# Scissor's congruence groups

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It has been known since ancient times that two polygons that have the same area can be divided into a finitely many pairwise congruent triangles. Hilbert, in his third problem at the International Congress of Mathematicians in 1900, asked for an example of two polyhedra of equal volume which cannot be divided into finitely many pairwise congruent tetrahedra. Within the same year, Dehn showed that the cube and the regular tetrahedron of equal volume indeed cannot be divided into finitely many pairwise congruent tetrahedra. Two polyhedra are said to be scissor's congruent if they can be divided into finitely many pairwise congruent tetrahedra. The question of how to parametrize the set of polyhedra up to scissor's congruence turns out to involve much of the modern mathematics developed in the twentieth century. We will discuss the solution to this question along with some the modern mathematical structures involved.

The present notes are partly based on Dupont's book [5] which we recommend for further reading. We have strived to supply proofs of some basic results in the theory which are not readily found in the literature.

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## 1. Hilbert's third problem

Two planar polygons P and P' are said to be *scissor's congruent* if they can be divided into finitely many pairwise congruent triangles. In this case,

$$\operatorname{area}(P) = \operatorname{area}(P')$$

The following theorem implies, in particular, that the opposite is true as well.

THEOREM 1.1. Every planar polygon P is scissor's congruent to a rectangle with one side of length 1.

PROOF. Since every planar polygon can be divided into triangles, we may assume that P is a triangle. First, as indicated by the following figure, every triangle is scissor's congruent to a parallelogram.



Next, as the following figure indicates, every parallelogram, in turn, is scissor's congruent to a rectangle.



Moreover, by iterating the procedure indicated by the next figure, it follows that every rectangle is scissor's congruent to a rectangle, where one side is at least 1, and where the other side is a most 1.



As indicated by the following figure, every such rectangle, in turn, is scissor's congruent to a parallelogram, where the distance between two of the parallel edges is

equal to 1.



Finally, as the following diagram indicates, every such parallelogram is scissor's congruent to a rectangle with one side of length 1.



This completes the proof.

REMARK 1.2. This theorem was likely known to the ancient Chinese and Greek. It gives one possible way of defining the area of a planar polygon, namely, as the length of the other side in the resulting rectangle.

In a letter in 1844, Gauss noted that the proof that two pyramids with the same base area and height have the same volume uses subdivision into an infinite number of pieces, and he asked for a proof of this which only uses subdivision into a finite number of pieces. Hilbert, however, did not think that this was possible and posed as his third problem at the International Congress of Mathematicians in Paris 1900 the following:

HILBERT'S THIRD PROBLEM. Show that there exists two polyhedra of equal volume which are not scissor's congruent.

Already the same year, Dehn proved that the cube and the regular tetrahedron of equal volume are not scissor's congruent. We will give a modern formulation of Dehn's proof in the next section. To begin, we make our definitions precise.

A k-simplex in  $\mathbb{R}^n$  is a tuple

$$\sigma = (a_0, a_1, \ldots, a_k)$$

of k + 1 points in  $a_i \in \mathbb{R}^n$ . The associated geometric k-simplex  $|\sigma|$  is defined to be the convex hull of the k + 1 points,

$$|\sigma| = \operatorname{conv}(\{a_0, \dots, a_k\}) = \left\{ \sum_{i=0}^k t_i a_i \mid t_i \in [0, 1], \ t_0 + \dots + t_k = 1 \right\}.$$

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In the case k = n, we say that  $\sigma$  is proper if  $|\sigma|$  is not contained in an affine hyperplane in  $\mathbb{R}^n$ . A sub-tuple  $\tau$  of  $\sigma$  is called a face of  $\sigma$ , and the subset  $|\tau| \subset |\sigma|$  is called a face of  $|\sigma|$ . We allow  $\tau = \emptyset$  and  $\tau = \sigma$ . A polytope in  $\mathbb{R}^n$  is a subset

$$P \subset \mathbb{R}^n$$

with the property that there exists a finite set  $\{\sigma_1, \ldots, \sigma_m\}$  of proper pairwise distinct *n*-simplices in  $\mathbb{R}^n$  such that

$$P = \bigcup_{i=1}^{m} |\sigma_i|$$

and such that for all for all  $1 \leq i < j \leq m$ , the intersection

$$|\sigma_i| \cap |\sigma_j|$$

is a face of both  $|\sigma_i|$  and  $|\sigma_j|$ . A set of simplices  $\{\sigma_1, \ldots, \sigma_m\}$  as above is said to be a *triangulation* of the polytope  $P \subset \mathbb{R}^n$ . We note that the number m of n-simplices in a triangulation of the non-empty polytope  $P \subset \mathbb{R}^n$  is not bounded above. The following figure illustrates two triangulations with m = 2 and m = 4, respectively, of the same polytope.



In particular, every non-empty polytope  $P \subset \mathbb{R}^n$  admits an infinite number of triangulations.

We write d(a, a') for the euclidean distance between the points a and a' in  $\mathbb{R}^n$ . An *isometry* of  $\mathbb{R}^n$  is a map  $f : \mathbb{R}^n \to \mathbb{R}^n$  such that for all  $a, a' \in \mathbb{R}^n$ ,

$$d(f(a), f(a')) = d(a, a').$$

We recall that isometries preserve volume and angles.

REMARK 1.3. We briefly recall the structure of the *euclidean group* E(n) that is defined to be the set of isometries of  $\mathbb{R}^n$  with the group structure given by composition. For every  $v \in \mathbb{R}^n$ , the map

$$t_v(a) = a + v$$

is an isometry called a *translation*. The assignment  $v \mapsto t_v$  is an isomorphism of the additive group  $\mathbb{R}^n$  onto the subgroup  $T(n) \subset E(n)$  of translations; it is a normal subgroup. The subgroup  $O(n) \subset E(n)$  of orthogonal transformations consists of the isometries f such that f(0) = 0; it is not normal. An orthogonal transformation is a linear map and the representing matrix is an orthogonal  $n \times n$ -matrix. Every element  $f \in E(n)$  can be written uniquely as the composition

$$f = t_{f(0)} \circ f,$$

where  $\bar{f}$  is the orthogonal transformation defined by  $\bar{f}(a) = f(a) - f(0)$ . Hence, the euclidean group is equal to the semi-direct product

$$E(n) = O(n) \ltimes T(n)$$

of the subgroups T(n) and O(n).

DEFINITION 1.4. The polytopes  $P, P' \subset \mathbb{R}^n$  are said to be *scissor's congruent* if there exists triangulations  $\{\sigma_1, \ldots, \sigma_m\}$  of P and  $\{\sigma'_1, \ldots, \sigma'_m\}$  of P' and isometries  $f_1, \ldots, f_m \in E(n)$  such that  $f(|\sigma_i|) = |\sigma'_i|$  for all  $i = 1, \ldots, m$ .

LEMMA 1.5. Let  $S = \{\sigma_1, \ldots, \sigma_r\}$  and  $T = \{\tau_1, \ldots, \tau_s\}$  be two triangulations of the polytope  $P \subset \mathbb{R}^n$ . Then there is a triangulation  $U = \{v_1, \ldots, v_t\}$  of P such that every geometric simplex associated to a simplex in S or T is equal to a union of geometric simplices associated to simplices in U.

PROOF. Clearly, it suffices to show that if the intersection

$$Q = |\sigma| \cap |\tau| \subset \mathbb{R}^n$$

of two geometric *n*-simplices has non-empty interior, then Q is a polytope. To this end, we recall the main theorem of convex polytopes. A halfspace  $H \subset \mathbb{R}^n$  is a subset of the form  $\{x \in \mathbb{R}^n \mid \langle x, v \rangle \leq a\}$  for some  $v \in \mathbb{R}^n$  and  $a \in \mathbb{R}$ . Now, the main theorem of convex polytopes states that the subset  $S \subset \mathbb{R}^n$  is equal to the convex hull of a finite set of points if and only if it is bounded and equal to the intersection of a finite set of halfspaces. We refer to [16, Theorem 1.1] for a proof. The following figure illustrates the theorem.



A subset  $S \subset \mathbb{R}^n$  of this form is called a convex polytope. The dimension of the convex polytope  $S \subset \mathbb{R}^n$  is the smallest dimensions of an affine subspace of  $\mathbb{R}^n$  that contains S. (Exercise: Show that if the subset  $R \subset \mathbb{R}^n$  is both convex and a polytope, then it is a convex polytope of dimension n.) In the case at hand, we find that  $Q \subset \mathbb{R}^n$  is a convex polytope. Indeed, if  $S_1, S_2 \subset \mathbb{R}^n$  are bounded subsets equal to the intersections of two finite sets of hyperplanes then the same is true for  $S_1 \cap S_2 \subset \mathbb{R}^n$ . Moreover, since Q has an interior point, its dimension is n.

Now, let  $S \subset \mathbb{R}^n$  be a convex polytope of dimension n. We show, by induction on  $n \ge 0$  that  $S \subset \mathbb{R}^n$  is a polytope. The case n = 0 is trivial, so we assume that the case n = d-1 has been proved and prove the case n = d. The subset  $F \subset S$  is a face of S if there exists a halfspace  $H = \{x \in \mathbb{R}^d \mid \langle x, v \rangle \le a\}$  such that  $S \subset H$  and  $F = S \cap \partial H$ , where  $\partial H = \{x \in \mathbb{R}^d \mid \langle x, v \rangle = a\}$  is the boundary of H. We recall from [16, Proposition 2.3] that every face of a convex polytope is itself a convex polytope and that the intersection of two faces is itself a face. We let  $F_1, \ldots, F_k$  be the faces in S of dimension d - 1, and let  $x \in S$  be an interior point, which exists because  $S \subset \mathbb{R}^d$  has dimension d. We write S as the union

$$S = \bigcup_{i=1}^{k} \operatorname{conv}(\{x\} \cup F_i)$$

of the cones whose bases are the faces  $F_1, \ldots, F_k$  and whose cone point is the interior point x. The following figure illustrates the situation.



By the inductive hypothesis, each of the faces  $F_1, \ldots, F_k$  admits a triangulation. We let  $\{\sigma_{i,1}, \ldots, \sigma_{i,m_i}\}$  be a triangulation of  $F_i$  with  $\sigma_{i,j} = (a_{i,j,1}, \ldots, a_{i,j,d})$ , and define  $\bar{\sigma}_{i,j} = (x, a_{i,j,1}, \ldots, a_{i,j,d})$ . Then  $\{\bar{\sigma}_{i,1}, \ldots, \bar{\sigma}_{i,m_i}\}$  is a triangulation of the cone conv $(\{x\} \cup F_i)$ , and the union

$$\bigcup_{i=1}^k \{\bar{\sigma}_{i,1},\ldots,\bar{\sigma}_{i,m_i}\}$$

is the desired triangulation of S. This proves the induction step.

COROLLARY 1.6. Scissor's congruence defines an equivalence relation on the set of polytopes in  $\mathbb{R}^n$ .

PROOF. It is clear that scissor's congruence is reflexive and symmetric. We show that it is also transitive. So let  $P, P', P'' \subset \mathbb{R}^n$  be three polytopes and assume that P and P' are scissor's congruent and that P' and P'' are scissor's congruent. We must show that P and P'' are scissor's congruent. Since P and P'are scissor's congruent, we have triangulations  $\{\sigma_1, \ldots, \sigma_r\}$  of P and  $\{\sigma'_1, \ldots, \sigma'_r\}$ of P' and isometries  $f_1, \ldots, f_r$  of  $\mathbb{R}^n$  such that  $f_i(|\sigma_i|) = |\sigma'_i|$  for all  $i = 1, \ldots, r$ , and since P' and P'' are scissor's congruent, we have triangulations  $\{\tau'_1, \ldots, \tau'_s\}$  of P' and  $\{\tau''_1, \ldots, \tau''_s\}$  of P'' and isometries  $g_1, \ldots, g_s$  of  $\mathbb{R}^n$  such that  $g_j(|\tau'_j|) = |\tau''_j|$ for all  $j = 1, \ldots, s$ . By Lemma 1.5, there exists a triangulation  $\{v'_1, \ldots, v'_t\}$  of P'with the property that for every  $i = 1, \ldots, r$  and  $j = 1, \ldots, s$ ,

$$\begin{aligned} |\sigma'_i| &= \bigcup_{k \in I(i)} |v'_k| \\ |\tau'_j| &= \bigcup_{k \in J(j)} |v'_k|, \end{aligned}$$

where I(i) and J(j) are subsets of  $\{1, 2, \ldots, t\}$ . We note that  $\{1, 2, \ldots, t\}$  is equal both to the disjoint union of the subsets  $I(1), \ldots, I(r)$  and to the disjoint union of the subsets  $J(1), \ldots, J(s)$ . We now define triangulations  $\{v_1, \ldots, v_t\}$  of P and  $\{v_1'', \ldots, v_t''\}$  of P'' as follows. If  $k \in I(i)$ , then we choose an *n*-simplex  $v_k$  such that  $f_i(|v_k|) = |v_k'|$ . (The choice of  $v_k$  amounts to a choice of ordering of the vertices in  $|v_k|$ .) Similarly, if  $k \in J(j)$ , then we choose an *n*-simplex  $v_k''$  such that  $g_j(|v_k'|) = |v_k''|$ . Finally, if  $k \in I(i)$  and  $k \in J(j)$ , then  $h_k = g_j \circ f_i$  is an isometry of  $\mathbb{R}^n$  and  $h_k(|v_k|) = |v_k''|$ . This shows that P and P'' are scissor's congruent as desired.  $\Box$ 

# PROPOSITION 1.7. If the polytopes $P, P' \subset \mathbb{R}^n$ are scissor's congruent, then $\operatorname{vol}(P) = \operatorname{vol}(P').$

PROOF. Since the polytopes P and P' are scissor's congruent, we can find triangulations  $\{\sigma_1, \ldots, \sigma_m\}$  of P and  $\{\sigma'_1, \ldots, \sigma'_m\}$  of P' and isometries  $f_1, \ldots, f_m$  such that  $f_i(|\sigma_i|) = |\sigma'_i|$ . And since isometries preserve volume, we find

$$vol(P) = \sum_{i=1}^{m} vol(|\sigma_i|) = \sum_{i=1}^{m} vol(f_i(|\sigma_i|)) = \sum_{i=1}^{m} vol(|\sigma'_i|) = vol(P').$$

This completes the proof.

### 2. The Dehn invariant

The Dehn invariant D(P) is an invariant of polytopes  $P \subset \mathbb{R}^3$ . Like the volume, it has the property that if the polytopes  $P, P' \subset \mathbb{R}^3$  are scissor's congruent, then D(P) = D(P'). Therefore, it is possible to prove that P and P' are not scissor's congruent by showing that  $D(P) \neq D(P')$ . However, unlike the volume, the Dehn invariant D(P) is not a real number but rather an element of the (much larger) real vector space  $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$ . We devote the remainder of this section to the definition and study of this vector space.

In general, the *tensor product* of the abelian groups A and B is defined to be the abelian group  $A \otimes B$  that has one generator  $a \otimes b$  for every ordered pair  $(a, b) \in A \times B$  and with these generators subject to the relations that

$$(a + a') \otimes b = a \otimes b + a' \otimes b$$
$$a \otimes (b + b') = a \otimes b + a \otimes b'$$

for all  $a, a' \in A$  and  $b, b' \in B$ . The relations express that  $A \otimes B$  has the following universal property: If C is an abelian group and if

$$f: A \times B \to C$$

is a bi-additive map, then there is a unique additive map

$$\bar{f}: A \otimes B \to C$$

such that  $\overline{f}(a \otimes b) = f(a, b)$ . We prove two basic lemmas about tensor products.

LEMMA 2.1. Let A and B be abelian groups. For all  $a \in A$ ,  $b \in B$ , and  $n \in \mathbb{Z}$ ,

$$(na) \otimes b = n(a \otimes b) = a \otimes (nb).$$

PROOF. We prove the left-hand equality; the proof of the right-hand equality is analogous. Suppose first that  $n \ge 1$ . The case n = 1 is trivial, so we assume, inductively, that the case n = r - 1 has been proved and prove the case n = r. By using the first of the defining relations, we find

$$(ra) \otimes b = (a + (r - 1)a) \otimes b = a \otimes b + ((r - 1)a) \otimes b$$
$$= a \otimes b + (r - 1)(a \otimes b) = r(a \otimes b),$$

which proves the induction step. Next, we have

$$0 \otimes b + 0 \otimes b = (0+0) \otimes b = 0 \otimes b.$$

and subtracting  $0 \otimes b$  on both sides, we find  $0 \otimes b = 0$ . This proves the case n = 0. Finally, suppose that  $n \leq -1$ . By what was just proved, we find

$$(na) \otimes b + (-na) \otimes b = (na + (-na)) \otimes b = 0 \otimes b = 0.$$

Hence, since  $-n \ge 1$ , we find

$$(na) \otimes b = -((-na) \otimes b) = -(-n(a \otimes b)) = n(a \otimes b)$$

as desired.

