Analytic properties of multiple zeta-functions in several variables

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Abstract

We report several recent results on analytic properties of multiple zeta-functions, mainly in several variables, such as the analytic continuation, the asymptotic behaviour, the location of singularities, and the recursive structure. Some results presented in this paper have never been published before.

1 Euler-Zagier sums

Let r be a positive integer. We begin with the discussion on the Euler-Zagier r-fold sum

$$\begin{aligned}\zeta_{EZ,r}(s_1,\ldots,s_r) &= \sum\cdots\sum_{1\le m_1<\cdots<m_r<\infty} m_1^{-s_1}m_2^{-s_2}\cdots m_r^{-s_r}\\ &= \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} m_1^{-s_1}(m_1+m_2)^{-s_2}\cdots(m_1+\cdots+m_r)^{-s_r}, \qquad (1.1)\end{aligned}$$

where $s_1,\ldots,s_r$ are complex variables. This multiple series is convergent absolutely in the region

$$\{(s_1,\ldots,s_r)\in\mathbf{C}^r \mid \Re(s_{r-k+1}+\cdots+s_r)>k \quad (1\le k\le r)\}.$$

The case $r=2$ of (1.1) was already investigated by L. Euler in the eighteenth century, and the general r-fold case has recently been studied by Zagier [48] and others. In particular, their research on special values of $\zeta_{EZ,r}$ at positive integers shows the great importance of this function in various fields of mathematics and mathematical physics.

The meromorphic continuation of (1.1) to $\mathbf{C}^r$ has been achieved by various methods; see Arakawa and Kaneko [5], Zhao [49], Akiyama, Egami and Tanigawa [1], and the author [25]. It can also be regarded as a special case of Essouabri's general result [9]; see Section 3.

The method of [1] is based on the Euler-Maclaurin summation formula. Let n_1 be a positive integer, η be a real number, $\Re s > 1$, $\alpha \geq 0$, $B_j(x)$ the j-th Bernoulli polynomial, and $\tilde{B}_j(x) = B_j(x - [x])$. Akiyama and Ishikawa [2] proved a modified version of the Euler-Maclaurin formula, including a parameter η, which implies

$$\sum_{n > n_1 + \eta} \frac{1}{(n+\alpha)^s} = \sum_{j=-1}^{J} \frac{\tilde{B}_{j+1}(\eta)}{(j+1)!} \frac{(s)_j}{(n_1+\eta+\alpha)^{s+j}} - \frac{(s)_{J+1}}{(J+1)!} \int_{n_1+\eta}^{\infty} \frac{\tilde{B}_{J+1}(u)}{(u+\alpha)^{s+J+1}} du \tag{1.2}$$

for any positive integer J, where $(s)_j = \Gamma(s+j)/\Gamma(s)$ (see Lemma 1 of [2]). In [1], formula (1.2) (with $\alpha = \eta = 0$) is applied to the sum with respect to m_r on the second member of (1.1), and an expression of $\zeta_{EZ,r}(s_1, \ldots, s_r)$ as a sum involving $\zeta_{EZ,r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + j)$ $(-1 \leq j \leq J)$ is obtained. Hence the analytic continuation can be shown by induction on r, because the integral term on the right-hand side of (1.2) converges in a larger region of s when J becomes larger.

On the other hand, the basic tool of the author's method is the Mellin-Barnes integral formula

$$(1+\lambda)^s = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)} \lambda^z dz, \tag{1.3}$$

where $s, \lambda \in \mathbf{C}$, $\Re s > 0$, $|\arg \lambda| < \pi$, $\lambda \neq 0$, and $c \in \mathbf{R}$, $-\Re s < c < 0$. The path of integration is the vertical line from $c - i\infty$ to $c + i\infty$. The key point is to apply (1.3) (with $\lambda = m_r/(m_1 + \cdots + m_{r-1})$) to the factor

$$(m_1 + \cdots + m_r)^{-s_r} = ((m_1 + \cdots + m_{r-1})^{-s_r} \left(1 + \frac{m_r}{m_1 + \cdots + m_{r-1}}\right)^{-s_r}$$

on the last member of (1.1), and express $\zeta_{EZ,r}(s_1, \ldots, s_r)$ as an integral whose integrand includes $\zeta_{EZ,r-1}(s_1, \ldots, s_{r-2}, s_{r-1}+s_r+z)$ as a factor. Then the analytic continuation can be shown by shifting the path of integration suitably.

We can also see the location of possible singularities by both of the above methods. Akiyama, Egami and Tanigawa [1] considered this matter more carefully, and proved

Theorem 1 (Akiyama, Egami and Tanigawa [1]) *Singularities of* $\zeta_{EZ,r}(s_1, \ldots, s_r)$ *are located only on*

$$s_r = 1, \quad s_{r-1} + s_r = 2, 1, 0, -2, -4, -6, \ldots$$

and

$$s_{r-k+1} + s_{r-k+2} + \cdots + s_r = k - n \quad (3 \leq k \leq r,\ n \in \mathbf{N}_0),$$

where $\mathbf{N}_0$ *denotes the set of non-negative integers. All of the above sets are indeed singularity sets.*

In the same paper [1], they also studied the values of $\zeta_{EZ,r}(s_1,\ldots,s_r)$ at non-positive integers. This direction of research has been continued by Akiyama and Tanigawa [3], and Kamano [16].

It is an important problem to generalize the analytic theory of Euler-Zagier sums to more general situation. Akiyama and Ishikawa [2] studied the series

$$\begin{aligned}&\zeta_{EZ,r}(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r)\\&\quad=\sum\cdots\sum_{1\leq m_1<\cdots<m_r<\infty}(m_1+\alpha_1)^{-s_1}(m_2+\alpha_2)^{-s_2}\cdots(m_r+\alpha_r)^{-s_r}\end{aligned}\tag{1.4}$$

and

$$\begin{aligned}&L_{EZ,r}(s_1,\ldots,s_r;\chi_1,\ldots,\chi_r)\\&\quad=\sum\cdots\sum_{1\leq m_1<\cdots<m_r<\infty}\frac{\chi_1(m_1)}{m_1^{s_1}}\frac{\chi_2(m_2)}{m_2^{s_2}}\cdots\frac{\chi_r(m_r)}{m_r^{s_r}},\end{aligned}\tag{1.5}$$

where $0\leq\alpha_k<1$ $(1\leq k\leq r)$ and χ_k $(1\leq k\leq r)$ are Dirichlet characters of the same conductor. It is clear that (1.5) can be expressed as a linear combination of several series of the form (1.4). Akiyama and Ishikawa [2] applied (1.2) to the right-hand side of (1.4) to obtain an expression involving

$$\zeta_{EZ,r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+j;\alpha_1,\ldots,\alpha_{r-1})\quad(-1\leq j\leq J).$$

This expression gives the analytic continuation of (1.4). Akiyama and Ishikawa also discussed the location of singularities of (1.4) and (1.5). Ishikawa [13] further studied the location of poles of (1.5) in the special case $s_1=\cdots=s_r=s$, and applied the result to the evaluation of certain multiple character sums (Ishikawa [14]).

