1 Algebraic $K$-theory and the $p$-adic $L$-function

We let $p$ be an odd prime number and define $F_p$ to be the $p$-completion of the homotopy fiber of the cyclotomic trace map
\[
\text{tr}: K(\mathbb{Z}) \to TC(\mathbb{Z}; p).
\]
Then we have a long-exact sequence of $p$-adic homotopy groups
\[
\cdots \to \pi_q(F_p) \to K_q(\mathbb{Z}; \mathbb{Z}_p) \xrightarrow{\text{tr}_q} TC_q(\mathbb{Z}; p, \mathbb{Z}_p) \to \pi_{q-1}(F_p) \to \cdots
\]
We will argue that the following (1)–(3) hold:

1. The value $L_p(\mathbb{Q}, \omega^{-2k}, 1+2k)$ is non-zero if and only if $\pi_{4k+1}(F_p)$ is zero.
2. If $L_p(\mathbb{Q}, \omega^{-2k}, 1+2k) \neq 0$, then $\pi_{4k}(F_p)$ and $\pi_{4k-1}(F_p)$ are finite and
   \[
   |L_p(\mathbb{Q}, \omega^{-2k}, 1+2k)|_p = \#\pi_{4k-1}(F_p)/\#\pi_{4k}(F_p).
   \]
3. There is an exact sequence
   \[
   0 \to \mathbb{Z}_p \to \pi_{4k-2}(F_p) \to K_{4k-2}(\mathbb{Z}; \mathbb{Z}_p) \to 0.
   \]
Here and below, we write $|a|_p = p^{-v_p(a)}$ for the $p$-adic absolute value of $a$.

We briefly recall the $p$-adic $L$-function. The Dirichlet series associated with the Dirichlet character $\chi$ is defined by
\[
L(\mathbb{Q}, \chi, s) = \sum_{n=1}^{\infty} \chi(n)n^{-s} = \prod_p (1 - \chi(p)p^{-s})^{-1}.
\]
The series converges for $\text{Re}(s) > 1$ and admits a unique extension to the meromorphic function $L(\mathbb{Q}, \chi, s)$ called the Dirichlet $L$-function. The values of this function at negative integers are rational numbers. We remark that $L(\mathbb{Q}, 1, s) = \zeta(s)$ is the Riemann zeta function. Let $\omega$ be the Teichmüller character. The Kubota-Leopoldt $p$-adic $L$-function associated with $\chi$ is the unique continuous function
\[
L_p(\mathbb{Q}, \chi, \cdot): \mathbb{Z}_p \setminus \{1\} \to \mathbb{Q}_p
\]
with the property that
\[
L_p(\mathbb{Q}, \chi, 1-i) = (1 - (\chi \omega^{-i})(p)p^{i-1})L(\mathbb{Q}, \chi \omega^{-i}, 1-i)
\]
for all integers $i \geq 1$. In this sense, the $p$-adic $L$-function interpolates the values of the Dirichlet $L$-function with the Euler factor corresponding to the prime $p$ removed. In particular, $L_p(\mathbb{Q}, \omega^i, 1-i) = (1 - p^{i-1})\zeta(1-i)$ for $i \geq 1$.

We wish to understand the kernel and cokernel of the cyclotomic trace map
\[
\text{tr}_q: K_q(\mathbb{Z}, \mathbb{Z}_p) \to TC_q(\mathbb{Z}; p, \mathbb{Z}_p).
\]
Let \( f : \mathbb{Z} \to \mathbb{Z}_p \) be the canonical ring homomorphism. Then, in the diagram

\[
\begin{array}{ccc}
K_q(\mathbb{Z}, \mathbb{Z}_p) & \xrightarrow{f_q} & K_q(\mathbb{Z}_p, \mathbb{Z}_p) \\
\downarrow \text{tr}_q & & \downarrow \text{tr}_q \\
TC_q(\mathbb{Z}; p, \mathbb{Z}_p) & \xrightarrow{f_q} & TC_q(\mathbb{Z}_p; p, \mathbb{Z}_p),
\end{array}
\]

the right-hand vertical map \( \text{tr}_q \) and lower horizontal map \( f_q \) are isomorphisms. Hence, the kernel and cokernel of the left-hand vertical map \( \text{tr}_q \) are isomorphic to the kernel and cokernel, respectively, of the upper horizontal map \( f_q \). The latter, in turn, are isomorphic to the kernel and cokernel of the map

\[
K_q(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p) \xrightarrow{f_q} K_q(\mathbb{Q}_p, \mathbb{Z}_p).
\]

