On the Siegel-Tatuzawa theorem for a class of L-functions

Yumiko Ichihara*and Kohji Matsumoto

Abstract

Theorems of the Siegel-Tatuzawa type on exceptional zeros of L-functions are discussed in a general setting. We give certain conditions under which such theorems can be shown. In particular, we prove a theorem of the Siegel-Tatuzawa type for symmetric power L-functions under certain assumptions.

1 Introduction

Generally speaking, the value at $s = 1$ of twisted L-functions is deeply connected with real zeros of it. In the case of Dirichlet L-functions, the Siegel theorem shows that for any $\varepsilon > 0$, there exists a positive non-effective constant $C(\varepsilon)$ such that $L(1, \chi) > C(\varepsilon)d^{-\varepsilon}$, where χ is a real primitive Dirichlet character and d is the conductor of it (see [4]). This implies that $L(\sigma, \chi)$ does not have real zeros for $\sigma > 1 - C'(\varepsilon)d^{-\varepsilon}$, where the constant $C'(\varepsilon)$ is positive and non-effective. In 1951, Tatuzawa proved that $C(\varepsilon)$ can be effective except for at most one real character (see [16]).

Let f_1 be a Maass form with respect to the Hecke congruence subgroup $\Gamma_0(N)$, which is an eigenfunction of the Laplacian with the eigenvalue λ_1, and F_1 the adjoint square lift of f_1 in the sense of Gelbart-Jacquet [5]. Then it holds that

$$L(s, f_1 \otimes f_1) = \zeta(s)L_N(s)L(s, F_1),$$

where $L(s, f_1 \otimes f_1)$ is the Rankin-Selberg L-function associated with f_1, $L(s, F_1)$ is the L-function attached to F_1 and $L_N(s)$ atuzawais the product

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of bad Euler factors. iegelIn [8], Hoffstein and Lockhart proved the existence of an effective constant $c(\varepsilon) = c(\varepsilon, F_1) > 0$ for which

$$L(1, F_1) \geq c(\varepsilon)(\lambda_1 N)^{-\varepsilon}$$

holds for all F_1 with at most one exception. This is an analogue of the Siegel-Tatuzawa theorem. They mentioned that their method can be applied to holomorphic cusp forms of weight k. In fact, a modification of their method shows that an analogue of the Siegel-T theorem holds for the Rankin-Selberg L-function $L_{f\otimes f}$ associated with a holomorphic cusp form f. Recently it has been shown that a much stronger result, that is the non-existence of the Siegel zero, is true for a few types of L-functions (see the appendix of [8], and also [2], [9] and [14]). For instance, the above L-function $L_{f\otimes f}$ does not have the S zero.

In order to prove the Siegel-Tatuzawa type of theorems, a standard method is to introduce a suitable auxiliary Dirichlet series which is constructed by a product of several related L-functions. In the above cases, such auxiliary Dirichlet series (see [4] and [8]) have a simple pole at $s = 1$ and non-negative coefficients. These conditions have been essentially used in the known classical proofs of the Siegel-Tatuzawa type of theorems.

In general, it is often not difficult to find an auxiliary Dirichlet series constructed by a product of some L-functions and has non-negative coefficients, but it is not certain that the order of the pole of it at $s = 1$ is simple. In this paper we propose a condition ((H2) below) on the order of the pole, which is suitable for the purpose of discussing the matter of Siegel-Tatuzawa type of theorems. If the order of the pole satisfies (H2) and is odd, then our argument is basically similar to that of Hoffstein and Lockhart [8]. However, if the order of the pole is even, we have to introduce further new ideas to prove Siegel-Tatuzawa type of theorems.

In literature, some auxiliary series with poles of even order have been introduced to show Siegel type of theorems. For example, in the case of Rankin-Selberg L-functions $L_{f\otimes g}$ associated with two cusp forms $f \neq g$, the first author introduced an auxiliary Dirichlet series for the proof of the analogue of Siegel's theorem for $L_{f\otimes g}$ (see [10]). It has non-negative coefficients, but the order of the pole of it is two. (Recently, Ramakrishnan and Wang proved that $L_{f\otimes g}$ does not have the Siegel zero (see [14])). In the case of L-function L_f associated with a cusp form f, Golubeva and Fomenko introduced an auxiliary Dirichlet series for the proof of the analogue of Siegel's theorem for L_f. It also has positive coefficients and a double pole at $s = 1$ (see [6]). (In 1995 Hoffstein and Ramakrishnan proved L_f does not have the Siegel zero (see [9])). They also considered an auxiliary Dirichlet series which has positive coefficients and a pole of order 4 in [7]. However, those auxiliary series seem to be not suitable to handle Siegel-Tatuzawa type of theorems.