COROLLARY 2.2. The additive map

$$f \colon \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z}) \to \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$$

defined by  $f(x \otimes (y + \pi \mathbb{Z})) = x \otimes (y + \pi \mathbb{Q})$  is an isomorphism.

**PROOF.** To prove the corollary, we define an additive map

$$\bar{g} \colon \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q}) \to \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$$

and show that f and  $\overline{g}$  are each other's inverses. To this end, we claim that the formula  $g(x, y + \pi \mathbb{Q}) = x \otimes (y + \pi \mathbb{Z})$  gives a well-defined bi-additive map

$$q: \mathbb{R} \times (\mathbb{R}/\pi\mathbb{Q}) \to \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z}).$$

Indeed, suppose that the elements  $(x_1, y_1 + \pi \mathbb{Q})$  and  $(x_2, y_2 + \pi \mathbb{Q})$  of  $\mathbb{R} \times (\mathbb{R}/\pi \mathbb{Q})$  are equal. In this case, we have  $x_1 = x_2$  and  $y_1 - y_2 \in \pi \mathbb{Q}$ , so if we write  $x = x_1 = x_2$  and  $y_1 - y_2 = \pi \cdot (m/n)$ , then we find

$$x_1 \otimes (y_1 + \pi \mathbb{Z}) - x_2 \otimes (y_2 + \pi \mathbb{Z}) = x \otimes (y_1 - y_2 + \pi \mathbb{Z})$$
$$= x \otimes (\pi \cdot \frac{m}{n} + \pi \mathbb{Z}) = \frac{1}{n} \cdot (nx) \otimes (\pi \cdot \frac{m}{n} + \pi \mathbb{Z})$$
$$= \frac{1}{n} \cdot x \otimes (\pi \cdot m + \pi \mathbb{Z}) = 0$$

which shows that g is well-defined as claimed. Here the third and fourth equalities follow from Lemma 2.1. Moreover, the defining relations for the tensor product  $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$  show that g is a bi-additive map. Therefore, it gives rise to the desired additive map  $\bar{g}$  defined by the formula  $\bar{g}(x \otimes (y + \pi\mathbb{Q})) = x \otimes (y + \pi\mathbb{Z})$ . It is clear from this formula that the maps f and  $\bar{g}$  are each other's inverses. Hence, the map f is an isomorphism as stated.

We next recall that the direct sum  $B_1 \oplus B_2$  of the abelian groups  $B_1$  and  $B_2$  is defined to be the set of ordered pairs  $(b_1, b_2)$  with componentwise sum.

LEMMA 2.3. Let A,  $B_1$ , and  $B_2$  be abelian groups. Then the additive map

 $f \colon (A \otimes B_1) \oplus (A \otimes B_2) \to A \otimes (B_1 \oplus B_2)$ 

defined by  $f(a_1 \otimes b_1, a_2 \otimes b_2) = a_1 \otimes (b_1, 0) + a_2 \otimes (0, b_2)$  is well-defined and an isomorphism.

PROOF. We first argue that f is well-defined. To this end, it suffices to show that the additive maps  $f_1: A \otimes B_1 \to A \otimes (B_1 \oplus B_2)$  and  $f_2: A \otimes B_2 \to A \otimes (B_1 \oplus B_2)$ defined by  $f_1(a \otimes b_1) = a \otimes (b_1, 0)$  and  $f_2(a \otimes b_2) = a \otimes (0, b_2)$ , respectively, are well-defined. To show that  $f_1$  is well-defined, we must show that

$$f_1((a + a') \otimes b_1) = f_1(a \otimes b_1) + f_1(a' \otimes b_1)$$
  
$$f_1(a \otimes (b_1 + b'_1)) = f_1(a \otimes b_1) + f_1(a \otimes b'_1).$$

By the definition of the map  $f_1$ , this amounts to showing that

$$(a + a') \otimes (b_1, 0) = a \otimes (b_1, 0) + a' \otimes (b_1, 0)$$
$$a \otimes (b_1 + b'_1, 0) = a \otimes (b_1, 0) + a \otimes (b'_1, 0),$$

which follows from the defining relations of the tensor product  $A \otimes (B_1 \oplus B_2)$ . So the map  $f_1$  is well-defined. The proof that  $f_2$  is well-defined is entirely similar, so f is well-defined as stated.

To prove that f is an isomorphism, we let

$$g: A \otimes (B_1 \oplus B_2) \to (A \otimes B_1) \oplus (A \otimes B_2)$$

to be the additive map defined by  $g(a \otimes (b_1, b_2)) = (a \otimes b_1, a \otimes b_2)$ . We leave it to the reader to verify that the map g is well-defined. Finally, we check that  $f \circ g$  and  $g \circ f$  are equal to the respective identity maps. First,

$$(f \circ g)(a \otimes (b_1, b_2)) = f(a \otimes b_1, a \otimes b_2)$$
  
=  $a \otimes (b_1, 0) + a \otimes (0, b_2)$   
=  $a \otimes (b_1, b_2),$ 

where the last equality follows from the defining relations of the tensor product  $A \otimes (B_1 \oplus B_2)$ . Second,

$$(g \circ f)(a_1 \otimes b_1, a_2 \otimes b_2) = g(a_1 \otimes (b_1, 0) + a_2 \otimes (0, b_2))$$
  
=  $g(a_1 \otimes (b_1, 0)) + g(a_2 \otimes (0, b_2))$   
=  $(a_1 \otimes b_1, 0) + (0, a_2 \otimes b_2)$   
=  $(a_1 \otimes b_1, a_2 \otimes b_2)$ ,

where the second equality follows from g being additive. This shows that f is an isomorphism and that g is the inverse map of f.

More generally, let  $B_i$   $(i \in I)$  be a (possibly infinite) family of abelian groups. We recall that the direct sum  $\bigoplus_{i \in I} B_i$  is defined to be the abelian group given by the set of all tuples  $(b_i \mid i \in I)$  such that all but finitely many of the components  $b_i$ are equal to zero with componentwise addition. The map

$$\operatorname{in}_j \colon B_j \to \bigoplus_{i \in I} B_i$$

that to  $b \in B_j$  associates the tuple  $(b_i \mid i \in I)$  with  $b_i = b_j$ , if i = j, and  $b_i = 0$ , otherwise, is an additive map.

ADDENDUM 2.4. Let A and  $B_i$   $(i \in I)$  be abelian groups. Then the map

$$f: \bigoplus_{i \in I} (A \otimes B_i) \to A \otimes (\bigoplus_{i \in I} B_i)$$

defined by  $f(a_i \otimes b_i \mid i \in I) = \sum_{i \in I} a \otimes in_i(b_i)$  is well-defined and an isomorphism.

PROOF. The proof is similar to the proof of Lemma 2.3; we leave it as an exercise to the reader to write out the details.  $\Box$ 

The abelian group  $\mathbb{R}/\pi\mathbb{Q}$  has a natural structure of  $\mathbb{Q}$ -vector space with the scalar multiple of  $y + \pi\mathbb{Q} \in \mathbb{R}/\pi\mathbb{Q}$  by  $q \in \mathbb{Q}$  defined by  $q \cdot (y + \pi\mathbb{Q}) = qy + \pi\mathbb{Q}$ . Similarly, the abelian group  $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$  has a natural structure of  $\mathbb{R}$ -vector space with the scalar multiple of  $x \otimes (y + \pi\mathbb{Q}) \in \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$  by  $a \in \mathbb{R}$  defined by  $a \cdot x \otimes (y + \pi\mathbb{Q}) = ax \otimes (y + \pi\mathbb{Q})$ . We recall that, in general, if V is a vector space over the field k, then there exists a basis  $S \subset V$ . We also recall that the subset  $S = \{v_i \mid i \in I\} \subset V$  is a basis if and only if the k-linear map  $f \colon \bigoplus_{i \in I} k \to V$  defined by  $f(a_i \mid i \in I) = \sum_{i \in I} a_i v_i$  is an isomorphism.

PROPOSITION 2.5. If the subset  $S = \{y_i + \pi \mathbb{Q} \mid i \in I\}$  is a basis of the  $\mathbb{Q}$ -vector space  $\mathbb{R}/\pi\mathbb{Q}$ , then the subset  $T = \{1 \otimes (y_i + \pi\mathbb{Q}) \mid i \in I\}$  is a basis of the  $\mathbb{R}$ -vector space  $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$ .

PROOF. If  $S = \{y_i + \pi \mathbb{Q} \mid i \in I\}$  is a basis of the  $\mathbb{Q}$ -vector space  $\mathbb{R}/\pi\mathbb{Q}$ , then the  $\mathbb{Q}$ -linear map  $g \colon \bigoplus_{i \in I} \mathbb{Q} \to \mathbb{R}/\pi\mathbb{Q}$  given by  $g(q_i \mid i \in I) = \sum_{i \in I} (q_i y_i + \pi\mathbb{Q})$  is an isomorphism. It follows that the  $\mathbb{R}$ -linear map

$$h\colon \mathbb{R}\otimes (\bigoplus_{i\in I}\mathbb{Q})\to \mathbb{R}\otimes (\mathbb{R}/\pi\mathbb{Q})$$

defined by  $h(a \otimes (q_i \mid i \in I)) = a \otimes g(q_i \mid i \in I)$  is an isomorphism. We further recall from Addendum 2.4 that the map

$$f \colon \bigoplus_{i \in I} \left( \mathbb{R} \otimes \mathbb{Q} \right) \to \mathbb{R} \otimes \left( \bigoplus_{i \in I} \mathbb{Q} \right)$$

defined by  $f(a \otimes (q_i \mid i \in I)) = \sum_{i \in I} a_i \otimes in_i(q_i)$  is an isomorphism. Moreover, we claim that the formula  $e(a) = a \otimes 1$  defines an  $\mathbb{R}$ -linear isomorphism

$$e \colon \mathbb{R} \to \mathbb{R} \otimes \mathbb{Q}.$$

Indeed, by writing q = m/n and using Lemma 2.1, we see as in the proof of Corollary 2.2 that the map  $d: \mathbb{R} \otimes \mathbb{Q} \to \mathbb{R}$  defined by  $d(a \otimes q) = aq$  is well-defined and that  $d \circ e$  and  $e \circ d$  are equal to the respective identity maps. Finally, we leave it as an exercise to the reader to verify that the composition

$$c = h \circ f \circ (\bigoplus_{i \in I} e) \colon \bigoplus_{i \in I} \mathbb{R} \to \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$$

is given by the formula

$$c(a_i \mid i \in I) = \sum_{i \in I} a_i \cdot (1 \otimes (y_i + \pi \mathbb{Q})).$$

This shows that the subset  $T = \{1 \otimes (y_i + \pi \mathbb{Q}) \mid i \in I\}$  is a basis of the  $\mathbb{R}$ -vector space  $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$  as stated.  $\Box$ 

REMARK 2.6. It follows, in particular, from Proposition 2.5 that the Q-vector space  $\mathbb{R}/\pi\mathbb{Q}$  and the R-vector space  $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$  have the same dimension. (The dimension of a vector space is defined to be the cardinality of a basis. It is a theorem that any two bases of the same vector space have the same cardinality.) The common dimension is necessarily uncountably infinite. For on the one hand, every countably infinite dimensional Q-vector space is itself countable, and on the other hand, the set  $\mathbb{R}/\pi\mathbb{Q}$  is uncountable. In effect, it is not difficult to show that

$$\dim_{\mathbb{R}}(\mathbb{R}\otimes(\mathbb{R}/\pi\mathbb{Q})) = \dim_{\mathbb{Q}}(\mathbb{R}/\pi\mathbb{Q}) = \operatorname{card}(\mathbb{R}) = 2^{\aleph_0}.$$

COROLLARY 2.7. Suppose that  $\theta + \pi \mathbb{Q}$  is a non-zero element of  $\mathbb{R}/\pi \mathbb{Q}$ . Then  $1 \otimes (\theta + \pi \mathbb{Q})$  is a non-zero element of  $\mathbb{R} \otimes (\mathbb{R}/\pi \mathbb{Q})$ .

PROOF. Let  $S = \{y_i + \pi \mathbb{Q} \mid i \in I\}$  be a basis of the  $\mathbb{Q}$ -vector space  $\mathbb{R}/\pi\mathbb{Q}$ , and let  $T = \{1 \otimes (y_i + \pi\mathbb{Q}) \mid i \in I\}$  be the corresponding basis of the  $\mathbb{R}$ -vector space  $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$  given by Proposition 2.5. We write  $\theta + \pi\mathbb{Q}$  as a  $\mathbb{Q}$ -linear combination

$$\theta + \pi \mathbb{Q} = \sum_{i \in I} q_i (y_i + \pi \mathbb{Q}).$$

Since  $\theta + \pi \mathbb{Q}$  is assumed to be non-zero, it follows that at least one of the coordinates  $q_i \in \mathbb{Q}$  is non-zero. Now, we claim that

$$1 \otimes (\theta + \pi \mathbb{Q}) = \sum_{i \in I} 1 \otimes q_i(y_i + \pi \mathbb{Q}) = \sum_{i \in I} q_i \otimes (y_i + \pi \mathbb{Q}) = \sum_{i \in I} q_i(1 \otimes (y_i + \pi \mathbb{Q})).$$

Indeed, the first equality follows from the defining relations of the tensor product  $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$ ; the second equality follows from Lemma 2.1 upon writing  $q_i = m_i/n_i$ ; and the last equality follows from the definition of the  $\mathbb{R}$ -vector space structure on  $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Q})$ . We conclude that at least one of the coordinates of the element  $1 \otimes (\theta + \pi\mathbb{Q})$  with respect to the basis T is non-zero. Therefore this element is non-zero as stated.

REMARK 2.8. We may also consider  $\mathbb{R}/\pi\mathbb{Z}$  and  $\mathbb{R}/\pi\mathbb{Q}$  as topological spaces with the quotient topology induced by the standard metric topology on the real line  $\mathbb{R}$ . The first space is easy to visualize as a circle of radius  $\pi$ . However, the second space is not easily visualized. It is a totally disconnected space which means that the only connected subsets are the empty set and the subsets that consist of a single point.

Now, let  $\sigma$  be a proper 3-simplex in  $\mathbb{R}^3$ , and let e be an edge (1-face) of  $\sigma$ . We define the *dihedral angle* of  $\sigma$  at e to be the interior angle

## $\theta(\sigma, e) \in \mathbb{R}/\pi\mathbb{Z}$

between two normal vectors to |e| that lie in the two 2-faces  $|\tau|$  and  $|\tau'|$  of  $|\sigma|$  that intersect in |e| and that point into these faces.



Let also  $\ell(e) \in \mathbb{R}$  be the length of the geometric edge |e|. The *Dehn invariant* of the proper 3-simplex  $\sigma$  in  $\mathbb{R}^3$  is defined to be the sum

$$D(\sigma) = \sum_{e \subset \sigma} \ell(\sigma, e) \otimes \theta(\sigma, e) \in \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$$

which ranges over the (six) edges e of  $\sigma$ .

DEFINITION 2.9. The Dehn invariant of the polytope  $P \subset \mathbb{R}^3$  is the sum

$$D(P) = \sum_{i=1}^{m} D(\sigma_i) \in \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$$

of the Dehn invariants of the simplices in a triangulation  $\{\sigma_1, \ldots, \sigma_m\}$  of P.

It is clear from this definition that if  $P, P' \subset \mathbb{R}^3$  are scissor's congruent polytopes, then D(P) = D(P'). However, it is not at all clear that the Dehn invariant is well-defined. We need the following result.