Indeed, this follows immediately from the localization sequence in \( K \)-theory. To understand this map, we consider the induced map of the spectral sequences from motivic cohomology to algebraic \( K \)-theory,

\[
\begin{align*}
E^2_{s,t} &= H^{t-s}(\text{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(t)) \Rightarrow K_{s+t}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p) \\
E^2_{s,t} &= H^{t-s}(\text{Spec} \mathbb{Q}_p, \mathbb{Z}_p(t)) \Rightarrow K_{s+t}(\mathbb{Q}_p, \mathbb{Z}_p).
\end{align*}
\]

The motivic cohomology groups in the \( E^2 \)-terms are isomorphic to the corresponding étale cohomology groups, if \( s \leq t \), and are zero, otherwise. Indeed, this is the statement of the affirmed Beilinson-Lichtenbaum conjectures. We will not distinguish notationally between motivic cohomology groups and the associated étale cohomology groups.

Let \( i \neq 0, 1 \) be an integer which may be positive or negative. Then the étale cohomology groups \( H^q(\text{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i)) \) and \( H^q(\text{Spec} \mathbb{Q}_p, \mathbb{Z}_p(i)) \) are finitely generated \( \mathbb{Z}_p \)-modules which are non-zero for \( q = 1, 2 \) only [4, §1, Satz 5; §2, Satz 2; §3, Satz 4; §4, Lemmas 2 and 5]. Hence, for \( i > 1 \) and \( j = 1, 2 \), we have a commutative diagram

\[
\begin{array}{ccc}
K_{2i-j}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p) & \xrightarrow{f_{2i-j}} & K_{2i-j}(\mathbb{Q}_p, \mathbb{Z}_p) \\
\uparrow & & \uparrow \\
H^j(\text{Spec} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i)) & \xrightarrow{f_{2i-j}} & H^j(\text{Spec} \mathbb{Q}_p, \mathbb{Z}_p(i))
\end{array}
\]
where the vertical maps are isomorphisms. The lower horizontal map $f_{2i-j}$ appears in the Tate-Poitou duality sequence which takes the following form [4, §2, Satz 5].

\[
0 \rightarrow H^2(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \\
\rightarrow H^1(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i))^* f_{2i-1} \rightarrow H^1(\text{Spec } \mathbb{Q}_p, \mathbb{Z}_p(i))^* \\
\rightarrow H^1(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \\
\rightarrow H^2(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i))^* f_{2i-2} \rightarrow H^2(\text{Spec } \mathbb{Q}_p, \mathbb{Z}_p(i))^* \\
\rightarrow H^0(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \rightarrow 0
\]

Here the asterisk indicates the Pontryagin dual. Hence, the kernel and cokernel of the cyclotomic trace map is governed by the groups $H^q(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ with $i > 1$. The following conjecture was made by Lichtenbaum [2, Conjecture 9.1] for $i < 0$ and by Schneider [4, p. 192] in general.

**Conjecture.** The group $H^2(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ is zero for every integer $i \neq 0$.

The conjecture was proved for $i < 0$ by Soulé [5, Théorème 5] but it remains open for $i > 0$. The conjecture is equivalent to the statement that for every integer $i \neq 0$, the group $H^2(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(1-i))$ is finite. Moreover, by the discussion above, the conjecture for $i > 1$ is equivalent to the statement that the map

\[ f_{2i-1} : K_{2i-1}(\mathbb{Z}, \mathbb{Z}_p) \rightarrow K_{2i-1}(\mathbb{Z}_p, \mathbb{Z}_p) \]

is injective.

We next recall the precise relationship between the values of the $p$-adic $L$-function and the étale cohomology groups implied by the affirmed Main Conjecture of Iwazawa theory [3,6]. It was proved by Bayer and Neukirch [1, Theorem 6.1] that if $i \neq 1$ is an odd integer and if $L_p(\mathbb{Q}, \omega^{1-i}, i) \neq 0$, then the groups

\[ H^q(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(1-i)) \]

are finite for all $q \geq 0$, zero for $q \neq 1, 2$, and

\[
|L_p(\mathbb{Q}, \omega^{1-i}, i)|_p = \frac{\#H^1(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(1-i))}{\#H^2(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(1-i))} = \frac{\#H^0(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))}{\#H^1(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))}.
\]

We remark that in loc. cit., the equality above is stated for the so-called Iwazawa zeta function $\zeta_i(\mathbb{Q}, \omega^{1-i}, i)$ which is introduced in op. cit., §5 and which depends on a choice of topological generator $q \in \mathbb{Z}_p^*$. However, by the affirmed Main Conjecture of Iwazawa theory, it follows that

\[ L_p(\mathbb{Q}, \omega^{1-i}, i) = \zeta_i(\mathbb{Q}, \omega^{1-i}, i) \cdot u(q^{1-i} - 1) \]
where \( u(T) \in \mathbb{Z}_p[[T]]^* \). We also remark that for \( i < 0 \) odd, the \( p \)-adic \( L \)-function interpolates the values of the Riemann zeta function at negative integers. Hence, in this case, Lichtenbaum’s conjecture

\[
|\zeta(i)|_p = \frac{\#H^0(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))}{\#H^1(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))}
\]

ensues; see [1, Theorem 6.2].