In this paper, we introduce a new type of auxiliary Dirichlet series and develop a method of proving the Siegel-Tatuzawa type of theorem for general L-functions.

Let $s = \sigma + it$, χ be a real Dirichlet character of the modulus d, k be a positive integer, and consider the general L-functions $L_k(s, \chi)$ defined by the Euler product and satisfying assumptions (A1)-(A3) below. We put

$$L_k(s,\chi) = \prod_{p:\text{prime}} \prod_{j=1}^{J(k)} \left(1 - \frac{a_k(j,p)\chi(p)}{p^s}\right)^{-1},$$

where $J(k)$ is a natural number and $a_k(j,p)$ are complex numbers with $|a_k(j,p)| \le 1$, and further, $|a_k(j,p)| = 1$ for almost all prime numbers. This L-function is convergent absolutely for $\sigma > 1$. The assumptions of the L-function $L_k(s,\chi)$ are as follows.

(A1) The L-function $L_k(s,\chi)$ can be continued meromorphically to the whole plane. It has a possible pole at $s = 1$ if the character χ is principal, while it is entire if χ is non-principal.

(A2) There exists an absolute constant $0 < \delta_k < 1/2$ such that for any $\varepsilon > 0$

$$L_k(s,\chi) \ll \exp\left(\exp(\varepsilon|t|)\right), \qquad |t| \to \infty,$$

uniformly in $-\delta_k \le \sigma \le 1 + \delta_k$.

(A3) Let χ be primitive. Then there exists a natural number $N(k)$, real numbers $\alpha_\nu(k) > 0$ and complex numbers $\beta_\nu(k,\chi)$ $(1 \le \nu \le N(k))$ such that

$$\widetilde{L_k}(s,\chi) = W_{k,\chi}\widetilde{L_k}(1-s,\chi),$$

where

$$\widetilde{L_k}(s,\chi) = Q_{k,\chi}^s \prod_{\nu=1}^{N(k)} \Gamma\left(\alpha_\nu(k)s + \beta_\nu(k,\chi)\right) L_k(s,\chi).$$

Here $Q_{k,\chi}$ is a real number satisfying $Q_{k,\chi} \ll d^{\gamma(k)}$ ($\gamma(k)$ is a natural number) and $W_{k,\chi}$ is a complex number with $|W_{k,\chi}| = 1$.

Remark 1. Here, we assume the Ramanujan type of condition on the magnitude of $a_k(j,p)$. But it seems that this is not always necessary. In this paper, we use (3) which has been proved by Carletti, Monti Bragadin and Perelli in [3]. They proved it under the Ramanujan condition, but Molteni proved it under a different type of assumption instead of the Ramanujan condition (see [11]).

The plan of this paper is as follows. In Section 2 we give the statement of our main theorem (Theorem 1). For the preparation of the proof of it, we show some lemmas in Section 3, and we prove the main theorem in Section 4. In Section 5, we apply the main theorem to the case of symmetric power L-functions to obtain the Siegel-Tatuzawa theorem for them (Theorem 2).

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2 The statement of the main theorem

We fix an L-function $L_1(s,\chi)$ satisfying assumptions (A1)-(A3), where χ is a real Dirichlet character. Our purpose is to prove an analogue of Siegel-Tatuzawa theorem for this L-function $L_1(s,\chi)$. We assume that there exist suitable L-functions $L_k(s,\chi)$ $(2 \le k \le K)$ satisfying assumptions (A1)-(A3) and natural numbers e_k such that the function

$$\Lambda(s,\chi) = \prod_{k=1}^{K} L_k(s,\chi)^{e_k}$$

satisfy hypotheses (H1) and (H2) below. For $\sigma > 1$, we have

$$\log \Lambda(s,\chi) = \sum_{p:\text{prime}} \sum_{h=1}^{\infty} \frac{\chi^h(p)}{hp^{hs}} \sum_{k=1}^{K} \sum_{j=1}^{J(k)} a_k(j,p)^h e_k. \qquad (1)$$

We put $b(h,p) = \sum_{k=1}^{K} \sum_{j=1}^{J(k)} a_k(j,p)^h e_k$. Then

(H1) The coefficients $b(h,p)$ are non-negative.