LEMMA 2.10. The Dehn invariant D(P) of the polytope  $P \subset \mathbb{R}^3$  is independent of the choice of triangulation  $\{\sigma_1, \ldots, \sigma_m\}$  of P. PROOF. In view of Lemma 1.5, it suffices to show that if  $\{\sigma'_1, \ldots, \sigma'_m\}$  is a triangulation of the geometric 3-simplex  $|\sigma|$ , then

$$D(\sigma) = \sum_{i=1}^{m} D(\sigma'_i),$$

or equivalently,

$$\sum_{e \subset \sigma} \ell(e) \otimes \theta(\sigma, e) = \sum_{i=1}^{m} \sum_{e' \subset \sigma'_i} \ell(e') \otimes \theta(\sigma'_i, e').$$

To this end, we rewrite the right-hand side as

$$\sum_{e' \in E} \sum_{i \in I(e')} \ell(e') \otimes \theta(\sigma'_i, e') = \sum_{e' \in E} \ell(e') \otimes \big(\sum_{i \in I(e')} \theta(\sigma'_i, e')\big),$$

where E is the set of 1-simplices e' that are contained in one of  $\sigma'_1, \ldots, \sigma'_m$ , and where  $I(e') = \{i \mid e' \subset \sigma'_i\}$ . The set E is the disjoint union

$$E = E_1 \sqcup E_2 \sqcup E_3$$

where  $e' \in E_1$  if |e'| is contained in an edge |e| of  $|\sigma|$ , where  $e' \in E_2$  if the interior of |e'| is contained in the interior of  $|\sigma|$ , and where  $e' \in E_3$  if the interior of |e'| is contained in the interior of a face of  $|\sigma|$ . If  $e' \in E_2$  then  $\sum_{i \in I(e')} \theta(\sigma'_i, e') = 2\pi + \pi \mathbb{Z}$ , and if  $e' \in E_3$  then  $\sum_{i \in I(e')} \theta(\sigma'_i, e') = \pi + \pi \mathbb{Z}$ . Therefore,

$$\sum_{e' \in E} \ell(e') \otimes \left(\sum_{i \in I(e')} \theta(\sigma'_i, e')\right) = \sum_{e' \in E_1} \ell(e') \otimes \left(\sum_{i \in I(e')} \theta(\sigma'_i, e')\right)$$

Finally, we write the set  $E_1$  as the disjoint union

$$E_1 = \coprod_{e \subset \sigma} E_1(e)$$

where  $E_1(e) \subset E_1$  is the subset whose elements are the e' with  $|e'| \subset |e|$ . Since the geometric edges |e'| with  $e' \in E_1(e)$  subdivide the geometric edge |e|, we have

$$\ell(e) = \sum_{e' \in E_1(e)} \ell(e').$$

Moreover, for every edge  $e \subset \sigma$  and every  $e' \in E_1(e)$ , we have

$$\theta(\sigma, e) = \sum_{i \in I(e')} \theta(\sigma'_i, e')$$

Therefore, we find that

$$\sum_{e \subset \sigma} \ell(e) \otimes \theta(\sigma, e) = \sum_{e \subset \sigma} \sum_{e' \in E_1(e)} \ell(e') \otimes \left(\sum_{i \in I(e')} \theta(\sigma'_i, e')\right)$$
$$= \sum_{e' \in E_1} \ell(e') \otimes \left(\sum_{i \in I(e')} \theta(\sigma'_i, e')\right)$$

as desired. This completes the proof.

REMARK 2.11. Let  $P \subset \mathbb{R}^3$  be a polytope and let  $\{\sigma_1, \ldots, \sigma_m\}$  be a triangulation of P. We see as in the proof of Lemma 2.10 that

$$D(P) = \sum_{i=1}^{m} \sum_{e \subset \sigma_i} \ell(e) \otimes \theta(\sigma_i, e) = \sum_{e \in E} \ell(e) \otimes \left(\sum_{i \in I(e)} \theta(\sigma_i, e)\right),$$

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where E is the set of 1-simplices that are contained in one of the  $\sigma_1, \ldots, \sigma_m$ , and where  $I(e) = \{i \mid e \subset \sigma_i\}$ . Again, if the interior of |e| is contained either in the interior of P or in the interior of a face (we have not defined what this means in general) of P, then  $\sum_{i \in I(e)} \theta(\sigma_i, e)$  is zero in  $\mathbb{R}/\pi\mathbb{Z}$ . Therefore, when we calculate the Dehn invariant D(P), we can ignore the edges e of this kind.

THEOREM 2.12 (Dehn [3]). The cube C and the regular tetrahedron T of equal volume are not scissor's congruent.

PROOF. It suffices to show that  $D(C) \neq D(T)$ . Let  $\ell_C$  and  $\ell_T$  are the length of the edges in C and T. Using Remark 2.11, we find that

$$D(C) = 12 \cdot \ell_C \otimes \left(\frac{\pi}{2} + \pi \mathbb{Z}\right) = 0$$
$$D(T) = 6 \cdot \ell_T \otimes \left(\theta + \pi \mathbb{Z}\right)$$

where the angle  $0 \leq \theta \leq \pi$  is determined by

$$\cos(\theta) = \frac{1}{3}.$$

We claim that  $\theta \notin \pi \mathbb{Q}$ . Granting this, we see that D(T) is non-zero in  $\mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$ . Indeed, the claim shows that  $\theta + \pi \mathbb{Q}$  is non-zero in  $\mathbb{R}/\pi \mathbb{Q}$ , and hence, Corollary 2.7 shows that  $1 \otimes (\theta + \pi \mathbb{Q})$  is non-zero in  $\mathbb{R} \otimes (\mathbb{R}/\pi \mathbb{Q})$ . Finally, Corollary 2.2 shows that  $1 \otimes (\theta + \pi \mathbb{Z})$  and hence D(T) is non-zero in  $\mathbb{R} \otimes (\mathbb{R}/\pi \mathbb{Z})$ .

To prove the claim, we first show that if  $\cos(\theta) = 1/3$ , then for every positive integer  $\cos(n\theta) = a_n/3^n$ , where  $a_n$  is an integer not divisible by 3. We proceed by induction beginning from the case n = 1 which is trivial. So we assume that the cases  $n \leq r$  hold and prove the case n = r + 1. Now, as is well-known,

$$\cos((r+1)\theta) + \cos((r-1)\theta) = 2\cos(r\theta)\cos(\theta),$$

from which we conclude that

$$\cos((r+1)\theta) = 2\cos(r\theta)\cos(\theta) - \cos((r-1)\theta)$$
$$= \frac{2a_r}{3^{r+1}} - \frac{a_{r-1}}{3^{r-1}} = \frac{2a_r - 9a_{r-1}}{3^{r+1}}.$$

Since  $a_r$  and  $a_{r-1}$  are integers not divisible by 3, so is  $a_{r+1} = 2a_r - 9a_{r-1}$ . This proves the induction step. We conclude, by induction, that  $\cos(n\theta) = a_n/3^n$  with  $a_n$  an integer not divisible by 3 for all positive integers n. In particular, for every positive integer n, the rational number  $\cos(n\theta)$  is not an integer.

We can now prove the claim. For suppose, conversely, that

$$\theta = \frac{m}{n}\pi \in \pi\mathbb{Q}$$

with n a positive integer. In this case,  $n\theta = m\pi$ , and hence,

$$\cos(n\theta) = \cos(m\pi) = \pm 1$$

which is an integer. We conclude that  $\theta \notin \pi \mathbb{Q}$  as claimed.

 $\square$ 

In the next lecture, we will begin to formulate and answer the question of how to "parametrize" all polytopes  $P \subset \mathbb{R}^3$  "up to scissor's congruence."

## 3. The scissor's congruence group

We wish to parametrize polytopes in  $\mathbb{R}^n$  up to scissor's congruence. To make this problem precise, we introduce the scissor's congruence group.

DEFINITION 3.1. Suppose that  $P, Q \subset \mathbb{R}^n$  is a pair of polytopes with the property that also  $P \cup Q \subset \mathbb{R}^n$  is a polytope and that there exists a triangulation  $\{\sigma_1, \ldots, \sigma_k, \sigma_{k+1}, \ldots, \sigma_m\}$  of  $P \cup Q$  such that  $\{\sigma_1, \ldots, \sigma_k\}$  and  $\{\sigma_{k+1}, \ldots, \sigma_m\}$  are triangulations of P and Q. In this case, the polytope  $P \cup Q$  is denoted P + Q and called the polytope sum of P and Q.

The polytope sum P + Q is not defined for every pair of polytopes  $P, Q \subset \mathbb{R}^n$ . For example, the polytope sum P + P is not defined unless P is the empty polytope. However, the polytope sum P + Q is clearly defined whenever P and Q are disjoint. And if  $P, Q \subset \mathbb{R}^n$  is an arbitrary pair of polytopes, we can always find a translation  $t_v \colon \mathbb{R}^n \to \mathbb{R}^n$  such that P and  $Q' = t_v(Q)$  are disjoint.

DEFINITION 3.2 (Jessen [7]). The scissor's congruence group of  $\mathbb{R}^n$  is defined to be the abelian group given by the quotient

$$P(\mathbb{R}^n) = F(\mathbb{R}^n) / R(\mathbb{R}^n)$$

of the free abelian group  $F(\mathbb{R}^n)$  with one generator  $\langle P \rangle$  for every polytope  $P \subset \mathbb{R}^n$  by the subgroup  $R(\mathbb{R}^n) \subset F(\mathbb{R}^n)$  generated by the following elements (i)–(ii):

(i) For every pair of polytopes  $P,Q\subset \mathbb{R}^n$  for which the polytope sum P+Q is defined, the element

$$\langle P+Q\rangle - \langle P\rangle - \langle Q\rangle$$

is a generator of  $R(\mathbb{R}^n)$ .

(ii) For every polytope  $P \subset \mathbb{R}^n$  and every isometry  $f : \mathbb{R}^n \to \mathbb{R}^n$ , the element

$$\langle P \rangle - \langle f(P) \rangle$$

is a generator of  $R(\mathbb{R}^n)$ .

The class  $\langle P \rangle + R(\mathbb{R}^n) \in P(\mathbb{R}^n)$  that contains the polytope  $P \subset \mathbb{R}^n$  is denoted [P].

Let  $P, Q \subset \mathbb{R}^n$  be a pair of polytopes, and let Q' be a translation of Q such that P and Q' are disjoint. In this case, the polytope sum P + Q' is defined, and by relations (ii) and (i), respectively, we find that

$$[P] + Q] = [P] + [Q'] = [P + Q'].$$

In this way, we can interpret the sum of the classes [P] and [Q] in terms of polytope sum. It follows that every element in  $P(\mathbb{R}^n)$  can be written (non-uniquely) as a difference [P] - [Q] between the classes of two polytopes.

LEMMA 3.3. Let P and P' be two polytopes in  $\mathbb{R}^n$ . The following are equivalent.

- (1) The classes [P] and [P'] in  $P(\mathbb{R}^n)$  are equal.
- (2) There exists two scissor's congruent polytopes Q and Q' in  $\mathbb{R}^n$  such that the polytope sums P + Q and P' + Q' exist and are scissor's congruent.

PROOF. We first show that for  $R, R' \subset \mathbb{R}^n$  two scissor's congruent polytopes, we have [R] = [R']. By the definition of scissor's congruence, we have triangulations  $\{\sigma_1, \ldots, \sigma_m\}$  of R and  $\{\sigma'_1, \ldots, \sigma'_m\}$  of R' and isometries  $f_1, \ldots, f_m$  of  $\mathbb{R}^n$  such that  $f(|\sigma_i|) = |\sigma'_i|$  for all i = 1, ..., m. Therefore, by the definition of the scissor's congruence group, we find

$$[R] = \sum_{i=1}^{m} [|\sigma_i|] = \sum_{i=1}^{m} [f_i(|\sigma_i|)] = \sum_{i=1}^{m} [|\sigma'_i|] = [R']$$

as desired. Here the first and last equalities follow from the relation (i) and the second equality follows from the relation (ii).

Now, suppose that (2) holds. Then [Q] = [Q'] and [P+Q] = [P'+Q'] by what was just proved. Therefore, using Remark 3.1, we find that

$$[P] = [P] + [Q] - [Q] = [P + Q] - [Q]$$
$$= [P' + Q'] - [Q'] = [P'] + [Q'] - [Q'] = [P']$$

which proves (1).

Conversely, suppose that (1) holds. Then the difference  $\langle P \rangle - \langle P' \rangle$  is an element of  $R(\mathbb{R}^n) \subset F(\mathbb{R}^n)$ , and hence, can be written as a sum

$$\sum_{i=1}^{r} (\langle R_i + S_i \rangle - \langle R_i \rangle - \langle S_i \rangle) - \sum_{j=1}^{s} (\langle T_j + U_j \rangle - \langle T_j \rangle - \langle U_j \rangle) + \sum_{k=1}^{t} (\langle V_k \rangle - \langle W_k \rangle),$$

where the first and second summands are instances of the relation (i), and where the last summand are instances of the relation (ii). It follows that

$$\langle P \rangle + \sum_{i=1}^{r} (\langle R_i \rangle + \langle S_i \rangle) + \sum_{j=1}^{s} \langle T_j + U_j \rangle + \sum_{k=1}^{t} \langle W_k \rangle$$
  
=  $\langle P' \rangle + \sum_{i=1}^{r} \langle R_i + S_i \rangle + \sum_{j=1}^{s} (\langle T_j \rangle + \langle U_j \rangle) + \sum_{k=1}^{t} \langle V_k \rangle.$ 

This is an equality of two sums of generators in the free abelian group  $F(\mathbb{R}^n)$ . Such an equality holds if and only if the generators that appear on the left-hand side are equal, up to reordering, to the generators that appear on the right-hand side. From this equality, we find that the two polytopes defined by the following polytope sums are scissor's congruent. Here, we write (-)'' to indicate suitable translation to make the two polytope sums exist.

$$P + \sum_{i=1}^{r} (R_i'' + S_i'') + \sum_{j=1}^{s} (T_j + U_j)'' + \sum_{k=1}^{t} W_k''$$
$$P' + \sum_{i=1}^{r} (R_i + S_i)'' + \sum_{j=1}^{s} (T_j'' + U_j'') + \sum_{k=1}^{t} V_k''.$$

Now, if we define Q and Q' to the polytope sums

$$Q = \sum_{i=1}^{r} (R_i'' + S_i'') + \sum_{j=1}^{s} (T_j + U_j)'' + \sum_{k=1}^{t} W_k''$$
$$Q' = \sum_{i=1}^{r} (R_i + S_i)'' + \sum_{j=1}^{s} (T_j'' + U_j'') + \sum_{k=1}^{t} V_k'',$$

then P + Q and P' + Q' are scissor's congruent. But Q and Q' are also scissor's congruent. Indeed,  $R''_i + S''_i$  is scissor's congruent to  $R_i + S_i$  which is a translation

of  $(R_i + S_i)''$ ;  $(T_j + U_j)''$  is a translation of  $T_j + U_j$  which is scissor's congruent to  $T''_j + U''_j$ ; and  $W''_k$  is a translation of  $W_k$ , which is a translation of  $V_k$ , which, in turn, is a translation of  $V''_k$ . This proves (2).

THEOREM 3.4 (Zylev [17]). Let  $P, P', Q, Q' \subset \mathbb{R}^n$  be polytopes such that the polytope sums P + Q and P' + Q' exist and are equal. If Q and Q' are scissor's congruent, then also P and P' are scissor's congruent.

PROOF. We will use the following facts (a)–(b) without proof.

- (a) If  $A, B \subset \mathbb{R}^n$  are polytopes, then so is the subset  $A \cdot B \subset \mathbb{R}^n$  defined to be the closure of the intersection of the interior of A and the interior of B.
- (b) If  $A \subset B \subset \mathbb{R}^n$  are polytopes, then so is the subset  $B A \subset \mathbb{R}^n$  defined to be the closure of the complement of A in B, and B = A + (B A).