The author [26] [27] considered a further generalization of (1.4), that is the series

$$\begin{aligned}&\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha_1+m_1w_1)^{-s_1}(\alpha_2+m_1w_1+m_2w_2)^{-s_2}\\&\qquad\qquad\times\cdots\times(\alpha_r+m_1w_1+\cdots+m_rw_r)^{-s_r},\end{aligned}\tag{1.6}$$

where α_k, w_k $(1\leq k\leq r)$ are complex parameters. Let $-\pi<\theta\leq\pi$ and

$$H(\theta)=\{w\in\mathbf{C}\mid w\neq0,\ \theta-\pi/2<\arg w<\theta+\pi/2\}.$$

If we assume that $w_k\in H(\theta)$ $(1\leq k\leq r)$, then the series (1.6) is convergent absolutely when $\Re s_k$ $(1\leq k\leq r)$ are sufficiently large. Under the same assumption, the author proved the meromorphic continuation of (1.6) to $\mathbf{C}^r$ by using the Mellin-Barnes formula (1.3), and discussed the asymptotic behaviour with respect to w_r and the order estimate with respect to $\Im s_r$.

The aim of introducing the above generalized form (1.6) is to treat the Barnes multiple zeta-function

$$\zeta_{B,r}(s;\alpha;w_1,\ldots,w_r)=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha_r+m_1w_1+\cdots+m_rw_r)^{-s} \tag{1.7}$$

as a special case $s_1=\cdots=s_{r-1}=0$ and $s_r=s$. The author first considered the asymptotic behaviour of $\zeta_{B,2}(s;\alpha;w_1,w_2)$ in [22] by a different method (contour integration), and then by using (1.3) in [25]. These studies have applications to Hecke's zeta and L-functions attached to real quadratic fields; see Corrigendum and addendum of [22], and [23].

Multiple Dirichlet series of the Euler-Zagier type with general coefficients, of the form

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{a_1(m_1)}{m_1^{s_1}}\frac{a_2(m_2)}{(m_1+m_2)^{s_2}}\cdots\frac{a_r(m_r)}{(m_1+\cdots+m_r)^{s_r}}, \tag{1.8}$$

have been introduced and studied by Matsumoto and Tanigawa [30], under the assumption that the series $\sum_{m=1}^{\infty}a_k(m)m^{-s}$ $(1\le k\le r)$ have nice properties. The above (1.8) includes the multiple L-series of Arakawa and Kaneko [6]. In [30], the analytic continuation and a certain order estimate of (1.8) have been obtained by using the Mellin-Barnes integral (1.3).

It seems that the method of using the Mellin-Barnes integral is suitable to consider upper bound estimates of multiple zeta-functions. The case of the Euler-Zagier sum $\zeta_{EZ,r}(s_1,\ldots,s_r)$ was studied by Ishikawa and Matsumoto [15]; especially, non-trivial estimates in the cases $r=2$ and $r=3$ have been obtained. However it is still not clear how is the real order of magnitude of $\zeta_{EZ,r}(s_1,\ldots,s_r)$.

Recently, Matsumoto and Tsumura [31] introduced further generalized series

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{a_1(m_1)\cdots a_r(m_r)u^{-(m_1+\cdots+m_r)}}{(\alpha_1+m_1w_1)^{s_1}\cdots(\alpha_r+m_1w_1+\cdots+m_rw_r)^{s_r}}, \tag{1.9}$$

where $u\ge 1$, $\alpha_k,w_k\in\mathbf{R}$, $0<\alpha_k-\alpha_{k-1}\le w_k$, in connection with a study of certain generalized multiple polylogarithms. (As for multiple polylogarithms, see, for example, [7].)

2 Multiple series defined by linear forms

Let $A_{Nr}=(a_{nj})_{1\le n\le N,1\le j\le r}$ be an (N,r)-matrix, where a_{nj} are non-negative real numbers. Assume that all rows and all columns of A_{Nr} include at least one non-zero element. Let

$$\zeta_r(s_1,\ldots,s_N;A_{Nr})=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}(a_{11}m_1+\cdots+a_{1r}m_r)^{-s_1} \times\cdots\times(a_{N1}m_1+\cdots+a_{Nr}m_r)^{-s_N}. \tag{2.1}$$

The Euler-Zagier sum (1.1) is a special case of (2.1). Shintani [35], [36] considered the situation when all a_{nj} are positive (with characters and additional constant terms). Shintani actually treated the case $s_1 = \cdots = s_N$, but Hida [11] introduced multi-variable Shintani zeta-functions.

Other typical examples of (2.1) are the Mordell-Tornheim multiple series

$$\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1}) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} m_1^{-s_1}\cdots m_r^{-s_r}(m_1+\cdots+m_r)^{-s_{r+1}} \tag{2.2}$$

and the Apostol-Vu multiple series

$$\zeta_{AV,r}(s_1,\ldots,s_r;s_{r+1}) = \sum\cdots\sum_{1\leq m_1<\cdots<m_r<\infty} m_1^{-s_1}\cdots m_r^{-s_r}(m_1+\cdots+m_r)^{-s_{r+1}}. \tag{2.3}$$

Both of the above series (2.2) and (2.3) were introduced in the author's paper [28], though the history of some special cases goes back to Tornheim [38], Mordell [34], and Apostol and Vu [4]. The following theorem has been proved by the author in [24] for the case $r=2$, and in [28] for general r.

Theorem 2 *The series* (2.2) *and* (2.3) *can be continued meromorphically to* $\mathbf{C}^{r+1}$. *The possible singularities of* (2.2) *are located only on the subset of* $\mathbf{C}^{r+1}$ *defined by one of the following equations:*

$$\sum_{a=1}^{h} s_{j_a} + s_{r+1} = h - \ell\left(1-\left[\frac{h}{r}\right]\right) \quad (1\leq h\leq r,\ 1\leq j_1<\cdots<j_h\leq r,\ \ell\in\mathbf{N}_0).$$

Also, the possible singularities of (2.3) *are located only on the subset of* $\mathbf{C}^{r+1}$ *defined by one of the following equations:*

$$s_i+\cdots+s_{r+1} = r+1-i-\ell \quad (1\leq i\leq r,\ \ell\in\mathbf{N}_0).$$

The proof of Theorem 2 in [24] [28] is again based on (1.3). As in the case of the Euler-Zagier sum, using (1.3) we obtain

$$\begin{aligned}\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1}) &= \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\\ &\times\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+z)\zeta(s_r-z)dz,\end{aligned} \tag{2.4}$$

where $\zeta(\cdot)$ is the Riemann zeta-function. In the case of the Apostol-Vu series, we have

$$\begin{aligned}&\varphi_{j,r}(s_1,\ldots,s_j;s_{j+1},\ldots,s_r;s_{r+1})\\ &\quad=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\\ &\qquad\times\varphi_{j-1,r}(s_1,\ldots,s_{j-1};s_j-z,s_{j+1},\ldots,s_r;s_{r+1}+z)dz,\end{aligned} \tag{2.5}$$

where

$$\varphi_{j,r}(s_1,\ldots,,s_j;s_{j+1},\ldots,s_r;s_{r+1})=\sum\cdots\sum_{1\le m_1<\cdots<m_r<\infty} m_1^{-s_1}\cdots m_r^{-s_r}(m_1+\cdots+m_j)^{-s_{r+1}}. \tag{2.6}$$

