$\left.\begin{aligned} T(A[x]) &\xleftarrow{\sim} T(A) \wedge N^{cy}(\Pi) \\ N^{cy}(\Pi) &\xleftarrow{\sim} \bigvee_{i \in \mathbb{N}_0} N^{cy}(\Pi, i) \end{aligned}\right\}$$

$$\rho_{p^{n-1}}^{*} T(A[x])^{C_{p^{n-1}}} \xleftarrow{\sim} \bigvee_{i \in \mathbb{N}_0} \rho_{p^{n-1}}^{*} (T(A) \wedge N^{cy}(\Pi, i))^{C_{p^{n-1}}}$$

$$= \bigvee_{j \in \mathbb{N}_0} \rho_{p^{n-1}}^{*} (T(A) \wedge N^{cy}(\Pi, p^{n-1} j))^{C_{p^{n-1}}}$$

$$\vee \bigvee_{\substack{1 \le s < n \\ j \in I_p}} \rho_{p^{n-1}}^{*} (T(A) \wedge N^{cy}(\Pi, p^{n-1-s} j))^{C_{p^{n-1}}}$$

$$\leftarrow \bigvee_{j \in \mathbb{N}_0} \rho_{p^{n-1}}^{*} (T(A) \wedge N^{cy}(\Pi, p^{n-1} j))^{C_{p^{n-1}}}$$

$$\vee \bigvee_{\substack{1 \le s < n \\ j \in I_p}} \rho_{p^s}^{*} (\rho_{p^{n-1-s}}^{*} (T(A) \wedge N^{cy}(\Pi, p^{n-1-s} j))^{C_{p^{n-1-s}}})^{C_{p^s}}$$

Finally, there is an equivalence of $\mathbb{T}$-spectra given by the pairing

$$\rho_{p^v}^{*} T(A)^{C_{p^v}} \wedge \rho_{p^v}^{*} N^{cy}(\Pi, p^v j)^{C_{p^v}}$$

$$\xrightarrow{\sim} \rho_{p^v}^{*} (T(A) \wedge N^{cy}(\Pi, p^v j))^{C_{p^v}}$$

and the $\mathbb{T}$-equivariant homeomorphism

$$N^{cy}(\Pi, j) \xrightarrow{\Delta} \rho_{p^v}^{*} N^{cy}(\Pi, p^v j)^{C_{p^v}}$$

given by the composite

$$|N^{cy}(\Pi, j)[-]| \xrightarrow[\sim]{\Delta} |(sd_{p^v} N^{cy}(\Pi, p^v j)[-])^{C_{p^v}}|$$
$$\xrightarrow{\sim} |sd_{p^v} N^{cy}(\Pi, p^v j).|^{C_{p^v}} \xrightarrow[\sim]{D} |N^{cy}(\Pi, p^v j)[-]|^{C_{p^v}}.$$

Hence, we have an equivalence of Π-spectra

$$\rho^*_{p^{n-1}} T(A[x])^{C_{p^{n-1}}} \xleftarrow{\sim}$$
$$\bigvee_{j \in \mathbb{N}_0} \rho^*_{p^{n-1}} T(A)^{C_{p^{n-1}}} \wedge N^{cy}(\Pi, j)$$
$$\vee \bigvee_{\substack{1 \le s < n \\ j \in I_p}} \rho^*_{p^s} (\rho^*_{p^{n-1-s}} T(A)^{C_{p^{n-1-s}}} \wedge N^{cy}(\Pi, j))^{C_{p^s}}.$$

We first consider the restriction of this map to the top summands. This amounts to a map of Π-spectra

$$\rho^*_{p^{n-1}} T(A)^{C_{p^{n-1}}} \wedge N^{cy}(\Pi)$$
$$\longrightarrow \rho^*_{p^{n-1}} T(A[x])^{C_{p^{n-1}}}$$

which is multiplicative, if the domain is given the componentwise multiplication. Hence, the induced map of homotopy groups

$$\pi_*(\rho^*_{p^{n-1}} T(A)^{C_{p^{n-1}}} \wedge N^{cy}(\Pi))$$
$$\longrightarrow TR^n_*(A[x]; p)$$

identifies the domain with a sub-diff.-graded ring of the diff.-graded-ring in the target. We identify this.