By subdividing Q and Q', we can find polytopes  $Q_1, \ldots, Q_r$  and  $Q'_1, \ldots, Q'_r$  such that  $Q = Q_1 + \cdots + Q_r$  and  $Q' = Q'_1 + \cdots + Q'_r$ , and such that for all  $1 \leq i \leq r$ , the polytopes  $Q_i$  and  $Q'_i$  are scissor's congruent and their common volume strictly smaller than half the common volume of P and P'. The proof of the theorem is by induction on the number  $r \geq 0$  of summands. The case r = 0 is trivial, since Q and Q' are both empty. So we let r > 0 and assume that the theorem has been proved for smaller values of r. Since the volume of  $Q'_r$  is strictly smaller than the volume of  $P - (P \cdot Q'_r)$ , we can find pairwise disjoint polytopes  $R_1, \ldots, R_r \subset P - (P \cdot Q'_r)$  such that  $R_i$  and  $Q_i \cdot Q'_r$  are scissor's congruent for all  $1 \leq i \leq r$ . We now define  $\overline{Q}_1, \ldots, \overline{Q}_r$  and  $\overline{P}$  to be the polytopes

$$\bar{Q}_i = (Q_i - (Q_i \cdot Q'_r)) + R_i$$
  
$$\bar{P} = (P+Q) - Q'_r - (\bar{Q}_1 + \dots + \bar{Q}_{r-1})$$

Now, for all  $1 \leq i \leq r$ , the polytopes  $\bar{Q}_i$  and  $Q_i$  are scissor's congruent, and hence, scissor's congruent to  $Q'_i$ . Moreover,  $\bar{P}$  and P have the same volume. Since also

$$P' = (P' + Q') - Q'_r = (Q'_1 + \dots + Q'_{r-1}),$$

the inductive hypothesis shows that  $\overline{P}$  and P' are scissor's congruent. It remains to show that P and  $\overline{P}$  are scissor's congruent. To this end, we define

$$P = P - (R_1 + \dots + R_r) + (Q_1 \cdot Q'_r) + \dots + (Q_r \cdot Q'_r).$$

The polytopes P and  $\tilde{P}$  are scissor's congruent, since  $R_i$  and  $Q_i \cdot Q'_r$  are scissor's congruent for all  $1 \leq i \leq r$ . Finally,

$$\bar{P} = \tilde{P} - Q'_r + \bar{Q}_r$$

which shows that also  $\tilde{P}$  and  $\bar{P}$  are scissor's congruent. We conclude that P and  $\bar{P}$  are scissor's congruent, and hence, that P and P' are scissor's congruent. This completes the proof of the induction step.

COROLLARY 3.5. The following are equivalent for polytopes  $P, P' \subset \mathbb{R}^n$ .

- (a) The classes [P] and [P'] in  $P(\mathbb{R}^n)$  are equal.
- (b) The polytopes P and P' are scissor's congruent.

PROOF. It follows from Lemma 3.3 that (b) implies (a). Suppose, conversely, that [P] = [P']. By Lemma 3.3, there exists scissor's congruent polytopes Q and Q' such that the polytope sums P+Q and P'+Q' exist and are scissor's congruent. We define polytopes P'' and Q'' that are scissor's congruent to P' and Q', respectively,

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and for which P'' + Q'' = P + Q. Given these, Zylev's theorem shows that P and P'', and hence, P and P' are scissor's congruent as desired. Let  $\{\sigma_1, \ldots, \sigma_m\}$  be a triangulation of P + Q, let  $\{\sigma'_1, \ldots, \sigma'_m\}$  be a triangulations of P' + Q', and let  $f_1, \ldots, f_m$  be isometries of  $\mathbb{R}^n$  with  $f_i(|\sigma_i|) = |\sigma'_i|$ . Let  $U \subset P + Q$  be the union of the interiors of the geometric simplices  $|\sigma_1|, \ldots, |\sigma_m|$ , and let  $U' \subset P' + Q'$  be the union of the interiors of the geometric simplices  $|\sigma'_1|, \ldots, |\sigma'_m|$ . Let  $f: U \to U'$  be the union of the interiors of the geometric simplices  $|\sigma'_1|, \ldots, |\sigma'_m|$ . Let  $f: U \to U'$  be the union of the interiors of the geometric simplices  $|\sigma'_1|, \ldots, |\sigma'_m|$ . Let  $f: U \to U'$  be the homeomorphism whose restriction to the interior of  $|\sigma_i|$  is the isometry  $f_i$ . We now define  $P'' \subset \mathbb{R}^n$  to be the closure of  $f^{-1}(U' \cap P')$  and define  $Q'' \subset \mathbb{R}^n$  to be the closure of  $f^{-1}(U' \cap Q')$ . Using Lemma 1.5, we see that P'' and Q'' are polytopes and that Q'' is scissor's congruent to Q'. Finally, it is clear from the definition that the polytope sum P'' + Q'' exists and is equal to P + Q.

REMARK 3.6. The definition of the congruence group is an example of the general group-completion construction introduced by Grothendieck in his famous work on the Riemann-Roch theorem [1]. To briefly explain the construction, let M be an abelian monoid. We use additive notation and write x + y for the composition of the elements  $x, y \in M$ . The group completion (or Grothendieck group) of M is the abelian group K(M) defined by the quotient

$$K(M) = F(M)/R(M)$$

of the free abelian group F(M) with one generator  $\langle x \rangle$  for every  $x \in M$  by the subgroup  $R(M) \subset F(M)$  generated by the elements

$$\langle x+y\rangle - \langle x\rangle - \langle y\rangle$$

for all  $x, y \in M$ . Let  $[x] = \langle x \rangle + R(M) \in K(M)$ . There is a canonical monoid map

 $\tilde{\gamma} \colon M \to K(M)$ 

defined by  $\tilde{\gamma}(x) = [x]$ . This map has the following universal property: For every abelian group A and every monoid map  $\tilde{f}: M \to A$ , there exists a unique group homomorphism  $f: K(M) \to A$  such that  $\tilde{f} = f \circ \tilde{\gamma}$ . The map  $\tilde{\gamma}$  is injective if and only if for all  $x, y, z \in M$ , x + z = y + z implies that x = y. In this case we say that cancellation holds in M. We note that cancellation holds in K(M)because every element has an additive inverse. The group-completion construction introduces formal additive inverses of the elements in M. For example, if  $\mathbb{N}_0$  be the additive monoid of non-negative integers, then  $K(\mathbb{N}_0)$  is (canonically isomorphic to) the additive group  $\mathbb{Z}$  of all integers.

Now, let  $M(\mathbb{R}^n)$  denote the set of equivalence classes of polytopes  $P \subset \mathbb{R}^n$  under the equivalence relation of scissor's congruence, and let (P) denote the equivalence class that contains the polytope P. For all  $(P), (Q) \in M$ , we define

$$(P) + (Q) = (P + Q'),$$

where Q' is a translation of Q such that the polytope sum P + Q' is defined. We leave it as an exercise to the reader to verify that (P) + (Q) is well-defined. It follows from the definition of the scissor's congruence group  $P(\mathbb{R}^n)$  that the map

$$f: M(\mathbb{R}^n) \to P(\mathbb{R}^n)$$

defined by  $\tilde{f}((P)) = [P]$  is a monoid map and that it has the same universal property as the map  $\tilde{\gamma} \colon M(\mathbb{R}^n) \to K(M(\mathbb{R}^n))$ . Therefore, the induced map

$$f: K(M(\mathbb{R}^n)) \to P(\mathbb{R}^n)$$

is an isomorphism of abelian groups; the inverse is the unique group homomorphism  $\gamma \colon P(\mathbb{R}^n) \to K(M)$  induced by the monoid map  $\tilde{\gamma}$ . By Corollary 3.5, we see that cancellation holds in the monoid  $M(\mathbb{R}^n)$ . When he defined the scissor's congruence group in 1941, Jessen apparently did not realize the very general nature and importance of the construction. It was left for Grothendieck to discover this fifteen years later. Group-completion has played a central role in mathematics ever since.

# 4. The group $P(\mathbb{R}^3)$

We recall from Definition 3.2 that the scissor's congruence group  $P(\mathbb{R}^n)$  is defined to be the quotient  $F(\mathbb{R}^n)/R(\mathbb{R}^n)$  of the free abelian group  $F(\mathbb{R}^n)$  with one generator  $\langle P \rangle$  for every polytope  $P \subset \mathbb{R}^n$  by the subgroup  $R(\mathbb{R}^n)$  generated by the elements  $\langle P + Q \rangle - \langle P \rangle + \langle Q \rangle$ , for all polytopes  $P, Q \subset \mathbb{R}^n$  such that the polytope sum P + Q exists, together with the elements  $\langle P \rangle - \langle f(P) \rangle$ , for every polytope  $P \subset \mathbb{R}^n$  and every isometry  $f \colon \mathbb{R}^n \to \mathbb{R}^n$ . We define a group homomorphism

$$\operatorname{vol} = \operatorname{vol}_n \colon P(\mathbb{R}^n) \to \mathbb{R}$$

that to the generator  $[P] = \langle P \rangle + R(\mathbb{R}^n)$  associates the (*n*-dimensional) volume of P as follows. There is a unique group homomorphism vol':  $F(\mathbb{R}^n) \to \mathbb{R}$  that takes the generator  $\langle P \rangle$  to the volume of P; it maps the element  $\sum_P n_P \langle P \rangle$  of  $F(\mathbb{R}^n)$  to the real number  $\sum_P n_P \operatorname{vol}(P)$ . Now, if  $P, Q \subset \mathbb{R}^n$  are two polytopes such that the polytope sum P + Q exists, then  $\operatorname{vol}(P + Q) = \operatorname{vol}(P) + \operatorname{vol}(Q)$ , and therefore,  $\operatorname{vol}'(\langle P + Q \rangle - \langle P \rangle - \langle Q \rangle) = 0$ . Similarly, if  $P \subset \mathbb{R}^n$  is a polytope and  $f \colon \mathbb{R}^n \to \mathbb{R}^n$  an isometry, then  $\operatorname{vol}(P) = \operatorname{vol}(f(P))$ , and hence,  $\operatorname{vol}'(\langle P \rangle - \langle f(P) \rangle) = 0$ . It follows that  $\operatorname{vol}'(R(\mathbb{R}^n)) = 0$ , so the group homomorphism  $\operatorname{vol}': F(\mathbb{R}^n) \to \mathbb{R}$  induces a group homomorphism  $\operatorname{vol}: P(\mathbb{R}^n) \to \mathbb{R}$  as desired.

LEMMA 4.1. The group homomorphism vol:  $P(\mathbb{R}) \to \mathbb{R}$  is an isomorphism.

PROOF. Let  $F: \mathbb{R} \to P(\mathbb{R})$  be the map that takes a > 0 to the class [[0, a]] of the interval  $[0, a] \subset \mathbb{R}$ , that takes a < 0 to the opposite -[[a, 0]] of the class of the interval  $[a, 0] \subset \mathbb{R}$ , and that takes 0 to 0. We claim that F is a group homomorphism. Indeed, if 0 < a, b < a + b, then

$$F(a) + F(b) = [[0, a]] + [[0, b]] = [[0, a]] + [[a, a + b]]$$
  
= [[0, a + b]] = F(a + b);

if a < 0 < a + b < b, then

$$F(a) + F(b) = -[[a, 0]] + [[0, b]] = -[[a, 0]] + [[a, a + b]]$$
  
= [[0, a + b]] = F(a + b);

and similarly for b < 0 < a + b < a; if a < a + b < 0 < b, then

$$F(a) + F(b) = -[[a, 0]] + [[0, b]] = -[[a, 0]] + [[a, a + b]]$$
  
= -[[a + b, 0]] = F(a + b);

and similarly for b < a + b < 0 < a; and finally if a + b < a, b < 0, then

$$F(a) + F(b) = -[[a, 0]] - [[b, 0]] = -[[a, 0]] - [[a + b, a]]$$
  
= -[[a + b, 0]] = F(a + b);

this proves the claim. It follows immediately from definitions that  $\operatorname{vol} \circ F$  is equal to the identity map of  $\mathbb{R}$ . To prove that also  $F \circ \operatorname{vol}$  is equal to the identity map of  $P(\mathbb{R})$ , it will suffice to show that  $[P] = F(\operatorname{vol}([P]))$  for every polytope  $P \subset \mathbb{R}$ . Indeed, since  $F \circ \operatorname{vol}$  and  $\operatorname{id}_{P(\mathbb{R})}$  are group homomorphisms, it suffices to prove that they take the same value on a set of generators of the group  $P(\mathbb{R})$ . Now, for every polytope  $P \subset \mathbb{R}$  there exists  $a_1 < b_1 < a_2 < b_2 < \cdots < a_r < b_r$  such that

$$P = [a_1, b_1] + [a_2, b_2] + \dots + [a_r, b_r].$$

It follows that, if we write  $b_i = a_i + c_i$ , then

$$\begin{aligned} [P] &= [[a_1, b_1]] + \dots + [[a_r, b_r]] \\ &= [[0, c_1]] + [[c_1, c_1 + c_2]] + \dots + [[c_1 + \dots + c_{r-1}, c_1 + \dots + c_{r-1} + c_r]] \\ &= [[0, c_1 + \dots + c_r]] \\ &= (F \circ \text{vol})([P]) \end{aligned}$$

which shows that  $F \circ \text{vol} = \text{id}_{P(\mathbb{R})}$  as desired. This proves the lemma.

LEMMA 4.2. For every positive integer n, there is a group homomorphism

$$E: P(\mathbb{R}^n) \to P(\mathbb{R}^{n+1})$$

that to the class of P associates the class of the cylinder  $P \times [0,1]$ .

**PROOF.** We claim that if  $P \subset \mathbb{R}^n$  is a polytope, then so is the cylinder

$$P \times [0,1] \subset \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$$

To prove this, we may assume that  $P = |\sigma|$  is the geometric *n*-simplex associated with a proper *n*-simplex  $\sigma = (a_0, \ldots, a_n)$ . Now, if we let  $b_i = (a_i, 0) \in \mathbb{R}^n \times \mathbb{R}$  and  $c_i = (a_i, 1) \in \mathbb{R}^n \times \mathbb{R}$ , then the n + 1 proper (n + 1)-simplices

$$\sigma_0 = (b_0, c_0, c_1, \dots, c_{n-1}, c_n)$$
  

$$\sigma_1 = (b_0, b_1, c_1, \dots, c_{n-1}, c_n)$$
  

$$\vdots$$
  

$$\sigma_n = (b_0, b_1, \dots, b_{n-1}, b_n, c_n)$$

define a triangulation  $\{\sigma_0, \ldots, \sigma_n\}$  of  $|\sigma| \times [0, 1]$ , proving the claim. The following figure illustrates the triangulation for n = 1.



It follows that there is a group homomorphism  $E' \colon F(\mathbb{R}^n) \to P(\mathbb{R}^{n+1})$  that to the generator  $\langle P \rangle$  associates the class  $[P \times [0,1]]$ . It remains to show that E'maps the subgroup  $R(\mathbb{R}^n) \subset F(\mathbb{R}^n)$  to zero, and hence, induces the stated group homomorphism  $E \colon P(\mathbb{R}^n) \to P(\mathbb{R}^{n+1})$ . Now, if  $P, Q \subset \mathbb{R}^n$  are polytopes such that the polytope sum P + Q exists, then the polytope  $(P \times [0,1]) + (Q \times [0,1])$  exists and equal to  $(P+Q) \times [0,1]$ . Finally, if f is an isometry of  $\mathbb{R}^n$ , then  $f \times \operatorname{id}$  is an isometry of  $\mathbb{R}^n \times \mathbb{R}$  and  $(f \times \operatorname{id})(P \times [0,1]) = f(P) \times [0,1]$ . We conclude that E'maps the subgroup  $R(\mathbb{R}^n) \subset F(\mathbb{R}^n)$  to zero as desired.  $\Box$ 

We can now restate Theorem 1.1 as the following calculation of the scissor's congruence group of the plane.

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THEOREM 4.3. The group homomorphisms

$$P(\mathbb{R}) \xrightarrow{E} P(\mathbb{R}^2) \xrightarrow{\text{vol}} \mathbb{R}$$

are both isomorphisms.

PROOF. We note that  $\operatorname{vol}_2 \circ E = \operatorname{vol}_1 : P(\mathbb{R}) \to \mathbb{R}$  which is an isomorphism by Lemma 4.1. It follows that E is injective and that vol is surjective. Moreover, the image of E is a subgroup  $\operatorname{im}(E) \subset P(\mathbb{R}^2)$  and Theorem 1.1 shows that it contains the class [P] of every polytope  $P \subset \mathbb{R}^2$ . Since these classes generate  $P(\mathbb{R}^2)$ , it follows that  $\operatorname{im}(E) = P(\mathbb{R}^2)$ . Hence, the map E is also surjective and therefore an isomorphism. We conclude that also  $\operatorname{vol}_2$  is an isomorphism.  $\Box$ 

COROLLARY 4.4. The composition of the group homorphisms

$$P(\mathbb{R}^2) \xrightarrow{E} P(\mathbb{R}^3) \xrightarrow{\text{vol}} \mathbb{R}$$

is an isomorphism. In particular, the left-hand map E is injective.