Note that $\varphi_{r,r}=\zeta_{AV,r}$ and

$$\varphi_{1,r}(s_1;s_2,\ldots,s_r;s_{r+1})=\zeta_{EZ,r}(s_1+s_{r+1},s_2,\ldots,s_r).$$

The relations (2.4) and (2.5) imply the recursive sequences

$$\zeta_{MT,r}\to\zeta_{MT,r-1}\to\cdots\to\zeta_{MT,1}=\zeta$$

and

$$\zeta_{AV,r}=\varphi_{r,r}\to\varphi_{r-1,r}\to\cdots\to\varphi_{1,r}=\zeta_{EZ,r},$$

along which the proof of Theorem 2 goes inductively. The discussion in Section 1 implies another recursive sequence

$$\zeta_{EZ,r}\to\zeta_{EZ,r-1}\to\cdots\to\zeta_{EZ,1}=\zeta.$$

Thus we find a recursive structure in the family of multiple zeta-functions. This viewpoint is discussed in the last section of [28].

In [47], Maoxiang Wu introduced the χ-analogues of (2.2) and (2.3). Let $\chi_1,\ldots,\chi_r$ be Dirichlet characters of the same modulus q (≥ 2), and define

$$\begin{aligned} &L_{MT,r}(s_1,\ldots,s_r;s_{r+1};\chi_1,\ldots,\chi_r)\\ &\quad=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{\chi_1(m_1)\cdots\chi_r(m_r)}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_r)^{s_{r+1}}},\end{aligned}\tag{2.7}$$

$$\begin{aligned} &Ł_{AV,r}(s_1,\ldots,s_r;s_{r+1};\chi_1,\ldots,\chi_r)\\ &\quad=\sum\cdots\sum_{1\le m_1<\cdots<m_r<\infty}\frac{\chi_1(m_1)\cdots\chi_r(m_r)}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_r)^{s_{r+1}}}.\end{aligned}\tag{2.8}$$

These series are convergent absolutely for $\Re s_k>1$ $(1\le k\le r)$, $\Re s_{r+1}>0$. Wu proved the following two theorems.

Theorem 3 (Wu [47]) *The series* (2.7) *can be continued meromorphically to* $\mathbf{C}^{r+1}$. *If none of the characters* $\chi_1,\ldots,\chi_r$ *are principal, then* $L_{MT,r}$ *is entire. If there are* k *principal characters* $\chi_{j_1},\ldots,\chi_{j_k}$ *among them, then possible singularities are located only on the subsets of* $\mathbf{C}^{r+1}$ *defined by one of the following equations:*

$$\sum_{a=1}^{h}s_{j_{i(a)}}+s_{r+1}=h-\ell\left(1-\left[\frac{h}{r}\right]\right),$$

where $1\le h\le k$, $1\le i(1)<\cdots<i(h)\le k$, $\ell\in\mathbf{N}_0$.

Theorem 4 (Wu [47]) *The series* (2.8) *can be continued meromorphically to* $\mathbf{C}^{r+1}$*, and possible singularities are located only on the subsets of* $\mathbf{C}^{r+1}$ *defined by one of the following equations:*

$$\sum_{a=1}^{h} s_{r-a+1} + s_{r+1} = h - \ell,$$

where $1 \leq h \leq r$*,* $\ell \in \mathbf{N}_0$*.*

Since Wu [47] is unpublished, we briefly outline his proof of these two theorems here.

The proof of Theorem 3 is just a direct generalization of the argument developed in [28]. We omit the details, only noting that the basic formula, corresponding to (2.4), is

$$\begin{aligned} L_{MT,r}(s_1,\ldots,s_r;s_{r+1};\chi_1,\ldots,\chi_r) &= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})} \\ &\times L_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+z;\chi_1,\ldots,\chi_{r-1})L(s_r-z,\chi_r)dz, \end{aligned} \tag{2.9}$$

where $L(\cdot,\chi_r)$ is the Dirichlet L-function attached to χ_r.

To prove Theorem 4, we define

$$\begin{aligned} &\Phi_{j,r}(s_1,\ldots,s_j;s_{j+1},\cdots,s_r;s_{r+1};\chi_1,\ldots,\chi_r) \\ &\quad = \sum\cdots\sum_{1\leq m_1<\cdots<m_r<\infty} \frac{\chi_1(m_1)\cdots\chi_r(m_r)}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_j)^{s_{r+1}}}. \end{aligned} \tag{2.10}$$

Then corresponding to (2.5), we have

$$\begin{aligned} \Phi_{j,r}(s_1,\ldots,s_j;s_{j+1},\cdots,s_r;s_{r+1};\chi_1,\ldots,\chi_r) &= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})} \\ \times\Phi_{j-1,r}(s_1,\ldots,s_{j-1};s_j-z,s_{j+1},\cdots,s_r;s_{r+1}+z;\chi_1,\ldots,\chi_r)dz. \end{aligned} \tag{2.11}$$

Hence the induction argument goes along the sequence

$$L_{AV,r} = \Phi_{r,r} \longrightarrow \Phi_{r-1,r} \longrightarrow \cdots \longrightarrow \Phi_{1,r},$$

but

$$\Phi_{1,r}(s_1;s_2,\ldots,s_r;\chi_1,\ldots,\chi_r) = L_{EZ,r}(s_1+s_{r+1},s_2,\ldots,s_r;\chi_1,\ldots,\chi_r),$$

whose basic analytic properties has already been discussed by Akiyama and Ishikawa [2].

As explained in Section 1, $L_{EZ,r}(s_1,\ldots,s_r;\chi_1,\ldots,\chi_r)$ can be expressed in terms of $\zeta_{EZ,r}(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r)$, and the latter can be expressed as a sum involving

$$\zeta_{EZ,r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+j;\alpha_1,\ldots,\alpha_{r-1}) \quad (-1\leq j\leq J).$$

Using these expressions, Wu [47] proved (by induction) that both of the functions $\zeta_{EZ,r}(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r)$ and $L_{EZ,r}(s_1,\ldots,s_r;\chi_1,\ldots,\chi_r)$ are of polynomial order with respect to $|\Im s_1|,\ldots,|\Im s_r|$. Hence, using (2.11), we can show that $\Phi_{j,r}$ $(1\leq j\leq r)$ is also of polynomial order. Therefore it is possible to shift the path of integration on the right-hand side of (2.11) freely. The remaining part of the proof is the same as in [28].

