Algebraic Topology I

Lars Hesselholt

Object of the course: To every category $\mathscr{C}$, we associate a topological space $B\mathscr{C}$. The space $B\mathscr{C}$ is called the *classifying space* of $\mathscr{C}$. A functor $f\colon \mathscr{C} \to \mathscr{D}$ gives rise to a continuous map $Bf\colon B\mathscr{C} \to B\mathscr{D}$. Moreover, a natural transformation α from the functor $f\colon \mathscr{C} \to \mathscr{D}$ to the functor $g\colon \mathscr{C} \to \mathscr{D}$ gives rise to a homotopy $B\alpha\colon B\mathscr{C} \times [0,1] \to B\mathscr{D}$ from the map Bf to the map Bg. In short, the classifying space construction gives rise to a 2-functor from the 2-category of categories to the 2-category of topological spaces. In this way, properties of categories are reflected in the homotopy type of their classifying spaces.

The classifying space is constructed by gluing together simplices

$$\Delta[n] = \{(x_0, \dots, x_n) \in [0,1]^{n+1} \mid x_0 + \cdots + x_n = 1\}.$$

The general recipe for constructing a topological space by glying together simplices is called a *simplicial set*. The resulting topological space is called the *geometric realization* of the simplicial set. The first part of the course will focus on simplicial sets and their geometric realization along with the basic category theoretical notions of limits and colimits and adjoints functors which are needed to develop this theory.

The next part of the course focuses on homotopy theory. We introduce homotopy groups and define a continuous map between topological spaces to be a *weak equivalence* if it induces an isomorphism of the associated homotopy groups. The *homotopy category* of topological spaces to be the category obtained from the category of topological spaces and continuous maps by formally introducing an inverse map for every weak equivalence. The main techniques for studying the homotopy category are centered around two classes of maps called the *fibrations* and the *cofibrations*. The category of topological spaces together with the three classes of maps given by the weak equivalences, the fibrations, and the cofibrations form a *model category*. In homotopy theory, theorems live in the homotopy category, but their proofs live in the model category.

The final part of the course uses the techniques we have developed to define algebraic K-theory. We prove the so-called additivity theorem from which many of the basic properties of algebraic K-theory are readily derived.

Keywords: Homotopy theory, model categories, algebraic K-theory.

Required knowledge: An introductory course in algebraic topology including the fundamental group and covering spaces.

Text: The course lecture notes. The following texts are also useful:

Mark Hovey, *Model Categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society.

Daniel G. Quillen, *Homotopical Algebra*, Lecture Notes in Mathematics, vol. 43, Springer-Verlag, New York.

Friedhelm Waldhausen, *Algebraic K-theory of spaces*, Lecture Notes in Mathematics, vol. 1126, Springer-Verlag, New York.