PROOF. Indeed, we have  $\operatorname{vol}_3 \circ E = \operatorname{vol}_2 \colon P(\mathbb{R}^2) \to \mathbb{R}$  which is an isomorphism by Theorem 4.3.

The group  $P(\mathbb{R}^n)$  is generated by the classes of the proper geometric *n*-simplices. Indeed, if  $P \subset \mathbb{R}^n$  is a polytope and if  $\{\sigma_1, \ldots, \sigma_m\}$  is a triangulation of P, then

 $[P] = [|\sigma_1| + \dots + |\sigma_m|] = [|\sigma_1|] + \dots + [|\sigma_m|].$ 

We use this to make the following calculation.

LEMMA 4.5. The composition of the group homomorphisms

$$P(\mathbb{R}^2) \xrightarrow{E} P(\mathbb{R}^3) \xrightarrow{D} \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$$

is equal the zero map.

PROOF. It will suffice to show that for every triangle  $|\sigma| \subset \mathbb{R}^2$  in the plane, the cylinder  $|\sigma| \times [0,1] \subset \mathbb{R}^3$  has trivial Dehn invariant. As the following figure illustrates, there are a total of nine edges in  $|\sigma| \times [0,1]$ .



Six of these are of the form  $|\tau| \times \{0\}$  or  $|\tau| \times \{1\}$ , where  $|\tau|$  is an edge of  $|\sigma|$ , and the remaining three are of the form  $|v| \times [0, 1]$ , where |v| is a vertex of  $|\sigma|$ . The dihedral angles at the former six edges all are equal to  $\pi/2$ , and hence, these edges do not contribute to  $D(|\sigma| \times [0, 1])$ . The sum of the dihedral angles at the latter three edges is equal to  $\pi$ , and since the three edges all have the same length 1, they also do not contribute to  $D(|\sigma| \times [0, 1])$ . The lemma follows. The sequence of abelian groups and group homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is said to be exact if im(f) = ker(g). Here,

$$\operatorname{im}(f) = \{f(a) \mid a \in A\} \subset B$$
$$\operatorname{ker}(g) = \{b \in B \mid g(b) = 0\} \subset B.$$

We note that  $im(f) \subset ker(g)$  if and only if  $g \circ f$  is the zero homomorphism. For example, we have proved in Lemma 4.4 that the sequence

$$0 \longrightarrow P(\mathbb{R}^2) \xrightarrow{E} P(\mathbb{R}^3)$$

is exact. Indeed, the kernel of E is zero, since E is injective.

THEOREM 4.6 (Sydler [14]). The sequence

$$P(\mathbb{R}^2) \xrightarrow{E} P(\mathbb{R}^3) \xrightarrow{D} \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$$

 $is \ exact.$ 

We proved in Lemma 4.5 above that  $im(E) \subset ker(D)$ . The more difficult statement that im(E) is equal to ker(D) was proved by Sydler in 1965. We will outline a proof following Dupont [5] in the next section.

COROLLARY 4.7. The two polytopes  $P, P' \subset \mathbb{R}^3$  are scissor's congruent if and only if vol(P) = vol(P') and D(P) = D(P').

PROOF. Suppose first that  $P, P' \subset \mathbb{R}^3$  are two scissor's congruent polytopes. We proved in Proposition 1.7 that  $\operatorname{vol}(P) = \operatorname{vol}(P')$ , and a similar proof shows that D(P) = D(P'). Suppose conversely that  $P, P' \subset \mathbb{R}^3$  are two polytopes for which  $\operatorname{vol}(P) = \operatorname{vol}(P')$  and D(P) = D(P'). We conclude from Theorem 4.6 that

$$[P] - [P'] = E(\xi)$$

for some  $\xi \in P(\mathbb{R}^2)$ . But

$$\operatorname{vol}_2(\xi) = \operatorname{vol}_3(E(\xi)) = \operatorname{vol}_3([P] - [P']) = \operatorname{vol}_3([P]) - \operatorname{vol}_3([P']) = 0,$$

so by Theorem 4.3, we conclude that  $\xi = 0$ , and therefore, that [P] = [P']. Now, Corollary 3.5 shows that P and P' are scissor's congruent as stated.

It turns out that the Dehn invariant is not surjective. In the remainder of this section, we determine its image following Jessen [8]. We write  $\mathbb{R}^*$  for the multiplicative group of non-zero real numbers.

LEMMA 4.8. There is a left action of the group  $\mathbb{R}^*$  on  $P(\mathbb{R}^n)$  with  $\lambda \in \mathbb{R}^*$  acting through the group homomorphism  $\mu_{\lambda} \colon P(\mathbb{R}^n) \to P(\mathbb{R}^n)$  defined by

$$\mu_{\lambda}([P]) = \begin{cases} [\lambda P] & \text{if } \lambda > 0\\ -[(-\lambda)P] & \text{if } \lambda < 0 \end{cases}$$

PROOF. We first show that the group homomorphism  $\mu_{\lambda} \colon P(\mathbb{R}^n) \to P(\mathbb{R}^n)$  is well-defined. It will suffice consider the case where  $\lambda > 0$ . First, if  $P \subset \mathbb{R}^n$  is a polytope, then so is the subset  $\lambda P = \{\lambda x \mid x \in P\} \subset \mathbb{R}^n$  which we call the dilation of P by  $\lambda$ . So there is a group homomorphism  $\mu'_{\lambda} \colon F(\mathbb{R}^n) \to P(\mathbb{R}^n)$  that maps  $\langle P \rangle$ to  $[\lambda P]$ , and we must show that  $\mu'_{\lambda}$  maps  $R(\mathbb{R}^n)$  to zero. First, if  $P, Q \subset \mathbb{R}^n$  are polytopes such that the polytope sum P + Q is defined, then also the polytope sum  $\lambda P + \lambda Q$  is defined and is equal to  $\lambda (P + Q)$ , so  $\mu'_{\lambda}(\langle P + Q \rangle - \langle P \rangle - \langle Q \rangle) = 0$ . Second, let  $P \subset \mathbb{R}^n$  be a polytope and let  $f \colon \mathbb{R}^n \to \mathbb{R}^n$  be an isometry. We write f as the composition  $f = t_{f(0)} \circ \overline{f}$  of the orthogonal transformation  $\overline{f}$  and the translation  $t_{f(0)}$  as in Remark 1.3. Now, we have

$$\lambda f(P) = \lambda t_{f(0)}(\bar{f}(P)) = t_{\lambda f(0)}(\lambda \bar{f}(P)) = t_{\lambda f(0)}(\bar{f}(\lambda P))$$

which shows that also  $\mu'_{\lambda}(\langle P \rangle - \langle f(P) \rangle) = 0$ . This proves the group homomorphism  $\mu_{\lambda} \colon P(\mathbb{R}^n) \to P(\mathbb{R}^n)$  is well-defined. Finally, it remains to show that the group homomorphisms  $\mu_{\lambda} \colon P(\mathbb{R}^n) \to P(\mathbb{R}^n)$  with  $\lambda \in \mathbb{R}^*$  constitute a left action of the group  $\mathbb{R}^*$  on  $P(\mathbb{R}^n)$ . This means that for all  $\lambda, \mu \in \mathbb{R}^*$ ,  $\mu_{\lambda} \circ \mu_{\lambda'} = \mu_{\lambda\lambda'}$ . However, this follows immediately from the definition.

REMARK 4.9. If V is a real vector space, then we obtain a left action of  $\mathbb{R}^*$ on V by letting  $\lambda \in \mathbb{R}^*$  act through scalar multiplication by  $\lambda$ . However, the left action by  $\mathbb{R}^*$  on  $P(\mathbb{R}^n)$  does not extend to a real vector space structure on  $P(\mathbb{R}^n)$ except in the case n = 1. Indeed, for  $n \ge 2$ , the group homomorphisms

$$\mu_{\lambda+\lambda'}, \mu_{\lambda}+\mu_{\lambda'}: P(\mathbb{R}^n) \to P(\mathbb{R}^n)$$

are not equal. For instance, the figure



illustrates that for n = 2, the maps  $\mu_2$  and  $\mu_1 + \mu_1$  are different.

COROLLARY 4.10. The image of the Dehn invariant

$$D: P(\mathbb{R}^3) \to \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$$

is a subspace of the real vector space on the right-hand side.

PROOF. Since D is a group homomorphism,  $\operatorname{im}(D)$  is an additive subgroup. We must show that  $\operatorname{im}(D)$  is stable under scalar multiplication. It suffices to show that for every polytope  $P \subset \mathbb{R}^n$  and every positive real number  $\lambda$ ,  $\lambda D(P) \in \operatorname{im}(D)$ . But by the definition of the Dehn invariant, we have  $\lambda D(P) = D(\lambda P)$ .

**PROPOSITION 4.11.** The image of the Dehn invariant

$$D: P(\mathbb{R}^3) \to \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$$

is equal to the subspace generated by the elements

$$\cot \alpha \otimes \alpha + \cot \beta \otimes \beta - \cot(\alpha * \beta) \otimes (\alpha * \beta),$$

where  $\alpha, \beta \in (0, \pi/2)$  and  $\alpha * \beta \in (0, \pi/2)$  is the unique solution to the equation

$$\sin^2(\alpha * \beta) = \sin^2 \alpha \cdot \sin^2 \beta$$

PROOF. We recall from the discussion that precedes Lemma 4.5 that  $P(\mathbb{R}^3)$  is generated by the classes of all tetrahedra (proper geometric 3-simplices)  $T \subset \mathbb{R}^3$ . In effect, we claim that the following special type of tetrahedra generate  $P(\mathbb{R}^3)$ .



Indeed, every tetrahedron can be written as the polytope sum of six tetrahedra, three of which are of this special type and the remaining three of which are mirror reflections of tetrahedra of this special type. Now, every tetrahedron of the special type, in turn, is of the form  $\lambda T(\alpha, \beta)$ , where  $\lambda$  is a positive real number, and where  $T(\alpha, \beta) \subset \mathbb{R}^3$  is the tetrahedron with vertices

$$A = (0, 0, 0),$$
  

$$B = (\cot \alpha, 0, 0),$$
  

$$C = (\cot \alpha, \cot \alpha \cot \beta, 0),$$
  

$$D = (\cot \alpha, \cot \alpha \cot \beta, \cot \beta).$$

Here  $\alpha, \beta \in (0, \pi/2)$  and  $\alpha * \beta$  is defined as in the statement. It follows that the Dehn invariants of the tetrahedra  $T(\alpha, \beta)$  with  $\alpha, \beta \in (0, \pi/2)$  generate the subspace  $\operatorname{im}(D) \subset \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$ . By definition, we have

$$\theta(AC) = \theta(BC) = \theta(BD) = \frac{\pi}{2},$$

and by elementary geometry, we find

$$\theta(AB) = \alpha, \qquad \theta(CD) = \beta, \qquad \theta(AD) = \frac{\pi}{2} - \alpha * \beta$$
$$\ell(AD) = \cot(\alpha * \beta), \qquad \ell(AC) = \frac{\cot \alpha}{\sin \beta}, \qquad \ell(BD) = \frac{\cot \beta}{\sin \alpha}.$$

From this, we conclude that

$$D(T(\alpha,\beta)) = \cot \alpha \otimes \alpha + \cot \beta \otimes \beta - \cot(\alpha * \beta) \otimes (\alpha * \beta)$$

as stated. This completes the proof.

DEFINITION 4.12. The real vector space of absolute Kähler differentials of  $\mathbb{R}$  is the quotient real vector space

$$\Omega^1_{\mathbb{R}} = F/R,$$

where F is the real vector space with basis  $\{(a) \mid a \in \mathbb{R}\}$  and  $R \subset F$  is the subspace generated by the following elements (i)–(ii).

- (i) For all  $a, b \in \mathbb{R}$ , the element (a + b) (a) (b) is a generator of R.
- (ii) For all  $a, b \in \mathbb{R}$ , the element (ab) b(a) a(b) is a generator of R.

The universal derivation is the additive map

$$d\colon \mathbb{R}\to \Omega^1_{\mathbb{R}}$$

that to a associates da = (a) + R.

The relations (i)–(ii) imply the following identities (1)–(2), the second of which is called the Leibniz rule.

- (1) For all  $a, b \in \mathbb{R}$ , d(a+b) = da + db.
- (2) For all  $a, b \in \mathbb{R}$ , d(ab) = bda + adb.

Let V be a real vector space. The map  $D: \mathbb{R} \to V$  is said to be a derivation if it satisfies (1)–(2). The universal derivation  $d: \mathbb{R} \to \Omega^1_{\mathbb{R}}$  is a derivation. Moreover, if  $D: \mathbb{R} \to V$  is any derivation, then there exists a unique  $\mathbb{R}$ -linear map

$$h_D \colon \Omega^1_{\mathbb{R}} \to V$$

such that  $D = h_D \circ d$ . Indeed, since the elements da with  $a \in \mathbb{R}$  span  $\Omega^1_{\mathbb{R}}$ , and since we require that  $h_D(da) = D(a)$ , there exists a most one such map  $h_D$ . Conversely, let  $h'_D \colon F \to V$  be the unique  $\mathbb{R}$ -linear map that to (a) associates D(a). Since D is a derivation,  $h'_D$  maps the subspace  $R \subset F$  to zero, and hence, induces the desired  $\mathbb{R}$ -linear map  $h_D \colon \Omega^1_{\mathbb{R}} \to V$ .

The Leibniz rule implies that d(1) = 0, and since d is additive, we conclude that  $d(\mathbb{Z}) = 0$ . Let  $q \in \mathbb{Q}$  and write q = m/n. We have

$$0 = d(m) = d(n \cdot \frac{m}{n}) = \frac{m}{n}d(n) + nd(\frac{m}{n}) = nd(\frac{m}{n}),$$

and since  $n \neq 0$ , we find that  $d(\mathbb{Q}) = 0$ .

LEMMA 4.13. There is a surjective  $\mathbb{R}$ -linear map

$$C \colon \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z}) \to \Omega^1_{\mathbb{R}}$$

defined by the formula

$$C(\ell \otimes (\theta + \pi \mathbb{Z})) = \begin{cases} (\ell/\cos\theta)d(\sin\theta) & \text{if } \cos\theta \neq 0\\ 0 & \text{if } \cos\theta = 0. \end{cases}$$

**PROOF.** We first show that C is well-defined. If  $\theta_1 - \theta_2 \in \pi\mathbb{Z}$ , then

 $(\ell/\cos\theta_1)d(\sin\theta_1) = (\ell/\cos\theta_2)d(\sin\theta_2)$ 

as required. It is also clear that

$$C((\ell_1 + \ell_2) \otimes (\theta + \pi \mathbb{Z})) = C(\ell_1 \otimes (\theta + \pi \mathbb{Z})) + C(\ell_2 \otimes (\theta + \pi \mathbb{Z})),$$

so it remains to show that

$$C(\ell \otimes (\theta_1 + \theta_2 + \pi \mathbb{Z})) = C(\ell \otimes (\theta_1 + \pi \mathbb{Z})) + C(\ell \otimes (\theta_2 + \pi \mathbb{Z})).$$

To this end, we recall that

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$
$$\sin(\theta_1 + \theta_2) = \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2.$$

Moreover, since  $\cos^2 \theta + \sin^2 \theta = 1$ , we have

$$d(\cos\theta) = -\tan\theta d(\sin\theta),$$

and therefore,

$$d\sin(\theta_1 + \theta_2) = d(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)$$
  
=  $\sin\theta_2 d(\cos\theta_1) + \cos\theta_1 d(\sin\theta_2)$   
+  $\cos\theta_2 d(\sin\theta_1) + \sin\theta_1 d(\cos\theta_2)$   
=  $\frac{\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2}{\cos\theta_1} d(\sin\theta_1)$   
+  $\frac{\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2}{\cos\theta_2} d(\sin\theta_2)$   
=  $\frac{\cos(\theta_1 + \theta_2)}{\cos\theta_1} d(\sin\theta_1) + \frac{\cos(\theta_1 + \theta_2)}{\cos\theta_2} d(\sin\theta_2)$ 

The desired identity follows by multiplying both sides by  $\ell / \cos(\theta_1 + \theta_2)$ .