Special values of $\zeta_{MT,r}$, $\zeta_{AV,r}$ and their relatives have been studied by several mathematicians, including Tornheim, Mordell and Apostol and Vu themselves. Special values in the domain of absolute convergence have been further studied by Huard, Williams and Zhang [12], Subbarao and Sitaramachandrarao [37], and Tsumura's recent series of papers [39] [40] [41] [42] [43] [44]. In those papers, various relations among special values at integer arguments have been obtained. From an analytic point of view, however, it is important to reveal *whether those relations are valid only at integer points, or valid also at other values.* Tsumura [45] [46] discovered that some relations at integer points, proved in his previous articles [39] [40] [42], are actually valid continuously at other values. These relations of Tsumura may be regarded as functional relations among multiple zeta-functions.

Another functional relation has been found by the author [29], which implies, as a special case, a certain relation between $\zeta_{EZ,2}(s_1,s_2)$ and $\zeta_{EZ,2}(1-s_2,1-s_1)$. More generally, in [29] the author defined the double Hurwitz-Lerch zeta-function

$$\zeta_2(s_1,s_2;\alpha,\beta,w)=\sum_{m=0}^{\infty}(\alpha+m)^{-s_1}\sum_{n=1}^{\infty}e^{2\pi in\beta}(\alpha+m+nw)^{-s_2},$$

where $0<\alpha\leq 1$, $0\leq\beta\leq 1$, $w>0$, and proved a certain relation between $\zeta_2(s_1,s_2;\alpha,\beta,w)$ and $\zeta_2(1-s_2,1-s_1;1-\beta,1-\alpha,w)$. Note that the case $w=1$ of this function was already introduced by Katsurada [17] in his study on the mean square of Lerch zeta-functions. It is also possible to regard Proposition 1 of [29] as a double analogue of the functional equation of Hurwitz-Lerch zeta-functions.

3 Multiple series defined by polynomials

In [28], it has been shown that any multiple series of the form (2.1) can be continued meromorphically to the whole space $\mathbf{C}^n$, by the method of Mellin-Barnes integrals. It is in fact possible to prove a much more general result by the same method. Let

$$P_n(X_1,\ldots,X_r)=\sum_{k=1}^{K(n)}a_k(n)X_1^{p_1(k,n)}\cdots X_r^{p_r(k,n)}\qquad(1\leq n\leq N)$$

be polynomials, where $a_k(n) \in \mathbf{C}$, $p_j(k,n) \in \mathbf{N}_0$, and for any fixed j, at least one of $p_j(k,n)$ $(1 \le n \le N,\ 1 \le k \le K(n))$ is positive. We assume that $\Re a_k(n) > 0$ for all k and n. Hence

$$\theta_n = \max\{|\arg a_k(n)|\ ;\ 1 \le k \le K(n)\}$$

is smaller than $\pi/2$. Define

$$\zeta_r(s_1,\ldots,s_N;P_1,\ldots,P_N) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} P_1(\mathbf{m})^{-s_1}\cdots P_N(\mathbf{m})^{-s_N}, \tag{3.1}$$

where $\mathbf{m} = (m_1,\ldots,m_r)$ and $s_n = \sigma_n + it_n \in \mathbf{C}$ $(1 \le n \le N)$. It is clear that there exists a positive constant $\sigma_a = \sigma_a(P_1,\ldots,P_N)$ such that the series (3.1) is absolutely convergent when $\sigma_n > \sigma_a$ for $1 \le n \le N$.

By a *multiple strip* we mean a set of the form

$$\{(s_1,\ldots,s_N) \in \mathbf{C}^N \mid \sigma_{n1} \le \sigma_n \le \sigma_{n2} \quad (1 \le n \le N)\}, \tag{3.2}$$

where σ_{n1}, σ_{n2} $(1 \le n \le N)$ are any fixed real numbers with $\sigma_{n1} < \sigma_{n2}$. By $\mathcal{F}(\cdot)$ we denote a quantity, not necessarily the same at each occurrence, which is of polynomial order with respect to the indicated variables.

Theorem 5 *The multiple zeta-function (3.1) can be continued meromorphically to the whole space $\mathbf{C}^N$. The possible singularities of it are located only on hyperplanes of the form*

$$c_1s_1 + \cdots + c_Ns_N = u(c_1,\ldots,c_N) - \ell \qquad (\ell \in \mathbf{N}_0), \tag{3.3}$$

where $c_1,\ldots,c_N \in \mathbf{N}_0$ and $u(c_1,\ldots,c_N)$ is an integer determined by $c_1,\ldots,c_N$. Moreover, the estimate

$$\zeta_r(s_1,\ldots,s_N;P_1,\ldots,P_N) = O\left(\mathcal{F}(t_1,\ldots,t_N)\prod_{n=1}^{N} e^{\theta_n|t_n|}\right) \tag{3.4}$$

holds uniformly in any multiple strip (3.2), except in neighbourhoods of possible polar sets (3.3).

The case $N = 1$ of (3.1) was first studied by Mellin [32], [33]. The Mellin-Barnes integral (1.3) already appeared in those papers. After Mellin, many people including K. Mahler, P. Cassou-Noguès, and P. Sargos continued his research. The multi-variable form (3.1) was first discussed by Lichtin [18], [19], [20], [21], and he proved the continuation of (3.1) when polynomials are hypoelliptic. Then Essouabri [9], [10] introduced the condition $\mathrm{H_0S}$, under which he proved the continuation. Here we do not give the exact definition of $\mathrm{H_0S}$, but it is satisfied if all coefficients of polynomials have positive real parts. Moreover, though only the case $N = 1$ is discussed in [10], Essouabri mentioned in his thesis [9] that his result can be generalized to the multi-variable case. See also de Crisenoy [8], in which a twisted version of (3.1) (for general N) was studied.

Therefore, the meromorphic continuation of (3.1) was included, as a special case, in Essouabri's theorem. Nevertheless we give a proof of the above theorem here, because of several reasons. First, our method is quite different from Essouabri's and rather simple. Secondly, formula (3.5) below, which is the key of our proof, implies the recursive structure similar to those discussed in the preceding section. Thirdly, our method is suitable to obtain various explicit information, such as location of poles and order estimates, inductively. And finally, our method can be generalized to the case with general coefficients (similar to (1.8) and (1.9)).

Remark 1. When we write the (possible) polar sets of ζ_r in the form (3.3), we can choose $c_1, \ldots, c_N$ whose common greatest divisor is as small as possible. We call such tuples $(c_1, \ldots, c_N)$ *primitive*. Then, in the proof of Theorem 5 it will be shown that, for any fixed ζ_r, there are only finitely many primitive tuples $(c_1, \ldots, c_N)$ such that the (possible) polar sets of ζ_r are of the form (3.3).

Remark 2. For any fixed $c_1, \ldots, c_N$, there exists a positive integer $v(c_1, \ldots, c_N)$, by which the order of the singularity (3.3) is bounded uniformly for any ℓ.

Now we start the proof. We prove Theorem 5 with Remarks 1 and 2 by induction on

$$K(P_1, \ldots, P_N) = \prod_{n=1}^{N} K(n).$$

The argument is a generalization of the proof of Theorem 3 in [28].