The structure of the pointed $\mathbb{T}$-spaces

$$N^{cy}(\Pi, i) = |N^{cy}(\Pi, i)[-]|$$

is as follows. The space $N^{cy}(\Pi, 0)$ is the discrete space $\{0, 1\}$ with base-pt. 0. The cyclic set $N^{cy}(\Pi, i)[-]$, for $i > 0$, is generated by the single $(i-1)$-simpl. $x \wedge \dots \wedge x$ (i factors) modulo the relation that $t_{i-1}(x \wedge \dots \wedge x) = x \wedge \dots \wedge x$. This implies that there is a $\mathbb{T}$-equiv. homeomorphism

$$(\mathbb{T} \times_{C_i} \Delta^{i-1})_+ \xrightarrow{\sim} N^{cy}(\Pi, i),$$

and hence the inclusion of the barycenter of Δ^{i-1} gives a strong deformation retract of pointed $\mathbb{T}$-spaces

$$(\mathbb{T}/C_i)_+ \xrightarrow{\sim} N^{cy}(\Pi, i).$$

Moreover, the multiplication on $N^{cy}(\Pi)$ restricts to a pairing of the i-th and i'-th summands to the $(i+i')$-th summand, and there is a homotopy comm. diagr. of pointed $\mathbb{T}$-spaces

$$\begin{array}{ccc} N^{cy}(\Pi, i) \wedge N^{cy}(\Pi, i') & \xrightarrow{\mu} & N^{cy}(\Pi, i+i') \\ \uparrow \sim & & \uparrow \sim \\ (\mathbb{T}/C_i)_+ \wedge (\mathbb{T}/C_{i'})_+ & \longrightarrow & (\mathbb{T}/C_{i+i'})_+ \end{array}$$

$$(zC_i, z'C_{i'}) \longmapsto (z^i z'^{i'})^{1/(i+i')} C_{i+i'} .$$

<u>Lemma</u> The map of diff.-graded rings

$$\mathrm{TR}^n_*(A;p) \otimes \Omega^*_{\mathbb{Z}[x]} \longrightarrow \mathrm{TR}^n_*(A[x];p)$$

that takes $a \otimes 1$ to $f(a)$ and $1 \otimes x$ to $[x]_n$ is an isomorphism onto the sub-diff. gr. ring $\pi_*(\rho^*_{p^{n-1}} T(A)^{C_{p^{n-1}}} \wedge N^{cy}(\Pi))$.

<u>pf</u> We show that the maps lands in the stated summand. The map of components induced by the composite

$$\Pi \longrightarrow N^{cy}(\Pi) \xrightarrow{\Delta} \rho^*_{p^{n-1}} N^{cy}(\Pi)^{C_{p^{n-1}}}$$
$$\longrightarrow \rho^*_{p^{n-1}} T(A[x])^{C_{p^{n-1}}}$$

takes x^i to $[x^i]_n$. Indeed, by definition, the map

$$A[x] \xrightarrow{[-]_n} \mathrm{TR}^n_0(A[x];p)$$

is the map of components induced by

the composite

$$A[x] \to N^{cy}(A[x]) \xrightarrow{\Delta} \rho^*_{p^{n-1}} N^{cy}(A[x])^{C_{p^{n-1}}}$$
$$\to \rho^*_{p^{n-1}} T(A[x])^{C_{p^{n-1}}}.$$

And the composite map

$$\rho^*_{p^{n-1}} T(A)^{C_{p^{n-1}}} \xrightarrow[\sim]{\mathrm{can}} \rho^*_{p^{n-1}} T(A)^{C_{p^{n-1}}} \wedge N^{cy}(\Pi, 0)$$
$$\longrightarrow \rho^*_{p^{n-1}} T(A[x])^{C_{p^{n-1}}}$$

is equal to the map induced from the ring-homom. $f : A \to A[x]$ given by the inclusion of the constant polynomials. So the map of the statement lands in the indicated summand.