We next show that C is surjective. Since C is  $\mathbb{R}$ -linear, it suffices to show that for all  $a \in \mathbb{R}$ ,  $da \in \operatorname{im}(C)$ . If  $a \in \mathbb{Z}$ , then  $da = 0 \in \operatorname{im}(C)$ . And if  $a \notin \mathbb{Z}$ , then  $a \in (n, n + 1)$  for a unique  $n \in \mathbb{Z}$ . If n = 0, then we can write  $a = \sin \theta$ , so

$$da = C(1 \otimes (\theta + \pi \mathbb{Z}) \in \operatorname{im}(C).$$

If  $n \neq 0$ , then da = nd(a/n) which is in im(C) because  $d(a/n) \in im(C)$ .

LEMMA 4.14. The composition of the group homomorphisms

$$P(\mathbb{R}^3) \xrightarrow{D} \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z}) \xrightarrow{C} \Omega^1_{\mathbb{R}}$$

is equal to the zero map.

PROOF. For all  $a \in \mathbb{R}$ , we have  $d(a^2) = 2ada$ , so if  $a \neq 0$ , we find

$$\frac{d(a^2)}{a^2} = 2\frac{da}{a}.$$

We let  $\alpha$ ,  $\beta$ , and  $\alpha * \beta$  be as in the statement of Proposition 4.11 and write  $a = \sin^2 \alpha$  and  $b = \sin^2 \beta$  such that  $ab = \sin^2(\alpha * \beta)$ . We now have

$$C(\cot \alpha \otimes \alpha + \cot \beta \otimes \beta - \cot(\alpha * \beta) \otimes (\alpha * \beta))$$
  
=  $\frac{d(\sin \alpha)}{\sin \alpha} + \frac{d(\sin \beta)}{\beta} - \frac{d(\sin(\alpha * \beta))}{\sin(\alpha * \beta)}$   
=  $\frac{1}{2} \Big( \frac{d(\sin^2 \alpha)}{\sin^2 \alpha} + \frac{d(\sin^2 \beta)}{\sin^2 \beta} - \frac{d(\sin^2(\alpha * \beta))}{\sin^2(\alpha * \beta)} \Big)$   
=  $\frac{1}{2} \Big( \frac{da}{a} + \frac{db}{b} - \frac{d(ab)}{ab} \Big)$   
=  $\frac{1}{2ab} (bda + adb - d(ab)) = 0$ 

which proves the lemma.

THEOREM 4.15 (Jessen [8]). The sequence

$$P(\mathbb{R}^3) \xrightarrow{D} \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z}) \xrightarrow{C} \Omega^1_{\mathbb{R}}$$

is exact.

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The theorem amounts to the statement that the image of D, which we described in Proposition 4.11, is equal to the kernel of C. We proved in Lemma 4.14 that  $\operatorname{im}(D) \subset \operatorname{ker}(C)$ . In the next section, we outline a proof that equality holds. The theorems of Sydler and Jessen together with the ancient calculation of the scissor's congruence group  $P(\mathbb{R}^2)$  combine to give the following result, which constitutes the calculation of the scissor's congruence group  $P(\mathbb{R}^3)$ .

THEOREM 4.16 (Dehn-Sydler-Jessen). The following sequence of abelian groups and group homomorphism is exact.

$$0 \longrightarrow P(\mathbb{R}^2) \xrightarrow{E} P(\mathbb{R}^3) \xrightarrow{D} \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z}) \xrightarrow{C} \Omega^1_{\mathbb{R}} \longrightarrow 0$$

Moreover, the composition  $\operatorname{vol} \circ E \colon P(\mathbb{R}^2) \to \mathbb{R}$  is an isomorphism.

REMARK 4.17. Let  $S = \{a_i \mid i \in I\} \subset \mathbb{R}$  be a subset. Then the following are equivalent; for a proof see e.g. [11, Theorem 26.5].

(1) The subset  $\{a_i \mid i \in I\}$  is a transcendence basis of  $\mathbb{R}$  over  $\mathbb{Q}$ .

(2) The subset  $\{da_i \mid i \in I\}$  is a basis of the real vector space  $\Omega^1_{\mathbb{R}}$ .

The cardinality of a subset with these properties is equal to that of the real numbers. One may also show that the dimension of the subspace  $\operatorname{im}(D) \subset \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$  is equal to the cardinality of the real numbers.

## 5. Group homology

In this final section, we outline the homological proof of the Dehn-Sydler-Jessen theorem following Dupont [4] and Dupont-Sah [6].

We recall that  $P(\mathbb{R}^n)$  is defined to be the quotient abelian group

$$P(\mathbb{R}^n) = F(\mathbb{R}^n) / R(\mathbb{R}^n)$$

where  $R(\mathbb{R}^n) \subset F(\mathbb{R}^n)$  is the subgroup generated by the elements listed in (i)–(ii) of Definition 3.2. We now let  $R'(\mathbb{R}^n) \subset R(\mathbb{R}^n) \subset F(\mathbb{R}^n)$  be the subgroup generated by the elements listed in (i) only and define

$$P'(\mathbb{R}^n) = F(\mathbb{R}^n) / R'(\mathbb{R}^n).$$

We write  $[P]' = \langle P \rangle + R'(\mathbb{R}^n)$  for the class of the polytope  $P \subset \mathbb{R}^n$ . Suppose now that  $f \colon \mathbb{R}^n \to \mathbb{R}^n$  is an isometry. Since  $f(R'(\mathbb{R}^n)) = R'(\mathbb{R}^n)$ , there is a well-defined group automorphism

$$\rho(f)\colon P'(\mathbb{R}^n)\to P'(\mathbb{R}^n)$$

that takes the class [P]' to the class [f(P)]'. If  $f, g: \mathbb{R}^n \to \mathbb{R}^n$  are two isometries, then  $\rho(f \circ g) = \rho(f) \circ \rho(g)$ . It follows that there is a group homomorphism

$$\rho \colon E(n) \to \operatorname{Aut}(P'(\mathbb{R}^n))$$

that to the isometry f assigns the group automorphism  $\rho(f)$ . We say that the map  $\rho$  defines a structure of left E(n)-module on  $P'(\mathbb{R}^n)$ . In general, if the group homomorphism  $\rho: G \to \operatorname{Aut}(M)$  defines a structure of left G-module on the abelian group M, then the abelian group of *coinvariant* is defined to be the quotient

$$H_0(G, M) = M/N$$

of M by the subgroup  $N \subset M$  generated by the elements  $x - \rho(g)(x)$  with  $x \in M$ and  $g \in G$ . In the case at hand, we have the canonical isomorphism

$$H_0(E(n), P'(\mathbb{R}^n)) \xrightarrow{\sim} P(\mathbb{R}^n)$$

that takes the class of [P]' to [P]. We recall from Remark 1.3 that the subgroup of translations  $T(n) \subset E(n)$  is normal and that the quotient E(n)/T(n) is canonically identified with the subgroup  $O(n) \subset E(n)$  of orthogonal transformations. Since  $T(n) \subset E(n)$  is normal, the structure of left E(n)-module on  $P'(\mathbb{R}^n)$  induces a structure of left O(n)-module on the group of coinvariants  $H_0(T(n), P'(\mathbb{R}^n))$ , and we have canonical isomorphisms of abelian groups

$$H_0(O(n), H_0(T(n), P(\mathbb{R}^n)')) \xrightarrow{\sim} H_0(E(n), P'(\mathbb{R}^n)) \xrightarrow{\sim} P(\mathbb{R}^n).$$

To understand the left E(n)-module  $P'(\mathbb{R}^n)$ , we consider the chain complex  $\tilde{C}_*(\mathbb{R}^n)$ concentrated in degrees  $k \ge -1$ , where  $\tilde{C}_k(\mathbb{R}^n)$  is the free abelian group generated by the set consisting of all (not necessarily proper) k-simplices  $\sigma = (a_0, \ldots, a_k)$  in  $\mathbb{R}^n$ , and where the differential

$$d: \tilde{C}_k(\mathbb{R}^n) \to \tilde{C}_{k-1}(\mathbb{R}^n)$$

is the group homomorphism defined by

$$d(a_0, \dots, a_k) = \sum_{i=0}^k (-1)^i (a_0, \dots, \hat{a}_i, \dots, a_k).$$

In particular,  $\tilde{C}_{-1}(\mathbb{R}^n)$  is the free abelian group with the single generator ( ). The homology groups of  $\tilde{C}_*(\mathbb{R}^n)$  are zero in all degrees. We now define

$$\operatorname{Fil}_{n-1} \tilde{C}_k(\mathbb{R}^n) \subset \tilde{C}_k(\mathbb{R}^n)$$

to be the subgroup generated by the k-simplices  $\sigma = (a_0, \ldots, a_k)$  whose associated geometric simplex  $|\sigma|$  is contained in an affine hyperplane of  $\mathbb{R}^n$ . We remark that for  $k \leq n-1$ ,  $\operatorname{Fil}_{n-1} \tilde{C}_k(\mathbb{R}^n) = \tilde{C}_k(\mathbb{R}^n)$ . In general,  $\operatorname{Fil}_{n-1} \tilde{C}_*(\mathbb{R}^n) \subset \tilde{C}_*(\mathbb{R}^n)$  is a sub-chain complex, and we define  $\operatorname{gr}_n \tilde{C}_*(\mathbb{R}^n)$  to be the quotient chain complex such that we have the following short exact sequence of chain complexes.

$$0 \longrightarrow \operatorname{Fil}_{n-1} \tilde{C}_*(\mathbb{R}^n) \xrightarrow{i} \tilde{C}_*(\mathbb{R}^n) \xrightarrow{p} \operatorname{gr}_n \tilde{C}_*(\mathbb{R}^n) \longrightarrow 0$$

We remark that  $\operatorname{gr}_n \tilde{C}_n(\mathbb{R}^n)$  is a free abelian group and that a basis is given by the image by p of the subset of  $\tilde{C}_n(\mathbb{R}^n)$  that consists of all proper *n*-simplices. If  $\sigma = (a_0, \ldots, a_n)$  is a proper *n*-simplex and if  $a_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{R}^n$ , then the matrix  $(a_{ij} - a_{0j})_{1 \leq i,j \leq n}$  is invertible, and we define

$$\epsilon(\sigma) = \frac{\det(a_{ij} - a_{0j})}{|\det(a_{ij} - a_{0j})|} \in \{-1, +1\}$$

and call it the orientation of  $\sigma$ . The following figure illustrates the definition.



The following result was proved by Dupont [4, Theorem 2.3]. For a proof by purely algebraic means, we refer to Morelli [12, Proposition 1].

PROPOSITION 5.1. The group homomorphism

$$\varphi \colon H_n(\operatorname{gr}_n C_*(\mathbb{R}^n)) \to P'(\mathbb{R}^n)$$

that to the class of  $\sigma$  assigns  $\epsilon(\sigma) [|\sigma|]'$  is well-defined and an isomorphism.

The isometry  $f: \mathbb{R}^n \to \mathbb{R}^n$  induces an automorphism of abelian groups

$$\rho_k(f) \colon H_k(\operatorname{gr}_n \tilde{C}_*(\mathbb{R}^n)) \to H_k(\operatorname{gr}_n \tilde{C}_*(\mathbb{R}^n))$$

and by functoriality this defines a group homomorphism

$$\rho_k \colon E(n) \to \operatorname{Aut}(H_k(\operatorname{gr}_n C_*(\mathbb{R}^n))).$$

We define the canonical left E(n)-module structure on  $H_k(\operatorname{gr}_n \tilde{C}_*(\mathbb{R}^n))$  to be the left E(n)-module structure defined by the group homomorphism  $\rho_k$ . The isomorphism  $\varphi$  in Proposition 5.1, however, is not an isomorphism of left E(n)-modules. Now, in general, if  $\rho: E(n) \to \operatorname{Aut}(M)$  is any left E(n)-module, we define the associated twisted E(n)-module  $\rho^t: E(n) \to \operatorname{Aut}(M)$  by

$$\rho^t(f)(x) = \det(\bar{f})\rho(f)(x),$$

where  $f = t_{f(0)} \circ \overline{f}$  with  $\overline{f} \in O(n)$ . We write  $M^t$  for the twisted left E(n)-module associated with the left E(n)-module M.

ADDENDUM 5.2. There is an isomorphism of left E(n)-modules

$$\varphi \colon H_n(\operatorname{gr}_n \tilde{C}_*(\mathbb{R}^n))^t \to P'(\mathbb{R}^n)$$

that to the homology class of  $\sigma$  assigns  $\epsilon(\sigma)[|\sigma|]'$ .

PROOF. We let  $f \in E(n)$ , let  $\sigma = (a_0, \ldots, a_n)$  be a proper *n*-simplex in  $\mathbb{R}^n$ , and define  $f(\sigma) = (f(a_0), \ldots, f(a_n))$ . It is clear that  $|f(\sigma)| = f(|\sigma|)$  and it is immediate from the definition of orientation that  $\epsilon(f(\sigma)) = \det(\bar{f})\epsilon(\sigma)$ . Hence,

$$\varphi(\rho_n^t(f)(\sigma)) = \varphi(\det(f)(\rho_n(f)(\sigma))) = \det(f)\varphi(\rho_n(f)(\sigma))$$
$$= \det(\bar{f})\epsilon(f(\sigma))[|f(\sigma)|]' = \epsilon(\sigma)[f(|\sigma|)]'$$
$$= \rho(f)(\epsilon(\sigma)[|\sigma|]) = \rho(f)(\varphi(\sigma))$$

which shows that  $\varphi$  is a map of left E(n)-modules. Proposition 5.1 shows that it is an isomorphism.

We recall from Lemma 4.8 that there is a left action of the multiplicative group  $\mathbb{R}^*$  of non-zero real numbers on the scissor's congruence group  $P(\mathbb{R}^n)$ . There is also a left action of  $\mathbb{R}^*$  on the group  $P'(\mathbb{R}^n)$  defined in an entirely analogous manner. The action by  $\lambda \in \mathbb{R}^*$  is a map of left E(n)-modules  $\mu_{\lambda} \colon P'(\mathbb{R}^n) \to P'(\mathbb{R}^n)$ . Similarly, the scalar multiplication by  $\lambda \in \mathbb{R}^*$  defines map  $\mu_{\lambda} \colon \mathbb{R}^n \to \mathbb{R}^n$  which, in turn, induces a map of left E(n)-modules  $\mu_{\lambda} \colon H_n(\operatorname{gr}_n \tilde{C}_*(\mathbb{R}^n)) \to H_n(\operatorname{gr}_n \tilde{C}_*(\mathbb{R}^n))$ .

ADDENDUM 5.3. The isomorphism

$$\varphi \colon H_n(\operatorname{gr}_n \tilde{C}(\mathbb{R}^n))^t \xrightarrow{\sim} P'(\mathbb{R}^n)$$

is an isomorphism of left  $\mathbb{R}^*$ -modules.