First consider the case $K(P_1, \ldots, P_N) = 1$. Then $K(n) = 1$ $(1 \le n \le N)$, so all the P_n's are monomials and

$$\begin{aligned}\zeta_r(s_1, \ldots, s_N; P_1, \ldots, P_N) &= \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{n=1}^{N} \left(a_1(n) m_1^{p_1(1,n)} \cdots m_r^{p_r(1,n)} \right)^{-s_n} \\ &= a_1(1)^{-s_1} \cdots a_1(N)^{-s_N} \prod_{j=1}^{r} \zeta(p_j(1,1)s_1 + \cdots + p_j(1,N)s_N).\end{aligned}$$

Hence all the assertions of Theorem 5, Remarks 1 and 2 clearly hold.

Now consider the case $K(P_1, \ldots, P_N) \ge 2$. Let $\sigma_a^* \ge \sigma_a$, and at first assume that $(s_1, \ldots, s_N)$ is in the region

$$\mathcal{B}^* = \{(s_1, \ldots, s_N) \mid \sigma_n > 2\sigma_a^* (1 \le n \le N)\}.$$

Since at least one $K(n) \ge 2$, changing the parameters if necessary, we may assume that $K(N) \ge 2$. Then

$$P_N(\mathbf{m})^{-s_N} = \left(a_1(N) M_1(N) + \sum_{k=2}^{K(N)} a_k(N) M_k(N) \right)^{-s_N}$$

$$= \left(\sum_{k=2}^{K(N)} a_k(N)M_k(N)\right)^{-s_N} \left(1+\frac{a_1(N)M_1(N)}{\sum_{k=2}^{K(N)} a_k(N)M_k(N)}\right)^{-s_N},$$

where $M_k(N)=m_1^{p_1(k,N)}\cdots m_r^{p_r(k,N)}$. Hence, applying (1.3), we obtain

$$\zeta_r(s_1,\ldots,s_N;P_1,\ldots,P_N)=\frac{1}{2\pi i}\int_{(\gamma)}\frac{\Gamma(s_N+z)\Gamma(-z)}{\Gamma(s_N)}$$
$$\times\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}P_1(\mathbf{m})^{-s_1}\cdots P_{N-1}(\mathbf{m})^{-s_{N-1}}P_N^*(\mathbf{m})^{-s_N-z}P_N^{**}(\mathbf{m})^z dz, \quad (3.5)$$

where

$$P_N^*(\mathbf{m})=\sum_{k=2}^{K(N)}a_k(N)M_k(N), \qquad P_N^{**}(\mathbf{m})=a_1(N)M_1(N),$$

and we can choose γ as

$$-\sigma_N+\sigma_a<\gamma<-\sigma_a. \quad (3.6)$$

Then the multiple series on the right-hand side of (3.5) is absolutely convergent and is the zeta-function

$$\zeta_r(s_1,\ldots,s_{N-1},s_N+z,-z;P_1,\ldots,P_{N-1},P_N^*,P_N^{**}). \quad (3.7)$$

Since

$$K(P_1,\ldots,P_{N-1},P_N^*,P_N^{**})=K(1)\times\cdots\times K(N-1)\times(K(N)-1)\times 1$$
$$<\prod_{n=1}^{N}K(n)=K(P_1,\ldots,P_N),$$

by the induction assumption we see that (3.7) can be continued meromorphically to the whole space $\mathbf{C}^{N+1}$, and possible singularities are of the form

$$c_1s_1+\cdots+c_Ns_N+(c_N-c_{N+1})z=u(c_1,\ldots,c_{N+1})-\ell,$$

where $c_1,\ldots,c_{N+1},\ell\in\mathbf{N}_0$ and $u(c_1,\ldots,c_{N+1})\in\mathbf{Z}$. If $c_N=c_{N+1}$, then this is

$$c_1s_1+\cdots+c_Ns_N=u(c_1,\ldots,c_N)-\ell \quad (\ell\in\mathbf{N}_0), \quad (3.8)$$

which is irrelevant to z. If $c_N-c_{N+1}=d_0>0$, then

$$z=d_0^{-1}\{-c_1s_1-\cdots-c_Ns_N+u(c_1,\ldots,c_N,d_0)\}-d_0^{-1}\ell \quad (\ell\in\mathbf{N}_0), \quad (3.9)$$

and if $c_N-c_{N+1}=-e_0<0$, then

$$z=e_0^{-1}\{c_1s_1+\cdots+c_Ns_N-u(c_1,\ldots,c_N,e_0)\}+e_0^{-1}\ell \quad (\ell\in\mathbf{N}_0). \quad (3.10)$$

We write the first term on the right-hand side of (3.9) (resp. (3.10)) as $D(s_1,\ldots,s_N;\mathbf{c})$ (resp. $E(s_1,\ldots,s_N;\mathbf{c})$) for brevity, where $\mathbf{c}=(c_1,\ldots,c_N)$. Denote the set of all primitive tuples $\mathbf{c}=(c_1,\ldots,c_N)$ appearing in (3.8) (resp. (3.9), (3.10)) by T_0 (resp. T_D, T_E). These sets are finite because of Remark 1. The above (3.9) and (3.10) can be poles, with respect to z, of the integrand on the right-hand side of (3.5). The other poles of the integrand are

$$z=-s_N-\ell \quad (\ell\in\mathbf{N}_0) \tag{3.11}$$

and

$$z=\ell \quad (\ell\in\mathbf{N}_0). \tag{3.12}$$

We can assume that σ_a^* is so large that all the poles (3.9) and (3.11) are on the left of the line $\Re z=\gamma$, while all the poles (3.10) and (3.12) are on the right of $\Re z=\gamma$.

Now, let $(s_1^0,\ldots,s_N^0)$ be any point in the space $\mathbf{C}^N$, and we show that the right-hand side of (3.5) can be continued meromorphically to $(s_1^0,\ldots,s_N^0)$.

First, remove the singularities of the form (3.8) from the integrand. These singularities are cancelled by the factor

$$(c_1s_1+\cdots+c_Ns_N-u(c_1,\ldots,c_N)+\ell)^{v(c_1,\ldots,c_N)}$$

(by Remark 2 as a part of the induction assumption). Let L be a sufficiently large positive integer such that, if $\sigma_n\geq\Re s_n^0$ $(1\leq n\leq N)$,

$$c_1s_1+\cdots+c_Ns_N=u(c_1,\ldots,c_N)-L$$

does not hold for any $\mathbf{c}=(c_1,\ldots,c_N)\in T_0$. Define

$$\Phi(s_1,\ldots,s_N)=\prod_{\mathbf{c}\in T_0}\prod_{\ell=0}^{L-1}(c_1s_1+\cdots+c_Ns_N-u(c_1,\ldots,c_N)+\ell)^{v(c_1,\ldots,c_N)},$$

and rewrite (3.5) as

$$\zeta_r(s_1,\ldots,s_N;P_1,\ldots,P_N)=\Phi(s_1,\ldots,s_N)^{-1}J(s_1,\ldots,s_N), \tag{3.13}$$

where

$$\begin{aligned}J(s_1,\ldots,s_N)=&\frac{1}{2\pi i}\int_{(\gamma)}\frac{\Gamma(s_N+z)\Gamma(-z)}{\Gamma(s_N)}\Phi(s_1,\ldots,s_N)\\&\times\zeta_r(s_1,\ldots,s_{N-1},s_N+z,-z;P_1,\ldots,P_{N-1},P_N^*,P_N^{**})dz.\end{aligned} \tag{3.14}$$

Then the integrand on the right-hand side of (3.14) does not have singularities of the form (3.8) in the region $\sigma_n\geq\Re s_n^0$ $(1\leq n\leq N)$.