Since the reduced homology of $N^{cy}(\Pi)$ is torsion-free, the spectral sequence obtained from the skeleton filtration of $N^{cy}(\Pi)$ takes the form

$$E^2 = TR^n_*(A;p) \otimes \tilde{H}_*(N^{cy}(\Pi))$$
$$\Rightarrow \pi_*(\rho^*_{p^{n-1}} T(A)^{C_{p^{n-1}}} \wedge N^{cy}(\Pi)).$$

Let $x_i \in \tilde{H}_0(N^{cy}(\Pi, i))$ be the image of the generator of $\tilde{H}_0((\mathbb{T}/C_i)_+)$ given by the point C_i. We claim that the map of diff. graded rings

$$\Omega^*_{\mathbb{Z}[x]} \longrightarrow \tilde{H}_*(N^{cy}(\Pi))$$

that takes x to x_1 is an isom. The map on homology induced from the product

$$(\mathbb{T}/C_i)_+ \wedge (\mathbb{T}/C_{i'})_+ \to (\mathbb{T}/C_{ii'})_+$$

takes $C_i \otimes C_{i'}$ to $C_{ii'}$ so $x_i \cdot x_{i'} = x_{ii'}$. Hence the map in question is an isom. in degree zero, To prove that it is an isom. in degree one it suffices to show that $x^{i-1}dx$ is a generator of $\tilde{H}_1(N^{cy}(\Pi, i))$. But

$$i x^{i-1} dx = d(x^i) = d(x_i)$$

which is i times a generator. Indeed, the map induced from the projection

$$\tilde{H}_1(\mathbb{T}_+) \longrightarrow \tilde{H}_1((\mathbb{T}/C_i)_+)$$

takes $[\mathbb{T}]$ to $i \cdot [\mathbb{T}/C_i]$. This proves the claim.

The spectral sequence is then

$$E^2 = TR^n_*(A;p) \otimes \Omega^*_{\mathbb{Z}[x]}$$

$$\Rightarrow \pi_*(\rho^*_{p^{n-1}} T(A)^{C_{p^{n-1}}} \wedge N^{cy}(\Pi)) .$$

It is concentrated on the two lines $E^2_{0,*}$ and $E^2_{1,*}$ and hence amounts to the s.e.s. on the right in the following diagram.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
TR^n_*(A;p)\otimes \mathbb{Z}[x] \arrow[r, "\sim"] \arrow[d] & TR^n_*(A;p)\otimes \tilde{H}_0(N^{cy}(\Pi)) \arrow[d, "\text{edge}"] \\
TR^n_*(A;p)\otimes \Omega^*_{\mathbb{Z}[x]} \arrow[r] \arrow[d] & \pi_*(\rho^*_{p^{n-1}} T(A)^{C_{p^{n-1}}} \wedge N^{cy}(\Pi)) \arrow[d] \\
TR^n_*(A;p)\otimes \mathbb{Z}[x].dx \arrow[r] \arrow[d] & TR^n_*(A;p)\otimes \tilde{H}_1(N^{cy}(\Pi)) \arrow[d] \\
0 & 0
\end{tikzcd}$$

We must show that the lower horizontal map is an isom. The domain and range are both free $TR^n_*(A;p)\otimes \mathbb{Z}[x]$ - modules of rank 1. And since the map of the statement commutes with the differential, the generator $1\otimes dx = d(1\otimes x)$ on the left maps to the generator $d(1\otimes x) = 1\otimes dx$ on the right. //

The restriction of the equivalence

$$\rho^*_{p^{n-1}} T(A[x])^{C_{p^{n-1}}} \xleftarrow{\sim}$$

$$\bigvee_{j \in \mathbb{N}_0} \rho^*_{p^{n-1}} T(A)^{C_{p^{n-1}}} \wedge N^{cy}(\Pi, j)$$

$$\vee \bigvee_{\substack{1 \le s < n \\ j \in I_p}} \rho^*_{p^s} (\rho^*_{p^{n-1-s}} T(A)^{C_{p^{n-1-s}}} \wedge N^{cy}(\Pi, j))^{C_{p^s}}$$

to the bottom summand is understood, on homotopy groups, by means of the following general lemma.

<u>Lemma</u> Let T be a Π-spectrum, let $j \in I_p$, and let $\iota : C_j/C_j \to \Pi/C_j$ be the inclusion. Then the map

$$\pi_q(T) \oplus \pi_{q-1}(T) \xrightarrow{V^s_\iota + dV^s_\iota} \pi_q((T \wedge \Pi/C_j)^{C_{p^s}})$$

is an isom., for all integers q and non-negative integers s. //