For every integer  $q \ge 0$ , we introduce a chain complex of real vector spaces

$$C_*(\mathscr{T}(\mathbb{R}^n), \Lambda^q_{\mathbb{O}}(\mathfrak{g}))$$

that is concentrated in degrees  $k \ge -1$ . If  $k \ge 0$ , then we define

$$\tilde{C}_k(\mathscr{T}(\mathbb{R}^n), \Lambda^q_{\mathbb{Q}}(\mathfrak{g})) = \bigoplus_{U_0 \subset \cdots \subset U_k} \Lambda^q_{\mathbb{Q}}(U_0),$$

where the sum ranges over flags  $U_0 \subset \cdots \subset U_k$  of subspaces of  $\mathbb{R}^n$  of dimension at most n-1, and where  $\Lambda^q_{\mathbb{Q}}(V)$  is the *q*th exterior product of *V* considered as a rational vector space; and if k = -1, then we define

$$\hat{C}_{-1}(\mathscr{T}(\mathbb{R}^n), \Lambda^q_{\mathbb{O}}(\mathfrak{g})) = \Lambda^q_{\mathbb{O}}(\mathbb{R}^n).$$

If  $k \ge 1$  and if  $0 < i \le k$  (resp. if  $k \ge 1$  and i = 0, resp. if k = 0), then we let

$$d_i \colon \tilde{C}_k(\mathscr{T}(\mathbb{R}^n), \Lambda^q_{\mathbb{Q}}(\mathfrak{g})) \to \tilde{C}_{k-1}(\mathscr{T}(\mathbb{R}^n), \Lambda^q_{\mathbb{Q}}(\mathfrak{g}))$$

be the  $\mathbb{R}$ -linear map that maps the summand indexed by  $U_0 \subset \cdots \subset U_k$  to that indexed by  $U_0 \subset \cdots \subset \hat{U}_i \subset \cdots \subset U_k$  by the identity map  $\Lambda^q_{\mathbb{Q}}(U_0) \to \Lambda^q_{\mathbb{Q}}(U_0)$ (resp. by the map  $\Lambda^q_{\mathbb{Q}}(U_0) \to \Lambda^q_{\mathbb{Q}}(U_1)$  induced by the inclusion  $U_0 \to U_1$ ; resp. by the map  $\Lambda^q_{\mathbb{Q}}(U_0) \to \Lambda^q_{\mathbb{Q}}(\mathbb{R}^n)$  induced by the inclusion  $U_0 \to \mathbb{R}^n$ ). Now the differential of the chain complex  $\tilde{C}_*(\mathscr{T}(\mathbb{R}^n), \Lambda^q_{\mathbb{Q}}(\mathfrak{g}))$  is the  $\mathbb{R}$ -linear map defined by

$$d = \sum_{i=0}^{k} (-1)d_i \colon \tilde{C}_k(\mathscr{T}(\mathbb{R}^n), \Lambda^q_{\mathbb{Q}}(\mathfrak{g})) \to \tilde{C}_{k-1}(\mathscr{T}(\mathbb{R}^n), \Lambda^q_{\mathbb{Q}}(\mathfrak{g}))$$

The reader can easily verify that  $d \circ d$  is the zero homomorphism as required. The following result is [5, Proposition 3.11].

PROPOSITION 5.4 (Dupont). Let n be a positive integer.

- (1) The abelian group  $H_0(T(n), P'(\mathbb{R}^n))$  has a canonical  $\mathbb{R}$ -vector space structure such that for all  $\lambda \in \mathbb{R}^*$  and  $f \in O(n)$ , the maps  $\mu_{\lambda}$  and  $\rho(f)$  are  $\mathbb{R}$ -linear.
- (2) For every  $\lambda \in \mathbb{R}^*$ ,  $H_0(T(n), P'(\mathbb{R}^n))$  decomposes as a direct sum

$$H_0(T(n), P'(\mathbb{R}^n)) = \bigoplus_{q=1}^n H_0(T(n), P'(\mathbb{R}^n))^{\mu_\lambda = \lambda^q}$$

of eigenspaces for the dilation map  $\mu_{\lambda}$ , and moreover, the decomposition is independent of  $\lambda \in \mathbb{R}^*$ .

(3) For every  $1 \leq q \leq n$ , there a canonical isomorphism of left O(n)-modules

$$H_{n-1-q}(\tilde{C}_*\mathscr{T}(\mathbb{R}^n),\Lambda^q_{\mathbb{Q}}(\mathfrak{g})))^t \xrightarrow{\sim} H_0(T(n),P'(\mathbb{R}^n))^{\mu_{\lambda}=\lambda^q}$$

and the isomorphism is  $\mathbb{R}$ -linear.

PROOF. The proof uses the method introduced by Grothendieck of considering the two spectral sequences associated to a double complex; see e.g. [15, Section 5.6]. If  $V \subset \mathbb{R}^n$  is a real subspace, then we denote by T(V) the group of translations of V. For instance, we have  $T(\mathbb{R}^n) = T(n)$ . We now define a double complex of abelian groups  $A_{*,*}(\mathbb{R}^n)$  concentrated in bidegrees (p,q) with  $p,q \ge -1$ . It has

$$A_{p,q}(\mathbb{R}^n) = \begin{cases} H_0(T(\mathbb{R}^n), \tilde{C}_q(\mathbb{R}^n)) & \text{if } p = -1 \\ \bigoplus_{U_0 \subset \dots \subset U_p} H_0(T(U_0), \tilde{C}_q(U_0)) & \text{if } p \ge 0, \end{cases}$$

where the sum ranges over flags  $U_0 \subset \cdots \subset U_p$  of subspaces of  $\mathbb{R}^n$  of dimension at most n-1, and where the definition of the two differentials

$$d': A_{p,q}(\mathbb{R}^n) \to A_{p-1,q}(\mathbb{R}^n)$$
$$d'': A_{p,q}(\mathbb{R}^n) \to A_{p,q-1}(\mathbb{R}^n)$$

is similar to that of the differentials in  $\tilde{C}_*(\mathscr{T}(\mathbb{R}^n), \Lambda_{\mathbb{Q}}(\mathfrak{g}))$  and  $\tilde{C}_*(\mathbb{R}^n)$ , respectively, except that in the case of d'', we multiply by the factor  $(-1)^p$ . Associated with this double complex, we have two spectral sequences

$${}^{I}E^{2}_{s,t} = H_{s}(H_{t}(A_{*,*}(\mathbb{R}^{n}), d'), d'') \Rightarrow H_{s+t}(\operatorname{Tot}(A_{*,*}(\mathbb{R}^{n})))$$
$${}^{II}E^{2}_{s,t} = H_{s}(H_{t}(A_{*,*}(\mathbb{R}^{n}), d''), d') \Rightarrow H_{s+t}(\operatorname{Tot}(A_{*,*}(\mathbb{R}^{n})))$$

both of which converge to the homology of the associated total complex. In the first spectral sequence, we have

$${}^{I}\!E_{s,t}^{2} = \begin{cases} H_{t}(H_{0}(T(n), \operatorname{gr}_{n}\tilde{C}_{*}(\mathbb{R}^{n}))) & \text{if } s = -1\\ 0 & \text{if } s \ge 0, \end{cases}$$

so the spectral sequence collapses and shows that the edge homomorphism

$$H_t(H_0(T(n), \operatorname{gr}_n C_*(\mathbb{R}^n))) \to H_{t-1}(\operatorname{Tot}(A_{*,*}(\mathbb{R}^n)))$$

is an isomorphism. We also remark that the canonical map

$$H_0(T(n), H_t(\operatorname{gr}_n C_*(\mathbb{R}^n))) \to H_t(H_0(T(n), \operatorname{gr}_n C_*(\mathbb{R}^n)))$$

is also an isomorphism for  $t \leq n$  and that both groups are zero for t < n. Indeed, the complex  $\operatorname{gr}_n \tilde{C}_*(\mathbb{R}^n)$  is concentrated in degrees  $k \geq n$ . Therefore, the homology group  $H_k(\text{Tot}(A_{*,*}(\mathbb{R}^n)))$  of the total complex is zero for k < n-1 and is canonically isomorphic to  $H_0(T(n), P'(\mathbb{R}^n))^t$  for k = n-1. The isomorphism is compatible with the left actions by the groups O(n) and  $\mathbb{R}^*$ .

In the second spectral sequence  $E_{s,t}^r = {}^{II}E_{s,t}^r$ , we have

$$E^{1}_{*,t} = \begin{cases} \tilde{C}_{*}(\mathscr{T}(\mathbb{R}^{n}), \Lambda^{t}_{\mathbb{Q}}(\mathfrak{g})) & \text{if } t > 0\\ 0 & \text{if } t \leqslant 0. \end{cases}$$

and hence,

$$E_{*,t}^{2} = \begin{cases} H_{s}(\tilde{C}_{*}(\mathscr{T}(\mathbb{R}^{n}), \Lambda_{\mathbb{Q}}^{t}(\mathfrak{g}))) & \text{if } t > 0\\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Indeed, the complex  $\tilde{C}_*(V)$  is a free resolution of the trivial T(V)-module  $\mathbb{Z}$ . Hence, the homology group  $H_q(H_0(T(V), \tilde{C}_*(V)))$  is equal to the qth group homology group of V with coefficients in  $\mathbb{Z}$  if q > 0 and is zero if  $q \leq 0$ . But for every torsion free abelian group V, the qth group homology group of V with coefficients in  $\mathbb{Z}$ is canonically isomorphic to  $\Lambda^q_{\mathbb{Z}}(V)$ ; see e.g. [2, Theorem 6.4]. Now, for all  $r \geq 1$ , the groups  $E_{s,t}^r$  are real vector spaces. Indeed, this is true for r = 1, and since the differentials  $d^r : E_{s,t}^r \to E_{s-r,t+r-1}^r$  are  $\mathbb{R}$ -linear, the same holds for all  $r \geq 1$ . Now, the map  $\mu_{\lambda} : \mathbb{R}^n \to \mathbb{R}^n$  given by scalar multiplication by  $\lambda \in \mathbb{R}^*$  induces a map of spectral sequences  $\mu_{\lambda} : E_{s,t}^r \to E_{s,t}^r$  which is equal to multiplication by  $\lambda^t$ . It follows that  $d^r$  is zero for all  $r \geq 2$ . Indeed, if  $x \in E_{s,t}^r$ , then

$$\lambda^{t+r-1}d^r(x) = \mu_{\lambda}(d^r(x)) = d^r(\mu_{\lambda}(x)) = d^r(\lambda^t x) = \lambda^t d^r(x)$$

which shows that  $d^{r}(x) = 0$ . Finally, the spectral sequence gives a filtration

$$0 = F_{-2}H_m \subset F_{-1}H_m \subset \cdots \subset F_mH_m = H_m = H_m(\operatorname{Tot}(A_{*,*}(\mathbb{R}^n)))$$

and a canonical identification  $\operatorname{gr}_{s+t} = E_{s,t}^{\infty} = E_{s,t}^2$ , and both are compatible with the actions by the groups O(n) and  $\mathbb{R}^*$ . Now, for  $1 \leq q \leq n$ , we find

$$(H_{n-1})^{\mu_{\lambda}=\lambda^{q}} = (F_{n-1-q}H_{n-1})^{\mu_{\lambda}=\lambda^{q}} = (E_{n-1-q,q}^{2})^{\mu_{\lambda}=\lambda^{q}} = E_{n-1-q,q}^{2}$$

as left O(n)-modules. This completes the proof.

There is a map of chain complexes

$$\phi \colon \tilde{C}_*(\mathscr{T}(\mathbb{R}^n), \Lambda^q_{\mathbb{Q}}(\mathfrak{g})) \to \tilde{C}_*(\mathscr{T}(\mathbb{R}^n), \Lambda^q_{\mathbb{R}}(\mathfrak{g}))$$

compatible with the left actions of the groups O(n) and  $\mathbb{R}^*$ . Here the definition of the complex on the right-hand side is analogous to that of the complex on the left-hand side except that  $\Lambda^q_{\mathbb{Q}}(V)$  is replaced by  $\Lambda^q_{\mathbb{R}}(V)$ , and the chain map  $\phi$  is induced by the canonical projection  $\Lambda^q_{\mathbb{Q}}(V) \to \Lambda^q_{\mathbb{R}}(V)$ . If V is a finite dimensional real vector space with basis  $\{e_1, \ldots, e_k\}$ , then  $\Lambda^q_{\mathbb{R}}(V)$  is a finite dimensional real vector space with basis  $\{e_{i_1} \land \cdots \land e_{i_q} \mid 1 \leq i_1 < \cdots < i_q \leq k\}$ . By contrast, then the real vector space  $\Lambda^q_{\mathbb{Q}}(V)$  is uncountably infinite dimensional for all  $q \geq 2$ , unless V is the zero space. So the target of the chain map  $\phi$  is a much more manageable complex that is the domain. The injectivity part of the following result was proved by Jessen and Thorup [9, Theorem 2] and by Sah [13], and the surjectively part was proved by Dupont [4, Theorem 3.12] with the vanishing of the target homology group for p + q < n - 1 based on earlier work of Lusztig [10, §1]. We also refer the reader to [12, Theorem 2]. THEOREM 5.5. Let n and q be positive integers. The map of homology groups

$$\phi_* \colon H_p(C_*(\mathscr{T}(\mathbb{R}^n), \Lambda^q_{\mathbb{O}}(\mathfrak{g}))) \to H_p(C_*(\mathscr{T}(\mathbb{R}^n), \Lambda^q_{\mathbb{R}}(\mathfrak{g})))$$

induced by the chain map  $\phi$  is an isomorphism, if  $p + q \leq n - 1$ , and the common group is zero, if p + q < n - 1.

Now, for every  $1 \leq q \leq n$ , we define the real vector space

$$D^{q}(\mathbb{R}^{n}) = H_{n-1-q}(\tilde{C}_{*}(\mathscr{T}(\mathbb{R}^{n}), \Lambda^{q}_{\mathbb{R}}(\mathfrak{g}))).$$

It is a left O(n)-module, by functoriality, and the action by  $f \in O(n)$  is an  $\mathbb{R}$ -linear map. By Proposition 5.4 and Theorem 5.5, we have a canonical isomorphism

$$\bigoplus_{q=1}^{n} H_0(O(n), D^q(\mathbb{R}^n)^t) \xrightarrow{\sim} P(\mathbb{R}^n)$$

of abelian groups. We now show that roughly half of the summands vanish.

LEMMA 5.6. If 
$$n + q$$
 is odd, then  $H_0(O(n), D^q(\mathbb{R}^n)^t)$  is zero

PROOF. The canonical inclusion of the subgroup  $\{\pm id\}$  into O(n) induces a map of homology groups

$$H_0(\{\pm \operatorname{id}\}, D^q(\mathbb{R}^n)^t) \to H_0(O(n), D^q(\mathbb{R}^n)^t)$$

which clearly is surjective. Now, the map

$$\rho(-\mathrm{id})\colon D^q(\mathbb{R}^n)\to D^q(\mathbb{R}^n)$$

is equal to multiplication by  $(-1)^q$ , and hence, the map

$$\det(-\operatorname{id})\rho(-\operatorname{id})\colon D^q(\mathbb{R}^n)\to D^q(\mathbb{R}^n)$$

is equal to multiplication by  $(-1)^{n+q}$ . Therefore, we have

$$H_0(\{\pm \mathrm{id}\}, D^q(\mathbb{R}^n)^t) = D^q(\mathbb{R}^n)/(1 - (-1)^{n+q})D^q(\mathbb{R}^n).$$

If n+q is odd, then the right-hand side is equal to  $D^q(\mathbb{R}^n)/2D^q(\mathbb{R}^n)$ , which is zero as  $D^q(\mathbb{R}^n)$  is a real vector space.

We discuss one more general theorem, valid for all positive integers n. If V is a real vector space of dimension n and if  $1 \leq q \leq n$  is an integer, then we define the chain complex of real vector spaces  $\tilde{C}_*(\mathscr{T}(V), \Lambda^q_{\mathbb{R}}(\mathfrak{g}))$  in a manner entirely similar to the case  $V = \mathbb{R}^n$ .

LEMMA 5.7. Let V be a real vector space of dimension n and let  $1 \leq q \leq n$  be an integer. The homology group  $H_p(\tilde{C}_*(\mathscr{T}(V), \Lambda^q_{\mathbb{R}}(\mathfrak{g})))$  is zero unless p+q=n-1.

PROOF. For p + q < n - 1, the homology group vanishes by Theorem 5.5. Let

$$C^N_*(\mathscr{T}(V), \Lambda^q_{\mathbb{R}}(\mathfrak{g})) \to C_*(\mathscr{T}(V), \Lambda^q_{\mathbb{R}}(\mathfrak{g}))$$

be the chain map that, in degree  $k \ge 0$ , is given by the canonical inclusion

$$\bigoplus_{U_0 \subsetneq \cdots \subsetneq U_k} \Lambda^q_{\mathbb{R}}(U_0) \to \bigoplus_{U_0 \subset \cdots \subset U_k} \Lambda^q_{\mathbb{R}}(U_0)$$

of the summands indexed by all strict flags  $U_0 \subsetneq \cdots \subsetneq U_k$  of subspaces of V of dimension at most n-1, and that, in degree k = -1, is given by the identity map. It induces an isomorphism of all homology groups; see e.g. [15, Theorem 8.3.8]. Since the groups  $\tilde{C}_p^N(\mathscr{T}(V), \Lambda_{\mathbb{R}}^q(\mathfrak{g}))$  vanish for p+q > n-1, the same obviously is true for the homology groups.

If V is a real vector space and  $1 \leq q \leq n = \dim_{\mathbb{R}}(V)$  an integer, we write

$$D^{q}(V) = H_{n-1-q}(C_{*}(\mathscr{T}(V), \Lambda_{\mathbb{R}}^{q}(\mathfrak{g})))$$

for the unique non-zero homology group of the indicated chain complex.