Since $\Phi(s_1,\ldots,s_N)^{-1}$ is meromorphic in the whole space, in order to complete the proof of the continuation, our remaining task is to show the continuation of $J(s_1,\ldots,s_N)$. Let M be a positive integer, and $s_n^*=s_n^0+M$ $(1\leq n\leq N)$. We may choose M so large that $(s_1^*,\ldots,s_N^*)\in\mathcal{B}^*$. Let $\mathcal{I}_1$ be the set of all imaginary parts of the poles (3.9) and (3.11), and $\mathcal{I}_2$ be the set of all imaginary parts of the poles (3.10) and (3.12), for $(s_1,\ldots,s_N)=(s_1^*,\ldots,s_N^*)$.

Case 1. In the case $\mathcal{I}_1\cap\mathcal{I}_2=\emptyset$, we join $D(s_1^*,\ldots,s_N^*;\mathbf{c})$ and $D(s_1^0,\ldots,s_N^0;\mathbf{c})$ by the segment $S(D;\mathbf{c})$ which is parallel to the real axis. Similarly join $E(s_1^*,\ldots,s_N^*;\mathbf{c})$ and $E(s_1^0,\ldots,s_N^0;\mathbf{c})$ by the segment $S(E;\mathbf{c})$, and join $-s_N^*$ and $-s_N^0$ by the segment $S(N)$. Since $\mathcal{I}_1\cap\mathcal{I}_2=\emptyset$, we can deform the path $\Re z=\gamma$ to obtain a new path $\mathcal{C}$ from $\gamma-i\infty$ to $\gamma+i\infty$, such that all the segments $S(D;\mathbf{c})$ and $S(N)$ are on the left of $\mathcal{C}$, while all the segments $S(E;\mathbf{c})$ and the poles (3.12) are on the right of $\mathcal{C}$ (see Fig.1). Then we have

$$\begin{aligned} J(s_1,\ldots,s_N)=&\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\Gamma(s_N+z)\Gamma(-z)}{\Gamma(s_N)}\Phi(s_1,\ldots,s_N)\\ &\times\zeta_r(s_1,\ldots,s_{N-1},s_N+z,-z;P_1,\ldots,P_{N-1},P_N^*,P_N^{**})dz \end{aligned}\tag{3.15}$$

in a sufficiently small neighbourhood of $(s_1^*,\ldots,s_N^*)$. Next, on the right-hand side of (3.15), we move $(s_1,\ldots,s_N)$ from $(s_1^*,\ldots,s_N^*)$ to $(s_1^0,\ldots,s_N^0)$ with keeping the values of imaginary parts of each s_n. Since ζ_r in the integrand satisfies an estimate of the form (3.4) by the induction assumption, this procedure is possible; and, during this procedure, the path $\mathcal{C}$ does not cross any poles of the integrand. Hence the expression (3.15) gives the holomorphic continuation of $J(s_1,\ldots,s_N)$ to a neighbourhood of $(s_1^0,\ldots,s_N^0)$.

Case 2. Next consider the case $\mathcal{I}_1\cap\mathcal{I}_2\neq\emptyset$. Then the imaginary part of some member of $\{D(s_1^*,\ldots,s_N^*;\mathbf{c}),\ -s_N^*\mid \mathbf{c}\in T_D\}$ coincides with the imaginary part of some member of $\{E(s_1^*,\ldots,s_N^*;\mathbf{c}),\ 0\mid \mathbf{c}\in T_E\}$. We consider the case

$$\Im D(s_1^*,\ldots,s_N^*;\mathbf{c}_1)=\Im E(s_1^*,\ldots,s_N^*;\mathbf{c}_2)\tag{3.16}$$

for some $\mathbf{c}_1$ and $\mathbf{c}_2$, because other cases can be treated similarly. The associated poles are $D(s_1^*,\ldots,s_N^*;\mathbf{c}_1)-d_0^{-1}\ell_1$ and $E(s_1^*,\ldots,s_N^*;\mathbf{c}_1)+e_0^{-1}\ell_2$ ($\ell_1,\ell_2\in\mathbf{N}_0$). When $(s_1^*,\ldots,s_N^*)$ is moved to $(s_1^0,\ldots,s_N^0)$, these poles are moved to $D(s_1^0,\ldots,s_N^0;\mathbf{c}_1)-d_0^{-1}\ell_1$ and $E(s_1^0,\ldots,s_N^0;\mathbf{c}_1)+e_0^{-1}\ell_2$, respectively. In the case

$$\Re D(s_1^0,\ldots,s_N^0;\mathbf{c}_1)-d_0^{-1}\ell_1\neq\Re E(s_1^0,\ldots,s_N^0;\mathbf{c}_1)+e_0^{-1}\ell_2\tag{3.17}$$

for any ℓ_1 and ℓ_2, we modify the argument in Case 1 as follows. Let η be a small positive number, and consider the oriented polygonal path $S'(D;\mathbf{c}_1)$ joining the points $D(s_1^*,\ldots,s_N^*;\mathbf{c}_1)$, $D(s_1^*+i\eta,\ldots,s_N^*+i\eta;\mathbf{c}_1)$, $D(s_1^0+i\eta,\ldots,s_N^0+i\eta;\mathbf{c}_1)$, and then $D(s_1^0,\ldots,s_N^0;\mathbf{c}_1)$ in that order. Similarly define the path $S'(E;\mathbf{c}_2)$ which joins $E(s_1^*,\ldots,s_N^*;\mathbf{c}_2)$, $E(s_1^*+i\eta,\ldots,s_N^*+i\eta;\mathbf{c}_2)$, $E(s_1^0+i\eta,\ldots,s_N^0+i\eta;\mathbf{c}_2)$, and then $E(s_1^0,\ldots,s_n^0;\mathbf{c}_2)$. Then $S'(D;\mathbf{c}_1)$ lies on the lower side of the line

$$\mathcal{L}=\{z\mid \Im z=\Im D(s_1^*,\ldots,s_N^*;\mathbf{c}_1)=\Im E(s_1^*,\ldots,s_N^*;\mathbf{c}_2)\},$$

while $S'(E;\mathbf{c}_2)$ lies on the upper side of $\mathcal{L}$. Because of (3.17), we can define the path $\mathcal{C}'$, which is almost the same as $\mathcal{C}$, but near the line $\mathcal{L}$ we draw $\mathcal{C}'$ such that it separates

$$\bigcup_{\ell_1 \in \mathbf{N}_0} (S'(D;\mathbf{c}_1) - d_0^{-1}\ell_1) \qquad \text{and} \qquad \bigcup_{\ell_2 \in \mathbf{N}_0} (S'(E;\mathbf{c}_2) + e_0^{-1}\ell_2)$$