Now let \( i > 1 \) be odd and assume that \( L_p(\mathbb{Q}, \omega^{1-i}, i) \) is non-zero. Then we conclude from the theorem of Bayer-Neukirch and from the Tate-Poitou sequence that the group \( H^2(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \) is zero, that

\[
f_{2i-1} : K_{2i-1}(\mathbb{Z}, \mathbb{Z}_p) \to K_{2i-1}(\mathbb{Z}_p, \mathbb{Z}_p)
\]

is injective, and that the cokernel is related to the \( p \)-adic \( L \)-function as follows.

\[
|L_p(\mathbb{Q}, \omega^{1-i}, i)|_p = \frac{\#H^0(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))}{\#H^1(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i))} = \frac{\#H^2(\text{Spec } \mathbb{Q}_p, \mathbb{Z}_p(i))}{\#\text{coker}(f_{2i-1}) \cdot \#H^2(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i))} = \frac{\#K_{2i-2}(\mathbb{Z}_p, \mathbb{Z}_p)}{\#\text{coker}(f_{2i-1}) \cdot \#K_{2i-2}(\mathbb{Z}, \mathbb{Z}_p)}.
\]

The group \( K_{2i-2}(\mathbb{Z}, \mathbb{Z}_p) \) is zero for all odd integers \( i > 1 \) if and only if the Kummer-Vandiver conjecture holds for the prime number \( p \). The group \( K_{2i-2}(\mathbb{Z}_p, \mathbb{Z}_p) \) is finite cyclic of order \( p^v_p(i-1)+1 \) if \( p-1 \) divides \( i-1 \), and is zero otherwise. Therefore, if the Kummer-Vandiver conjecture holds for the prime \( p \), and if \( i > 1 \) is odd, we find

\[
\#\text{coker}(f_{2i-1}) = \begin{cases} p^{v_p(i-1)+1}|L_p(\mathbb{Q}, \omega^{1-i}, i)|_p^{-1} & \text{if } p-1 \text{ divides } i-1 \\ |L_p(\mathbb{Q}, \omega^{1-i}, i)|_p^{-1} & \text{otherwise,} \end{cases}
\]

or equivalently,

\[
\text{length}(\text{coker}(f_{2i-1})) = \begin{cases} v_p(L_p(\mathbb{Q}, \omega^{1-i}, i)) + v_p(i-1) + 1 & \text{if } p-1 \text{ divides } i-1 \\ v_p(L_p(\mathbb{Q}, \omega^{1-i}, i)) & \text{otherwise.} \end{cases}
\]

Next, we let \( i > 1 \) be even. Then it follows from [4, §4, Satz 6] that

\[
\text{rk}_{\mathbb{Z}_p} H^1(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i)) = \text{rk}_{\mathbb{Z}_p} H^2(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i)).
\]

But the right-hand group is finite by Soulé’s theorem [5, Théorème 5], and hence, also the left-hand group is finite. This, we claim, implies that

\[
H^2(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-i)) = 0.
\]

Indeed, by the Tate-Poitou sequence, the Pontryagin dual of this group is a subgroup of the finite group \( H^1(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i)) \), and hence, is finite. But it also a quotient of
the group $H^2(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p(i))$, and hence, a divisible group. Therefore, the group is zero as claimed. In addition, it follows that the boundary map

$$H^j(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(i)) \to H^{j+1}(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(i))$$

is an isomorphism for all $j \geq 0$. For every integer $i \neq 0$, we have

$$\#H^0(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(i)) = \max \{ p^v | [\mathbb{Q}(\mu_{p^v}) : \mathbb{Q}] \text{ divides } i \}$$
$$\#H^0(\text{Spec } \mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p(i)) = \max \{ p^v | [\mathbb{Q}_p(\mu_{p^v}) : \mathbb{Q}_p] \text{ divides } i \}$$

and both orders are equal to $p^{v_p(i)+1}$, if $p - 1$ divides $i$, and equal to 1, otherwise.

References