THEOREM 5.8. For all positive integers n and all integers  $1 \leq q \leq n$ , there is a canonical exact sequence of left O(n)-modules

$$0 \longrightarrow D^{q}(\mathbb{R}^{n}) \longrightarrow \bigoplus_{U_{n-1}} D^{q}(U_{n-1}) \longrightarrow \bigoplus_{U_{n-2}} D^{q}(U_{n-2}) \longrightarrow \cdots$$
$$\cdots \longrightarrow \bigoplus_{U_{q+1}} D^{q}(U_{q+1}) \longrightarrow \bigoplus_{U_{q}} \Lambda^{q}_{\mathbb{R}}(U_{q}) \longrightarrow \Lambda^{q}_{\mathbb{R}}(\mathbb{R}^{n}) \longrightarrow 0$$

where the sum  $\bigoplus_{U_d} D^q(U_d)$  ranges over all subspace  $U_d \subset \mathbb{R}^n$  of dimension d. The left action of  $f \in O(n)$  on this sum takes the summand indexed by  $U_d \subset \mathbb{R}^n$  to the summand indexed by  $f(U_d) \subset \mathbb{R}^n$  by the map  $D^q(U_d) \to D^q(f(U_d))$  induced by  $f: U_d \to f(U_d)$ .

PROOF. We define a filtration

$$0 = F_{q-2}\tilde{C}_{*}^{N}(q) \subset F_{q-1}\tilde{C}_{*}^{N}(q) \subset \cdots \subset F_{n-2}\tilde{C}_{*}^{N}(q) \subset F_{n-1}\tilde{C}_{*}^{N}(q) = \tilde{C}_{*}^{N}(q)$$

of the normalized chain complex  $\tilde{C}^N_*(q) = \tilde{C}^N_*(\mathscr{T}(\mathbb{R}^n), \Lambda^q_{\mathbb{R}}(\mathbb{R}^n))$  from the proof of Lemma 5.7. If  $q \leq s < n$ , then for  $k \geq 0$ , we define

$$F_s \tilde{C}_k^N(q) = \bigoplus_{\substack{U_0 \subsetneq \cdots \subsetneq U_k \\ \dim(U_k) \leqslant s}} \Lambda_{\mathbb{R}}^q(U_0) \subset \tilde{C}_k^N(q);$$

and for k = -1, we define  $F_s \tilde{C}_{-1}^N(q) = \Lambda_{\mathbb{R}}^q(\mathbb{R}^n)$ . Finally, we define  $F_{q-1} \tilde{C}_*^N(q)$  to be  $\Lambda_{\mathbb{R}}^q(\mathbb{R}^n)$  considered as a complex concentrated in degree k = -1. The filtration gives rise to a spectral sequence

$$E_{s,t}^1 = H_{s+t}(\operatorname{gr}_s \tilde{C}^N_*(q)) \Rightarrow H_{s+t}(\tilde{C}^N_*(q)).$$

By definition, we have for  $q \leqslant s < n$  a canonical identification

$$\operatorname{gr}_s \tilde{C}_k^N(q) = \bigoplus_{U_s} \tilde{C}_{k-1}^N(\mathscr{T}(U_s), \Lambda_{\mathbb{R}}^q(\mathfrak{g})),$$

where the sum ranges over subspaces  $U_s \subset \mathbb{R}^n$  whose dimension is equal to s. Therefore, by Lemma 5.7, we find that for  $q \leq s < n$ ,

$$E_{s,-q}^1 = \bigoplus_{U_s} D^q(U_s)$$

and that  $E_{s,t}^1 = 0$  for all other values of t. Similarly, we find immediately from the definition that  $E_{q-1,-q}^1 = \Lambda_{\mathbb{R}}^q(\mathbb{R}^n)$  and that  $E_{q-1,t}^1 = 0$  for all other values of t. Hence, for degree reasons, we have  $E_{s,t}^\infty = E_{s,t}^2$ , and the common group can be non-zero only if  $q-1 \leq s < n$  and t = -q. Now, the spectral sequence converges to  $H_{s+t}(\tilde{C}^N(q))$  which, by Lemma 5.7, is non-zero only if s+t = n-1-q. Therefore, we conclude that  $E_{n-1,-q}^2 = D^q(\mathbb{R}^n)$  is the only non-zero group in the  $E^2$ -term. This completes the proof.

The following result gives a simplified expression for the group homology of the group O(n) with coefficients in the twisted left O(n)-modules associated with the terms in the sequence in Theorem 5.8.

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LEMMA 5.9. Let n be a positive integer and let  $1 \leq q \leq n$  and  $q \leq s < n$  be integers. There is a canonical isomorphism of graded abelian groups

$$H_*(O(s), D^q(\mathbb{R}^s)^t) \otimes H_*(O(n-s), \mathbb{Z}^t) \xrightarrow{\sim} H_*(O(n), (\bigoplus_{U_s} D^q(U_s))^t).$$

Moreover, the common groups vanish unless n - s is even.

**PROOF.** There is a canonical isomorphism of O(n)-modules

$$\mathbb{Z}[O(n)] \otimes_{\mathbb{Z}[O(s) \otimes O(n-s)]} D^q(\mathbb{R}^s)^t \otimes \mathbb{Z}^t \xrightarrow{\sim} (\bigoplus_{U_s} D^q(U_s))^t.$$

Therefore, by Shapiro's lemma, we have a canonical isomorphism

$$H_*(O(s) \times O(n-s), D^q(\mathbb{R}^s)^t \otimes \mathbb{Z}^t) \xrightarrow{\sim} H_*(O(n), (\bigoplus_{U_s} D^q(U_s))^t).$$

Finally, the Eilenberg-Zilber theorem gives a canonical isomorphism

$$H_*(O(s), D^q(\mathbb{R}^s)^t) \otimes H_*(O(n-s), \mathbb{Z}^t) \xrightarrow{\sim} H_*(O(s) \times O(n-s), D^q(\mathbb{R}^s)^t \otimes \mathbb{Z}^t),$$

since the homology groups  $H_*(O(s), D^q(\mathbb{R}^s)^t)$  are torsion-free abelian groups. This proves the first statement. To prove the second statement, one uses that multiplication by the the central element  $\gamma \in O(s) \times O(n-s)$  defined by

$$\gamma(x_1,\ldots,x_s,x_{s+1},\ldots,x_n)=(x_1,\ldots,x_s,-x_{s+1},\ldots,-x_n)$$

induces an automorphism of the homology group in question which is both equal to the identity map and to multiplication by  $(-1)^{n-s}$ .

Finally, we specialize to the case n = 3. We have the canonical isomorphism

 $H_0(O(3), D^1(\mathbb{R}^3)^t) \oplus H_0(O(3), D^3(\mathbb{R}^3)^t) \xrightarrow{\sim} P(\mathbb{R}^3).$ 

The second summand is easily identified as follows. There is an isomorphism of left O(3)-modules vol:  $\Lambda^3_{\mathbb{R}}(\mathbb{R}^3) \to \mathbb{R}^t$  defined by  $\operatorname{vol}(a_1 \wedge a_2 \wedge a_3) = \det(a_{ij})$ . It induces the isomorphism of left O(3)-modules vol:  $\Lambda^3_{\mathbb{R}}(\mathbb{R}^3)^t \to \mathbb{R}$ , and hence,

$$H_0(O(3), D^3(\mathbb{R}^3)^t) = H_0(O(3), \Lambda^3_{\mathbb{R}}(\mathbb{R}^3)^t) \xrightarrow{\text{vol}} H_0(O(3), \mathbb{R}) = \mathbb{R}.$$

To identify the first summand is (a lot) more difficult. We use the general theory to prove the following theorem of Dupont [4, Corollary 1.2].

THEOREM 5.10 (Dupont). There is a canonical exact sequence

 $0 \to H_2(SO(3), \mathbb{R}^3) \to H_0(O(3), D^1(\mathbb{R}^3)^t) \to \mathbb{R} \otimes (\mathbb{R}/2\pi\mathbb{Z}) \to H_1(SO(3), \mathbb{R}^3) \to 0.$ where  $\mathbb{R}^3$  is the standard left SO(3)-module.

PROOF. In general, for G a group and  $C_*$  an exact sequence of left G-modules, there is a spectral sequence with  $E_{s,t}^1 = H_t(G, C_s)$  that converges to zero. In the case where  $C_*$  has three (non-zero) terms, the spectral sequence gives rise to the familiar long exact homology sequence. We apply the spectral sequence in the case of the following Lusztig exact sequence of left O(3)-modules

$$(\mathbb{R}^3)^t \longleftarrow (\bigoplus U_1)^t \longleftarrow (\bigoplus D^1(U_2))^t \longleftarrow D^1(\mathbb{R}^3)^t$$

from Theorem 5.8 which has four (non-zero) terms. We wish to understand the group  $E_{3,0}^1 = H_0(O(3), (D^1(\mathbb{R}^3)^t))$ . First, by Lemma 5.9, we have

$$E_{1,*}^1 = H_*(O(3), (\bigoplus U_1)^t) = H_*(O(1), (\mathbb{R}^1)^t) \otimes H_*(O(2), \mathbb{Z}^t).$$

Here  $\mathbb{R}^1$  is the standard left O(1)-module, so  $(\mathbb{R}^1)^t$  is the trivial O(1)-module  $\mathbb{R}$ . Since O(1) is finite and  $\mathbb{R}$  divisible, we have  $H_*(O(1), (\mathbb{R}^1)^t) = \mathbb{R}$  concentrated in degree zero. In particular, it will suffice to evaluate the groups  $H_*(O(2), \mathbb{Z}^t)$ modulo the Serre subcategory of torsion abelian groups. To this end, we consider the Hochschild-Serre spectral sequence

$$E_{p,q}^{2} = H_{p}(\{\pm 1\}, H_{q}(SO(2), \mathbb{Z}^{t})) \Rightarrow H_{p+q}(O(2), \mathbb{Z}^{t}).$$

Since the group SO(2) is abelian and since  $\mathbb{Z}^t$  is trivial as a left SO(2)-module, there is a canonical map of  $\{\pm 1\}$ -modules

$$f: \Lambda^q_{\mathbb{Z}}(SO(2))^t \to H_q(SO(2), \mathbb{Z}^t)$$

which is an isomorphism modulo the Serre category of torsion abelian groups. The generator -1 of  $\{\pm 1\}$  acts on the domain of f by the opposite of the map induced by inversion in SO(2), and this map is equal to multiplication by  $(-1)^{q+1}$ . We now compose the map of coinvariants induced by f with the edge homomorphism of the spectral sequence to obtain the canonical map

$$H_0(\{\pm 1\}, \Lambda^q_{\mathbb{Z}}(SO(2))^t) \to H_q(O(2), \mathbb{Z}^t)$$

which is an isomorphism modulo the Serre subcategory of torsion abelian groups since  $\{\pm 1\}$  is finite. Moreover, from the above description of the left  $\{\pm 1\}$ -module structure of  $\Lambda^q_{\mathbb{Z}}(SO(2))^t$  we find that

$$H_0(\{\pm 1\}, \Lambda^q_{\mathbb{Z}}(SO(2))^t) = \begin{cases} \Lambda^q_{\mathbb{Z}}(SO(2))/2\Lambda^q_{\mathbb{Z}}(SO(2)) & \text{if } q \text{ is even} \\ \Lambda^q_{\mathbb{Z}}(SO(2)) & \text{if } q \text{ is odd.} \end{cases}$$

Finally, we have the group isomorphism  $g: \mathbb{R}/2\pi\mathbb{Z} \to SO(2)$  defined by

$$g(\theta + 2\pi\mathbb{Z}) = \begin{pmatrix} \cos\theta & -\sin\theta\\ \sin\theta & \cos\theta \end{pmatrix}.$$

We conclude that the groups  $E_{1,q}^1$  in the hyperhomology spectral sequence are given, up to canonical isomorphism, by

$$E_{1,q}^{1} = \begin{cases} 0 & \text{if } q \text{ is even} \\ \mathbb{R} \otimes \Lambda_{\mathbb{Z}}^{q}(\mathbb{R}/2\pi\mathbb{Z}) & \text{if } q \text{ is odd.} \end{cases}$$

Similarly, by Lemma 5.9, we have

$$E_{2,*}^1 = H_*(O(3), (\bigoplus D^1(U_2))^t) = 0.$$

It remains to identify the groups  $E_{0,q}^1$ . The group O(3) is equal to the product group  $SO(3) \times \{\pm id\}$ . Moreover, as a left SO(3)-module  $(\mathbb{R}^3)^t = \mathbb{R}^3$ , and as a left  $\{\pm id\}$ -module,  $(\mathbb{R}^3)^t$  is the trivial module  $\mathbb{R}^3$ . Therefore, by the Eilenberg-Zilber theorem, we have a canonical isomorphism

$$H_*(SO(3), \mathbb{R}^3) \otimes H_*(\{\pm \operatorname{id}\}, \mathbb{Z}) \xrightarrow{\sim} H_*(O(3), (\mathbb{R}^3)^t),$$

since the first tensor factor is torsion free as an abelian group. Modulo the Serre subcategory of torsion abelian groups, the second tensor factor is isomorphic to  $\mathbb{Z}$  concentrated in degree zero. Therefore, we conclude that the map

$$H_q(SO(3), \mathbb{R}^3) \to H_q(O(3), (\mathbb{R}^3)^t)$$

induced by the canonical inclusion  $SO(3) \to O(3)$  is an isomorphism for all q. This identifies  $E_{0,q}^1$  with  $H_q(SO(3), \mathbb{R}^3)$ . The  $E^1$ -term of the hyperhomology spectral sequence now has been identified, up to canonical isomorphism, as follows.

:	÷	:	
$H_2(SO(3),\mathbb{R}^3)$	0	0	$H_2(O(3), D^1(\mathbb{R}^3)^t)$
$H_1(SO(3),\mathbb{R}^3)$	$\mathbb{R}\otimes (\mathbb{R}/2\pi\mathbb{Z})$	0	$H_1(O(3), D^1(\mathbb{R}^3)^t)$
$H_0(SO(3), \mathbb{R}^3)$	0	0	$H_0(O(3), D^1(\mathbb{R}^3)^t)$

Since the hyperhomology spectral sequence converges to zero, we find that there is an exact sequence as stated.  $\hfill \Box$ 

REMARK 5.11. Let us define the map  $D': P(\mathbb{R}^3) \to \mathbb{R} \otimes (\mathbb{R}/2\pi\mathbb{Z})$  to be the composition of the canonical projection  $P(\mathbb{R}^3) \to H_0(O(3), D^1(\mathbb{R}^3)^t)$  onto the weight 1 eigenspace of the dilation maps followed by the middle map in the exact sequence of Theorem 5.10. Then the composition of the map D' with the canonical projection  $\mathbb{R} \otimes (\mathbb{R}/2\pi\mathbb{Z}) \to \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$  is equal to the Dehn invariant. It is possible to define the refinement D' of the Dehn invariant geometrically as follows. We define the triangulation  $\{\sigma_1, \ldots, \sigma_m\}$  of the polytope  $P \subset \mathbb{R}^3$  to be oriented if  $\epsilon(\sigma_i) = +1$  for all  $1 \leq i \leq m$ . Given an oriented simplex  $\sigma$  and an edge  $e \subset \sigma$ , we can define a dihedral angle  $\theta'(\sigma, e) \in \mathbb{R}/2\pi\mathbb{Z}$ , since a normal plane to the edge e has an induced orientation. The definition of  $D'(\sigma)$  and D'(P) now proceeds as before. This refinement of course is inconsequential since the canonical projection  $\mathbb{R} \otimes (\mathbb{R}/2\pi\mathbb{Z}) \to \mathbb{R} \otimes (\mathbb{R}/\pi\mathbb{Z})$  is an isomorphism.

In conclusion we state without proof the following theorem of Dupont and Sah which completes the homological proof of the theorem of Dehn-Sydler-Jessen.

THEOREM 5.12 (Dupont-Sah [6]). The following (1)–(2) hold:

- (1) There is a canonical isomorphism  $\Omega^1_{\mathbb{R}} \xrightarrow{\sim} H_1(SO(3), \mathbb{R}^3)$ .
- (2) The group  $H_2(SO(3), \mathbb{R}^3)$  is zero.

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