(see Fig.2). Then the expression (3.15), with replacing $\mathcal{C}$ by $\mathcal{C}'$, is valid in a sufficiently small neighbourhood of $(s_1^*,\ldots,s_N^*)$. When $(s_1,\ldots,s_N)$ moves along

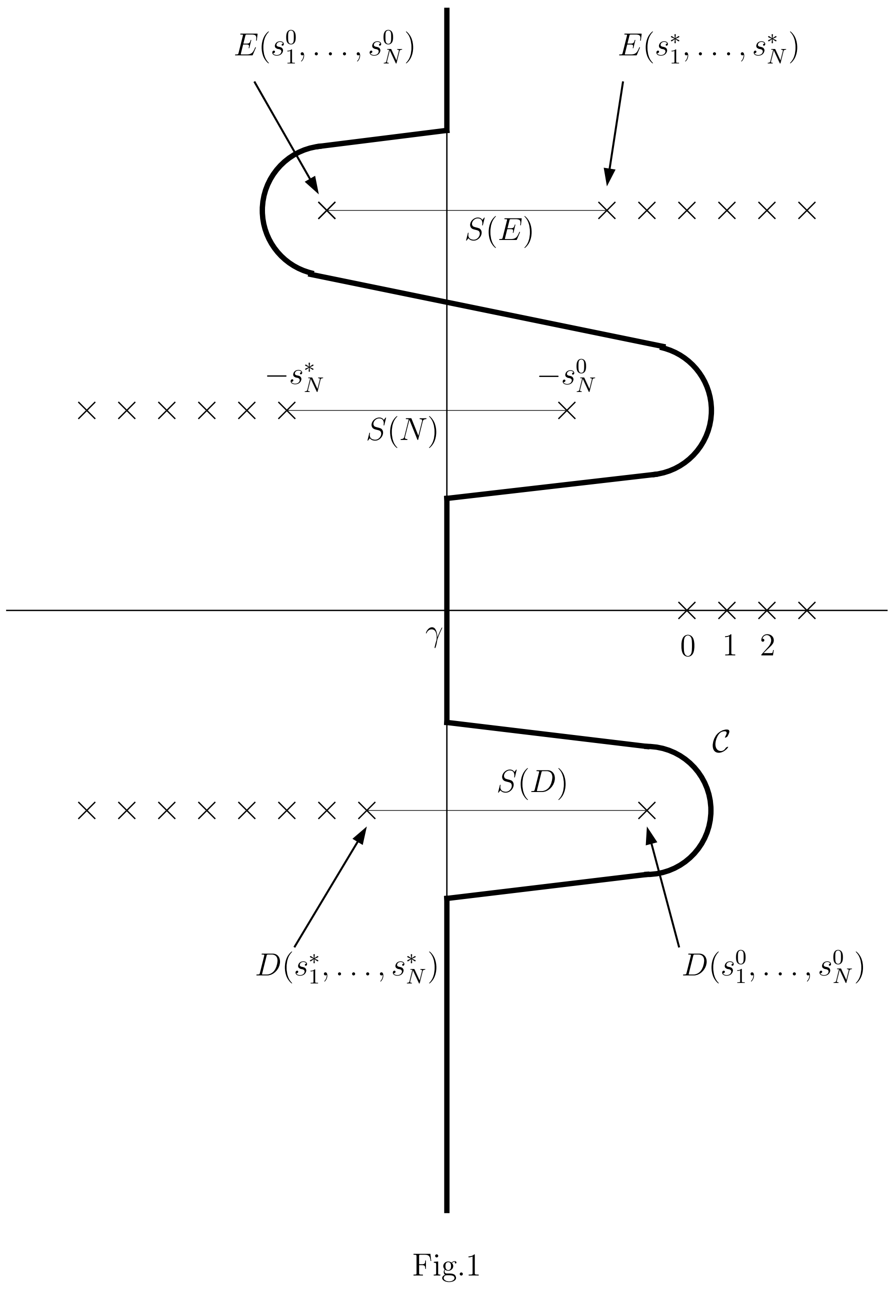

Fig.1

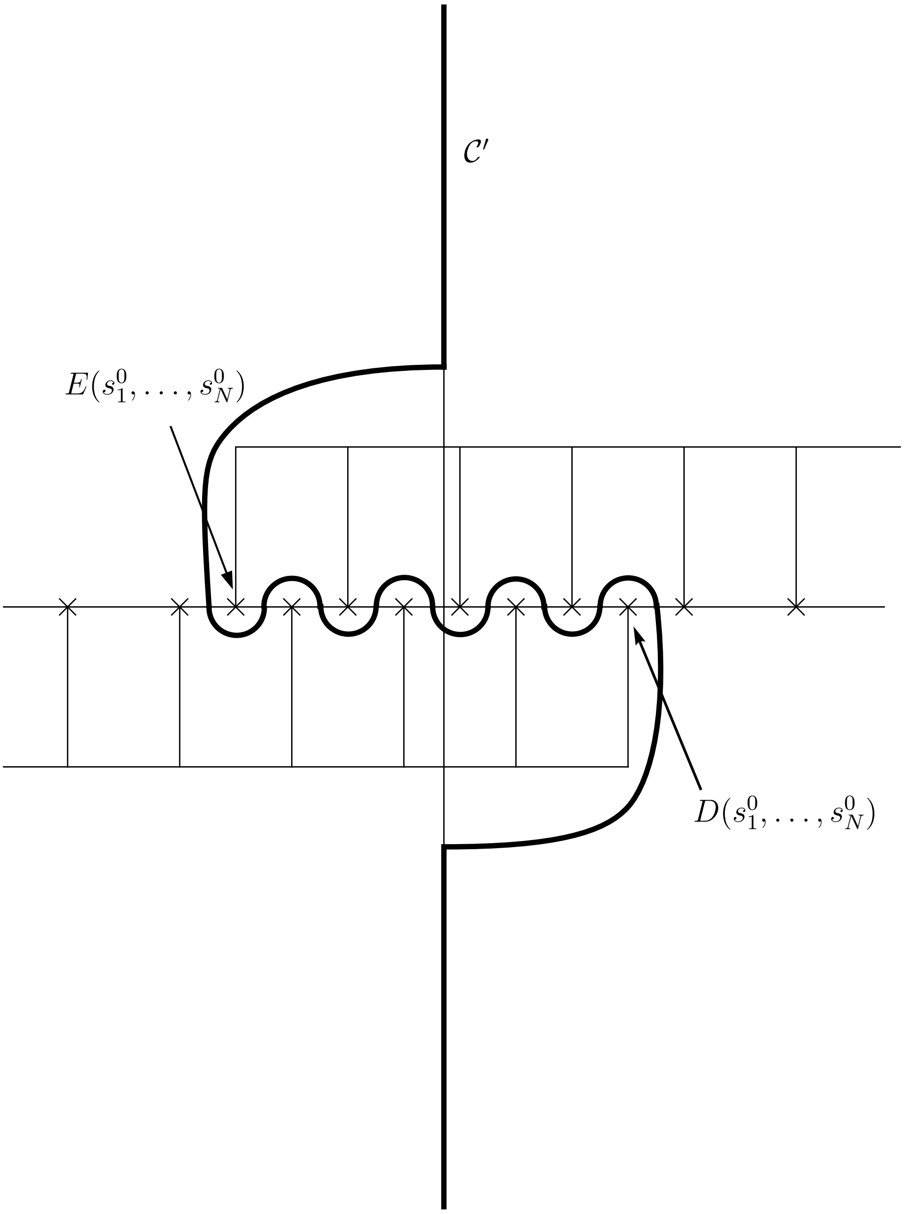

Fig.2

the polygonal path joining $(s_1^*,\ldots,s_N^*)$, $(s_1^*+i\eta,\ldots,s_N^*+i\eta)$, $(s_1^0+i\eta,\ldots,s_N^0+i\eta)$, and then $(s_1^0,\ldots,s_N^0)$ in that order, the path $\mathcal{C}'$ encounters no pole, hence we obtain the holomorphic continuation.

Case 3. The remaining case is that

$$D(s_1^0,\ldots,s_N^0;\mathbf{c}_1)-d_0^{-1}\ell_1=E(s_1^0,\ldots,s_N^0;\mathbf{c}_2)+e_0^{-1}\ell_2 \tag{3.18}$$

holds for some ℓ_1 and ℓ_2. Then this might hold for some other pairs of (ℓ_1,ℓ_2). In this case we consider the path $\mathcal{C}''$ which is almost the same as $\mathcal{C}$, but near the line $\mathcal{L}$ we only require that $S(D;\mathbf{c}_1)$ is on the left of $\mathcal{C}''$, and that the points

$E(s_1^*,\ldots,s_N^*;\mathbf{c}_2)+e_0^{-1}\ell_2$, $E(s_1^0,\ldots,s_N^0;\mathbf{c}_2)+e_0^{-1}\ell_2$ are not on $\mathcal{C}''$ for any ℓ_2. When we deform the path $\Re z=\gamma$ on the right-hand side of (3.14) to $\mathcal{C}''$, we might encounter several poles of the form (3.10). Then we move $(s_1,\ldots,s_N)$ from $(s_1^*,\ldots,s_N^*)$ to $(s_1^0,\ldots,s_N^0)$; again the path might encounter several poles of the same type. Hence, in a sufficiently small neighbourhood U of $(s_1^0,\ldots,s_N^0)$, the integral $J(s_1,\ldots,s_N)$ has the expression

$$
\begin{aligned}
&-R(s_1,\ldots,s_N)+\frac{1}{2\pi i}\int_{\mathcal{C}''}\frac{\Gamma(s_N+z)\Gamma(-z)}{\Gamma(s_N)}\Phi(s_1,\ldots,s_N)\\
&\times\zeta_r(s_1,\ldots,s_{N-1},s_N+z,-z;P_1,\ldots,P_{N-1},P_N^*,P_N^{**})dz,
\end{aligned}
\tag{3.19}
$$

where $R(s_1,\ldots,s_N)$ is the sum of residues of the above poles. Hence $R(s_1,\ldots,s_N)$ is a (finite) sum of residues of the form $\Gamma(s_N)^{-1}\Phi(s_1,\ldots,s_N)R(\ell_2)$, where

$$
\begin{aligned}
R(\ell_2)=&\frac{1}{(h-1)!}\frac{d^{h-1}}{dz^{h-1}}\Big\{(z-z(\ell_2))^h\Gamma(s_N+z)\Gamma(-z)\\
&\times\zeta_r(s_1,\ldots,s_{N-1},s_N+z,-z;P_1,\ldots,P_{N-1},P_N^*,P_N^{**})\Big\}\Big|_{z=z(\ell_2)}
\end{aligned}
\tag{3.20}
$$

with $z(\ell_2)=E(s_1,\ldots,s_N;\mathbf{c}_2)+e_0^{-1}\ell_2$, if the order of the pole is h. This implies that all possible singularities of $R(s_1,\ldots,s_N)$ are polar sets. Therefore expression (3.19) gives the meromorphic continuation of $J(s_1,\ldots,s_N)$ to U.

Now we have proved the meromorphic continuation of $\zeta_r(s_1,\ldots,s_N;P_1,\ldots,P_N)$. Next we show that all the possible polar sets of $\zeta_r(s_1,\ldots,s_N;P_1,\ldots,P_N)$ are of the form (3.3). This is clear for the polar sets of $\Phi(s_1,\ldots,s_N)^{-1}$. The polar sets of $J(s_1,\ldots,s_N)$ only appear in Case 3. Hence from condition (3.18) we see that those polar sets are also of the form (3.3). The assertions of Remarks 1 and 2 are easily verified from the above argument.

Lastly we show the assertion on the order of $\zeta_r(s_1,\ldots,s_N;P_1,\ldots,P_N)$. Since (3.7) satisfies an estimate of the form (3.4) by the induction assumption, the integral on the right-hand side of (3.15) clearly satisfies the same type of estimate with respect to $t_1,\ldots,t_{N-1}$. As for t_N, using Stirling's formula we find that the integral is

$$
\ll\int_{-\infty}^{\infty}\exp\left(\frac{\pi}{2}(|t_N|-|t_N+y|-|y|)+\theta_N(|t_N+y|+|y|)\right)\mathcal{F}(t_N,y)dy,\tag{3.21}
$$

which is $O(e^{\theta_N|t_N|}\mathcal{F}(t_N))$ by Lemma 4 of [26]. Hence we obtain the desired assertion in Case 1, and the treatment of Case 2 is similar.

In Case 3, we have to estimate $R(s_1,\ldots,s_N)$. Since

$$
R(\ell_2)=\frac{1}{2\pi i}\int_{\mathcal{K}}\Gamma(s_N+z)\Gamma(-z)\zeta_r(s_1,\ldots,s_{N-1},s_N+z,-z;P_1,\ldots,P_{N-1},P_N^*,P_N^{**})dz,
$$

where $\mathcal{K}$ is a small circle round the point $z(\ell_2)$, it is clear that $R(s_1,\ldots,s_N)$ satisfies an estimate of the form (3.4) with respect to $t_1,\ldots,t_{N-1}$. As for t_N, the relevant exponential factor is the same as the exponential factor in (3.21), hence we can obtain the desired estimate as above. The proof of Theorem 5 is now complete.

Acknowledgement. The author expresses his gratitude to Dr. Yasushi Komori for helping him to draw the figures, and to Professor Hirofumi Tsumura for pointing out several misprints in the original version of the manuscript.

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