(H2) In the case $\chi = \chi_0$ is the trivial character, we use the notation $\Lambda(s)$ in place of $\Lambda(s,\chi_0)$, and we denote by r the order of the pole of $\Lambda(s)$ at $s = 1$. Then $1 \le r \le e_1$.

These hypotheses are keys in the proof of the Siegel-Tatuzawa theorem for $L_1(s,\chi)$. Concerning $L_1(s,\chi)$, we add one more hypothesis:

(H3) There exists an effective positive constant C_1 such that $L_1(s,\chi)$ has no zeros in the region
$$\sigma > 1 - \frac{C_1}{\log d}$$
for $0 < |t| \le 1$.

Remark 2. The hypothesis (H3) is automatically satisfied in many cases. In fact, let
$$L_1^*(s,\chi) = \prod_{p:\mathrm{prime}} \prod_{j=1}^{J(1)} \left(1 - \frac{\overline{a}_1(j,p)\chi(p)}{p^s}\right)^{-1},$$
where $\overline{a}_1(j,p)$ is the complex conjugate of $a_1(j,p)$. If $L_1^*(s,\chi) = L_1(s,\chi)$, or more generally, if the function $\Lambda(s,\chi)$ has $L_1^*(s,\chi)$ as one of the factors, we can show that there is an effective positive constant C_1 such that $L_1(s,\chi) \ne 0$ for
$$\sigma > 1 - \frac{C_1}{\log\left(d(|t|+2)\right)}$$
except for the real axis, by using the ordinary method under hypotheses (H1) and (H2). Therefore especially we find that $L_1(s,\chi)$ satisfies hypothesis (H3). (See the end of Section 3.)

Our main theorem in the present paper can be stated as follows.

Theorem 1. *We denote by X the set of all real primitive Dirichlet characters. If we find L-functions $L_k(s,\chi)$ satisfying assumptions (A1)-(A3) and hypotheses (H1)-(H3), then for any $\varepsilon > 0$, there exists an effective positive constant $C(\varepsilon)$ such that*
$$|L_1(1,\chi)| > \frac{C(\varepsilon)}{d^{\varepsilon}} \tag{2}$$
for any $\chi \in X$, except for at most one possible element of X. Here d is the conductor of χ.

Remark 3. The important point in the theorem is that the constant $C(\varepsilon)$ is effective. If we do not require the effectiveness of the constant, the inequality of the form (2) can be shown for any $\chi \in X$ without exception. This claim is an analogue of Siegel's theorem. As is explained at the end of Section 4, this can be deduced easily from Theorem 1.

3 Preliminaries

First of all, we recall some known facts. Assumptions (A1)-(A3) imply that our $L_k(s,\chi)$ is an example of "general L-functions" in the sense of Carletti, Monti Bragadin and Perelli [3] if χ is primitive. Therefore we have

$$L_k^{(n)}(1,\chi) \ll d^{\varepsilon} \tag{3}$$

for any non-negative integer n and an arbitrarily small number $\varepsilon > 0$ where χ is a non-principal Dirichlet character of the modulus d. This can be easily reduced to the case when χ is primitive, and follows from Theorem 2 in [3] for primitive χ. For $-1 \le \sigma \le 2$ it holds uniformly that

$$-\frac{L_k'}{L_k}(s,\chi) = \frac{m(k)}{s-1} - \sum_{|t-\gamma|\le 1} \frac{1}{s-\rho} + O\left(\log(d(|t|+2))\right), \tag{4}$$

where the order of pole of $L_k(s,\chi)$ at $s=1$ is denoted by $m(k)$, $\rho = \beta + i\gamma$ runs over the zeros of $L_k(s,\chi)$ and χ is a Dirichlet character of the modulus d. This is again reduced to the primitive case, which is Lemma 4 in [3].

We put

$$\phi(s,\chi) = \Lambda(s)\Lambda(s,\chi),$$
$$\phi(s,\chi_1,\chi_2) = \Lambda(s)\Lambda(s,\chi_1)\Lambda(s,\chi_2)\Lambda(s,\chi_1\chi_2),$$

where the Dirichlet characters χ, χ_1 and χ_2 are primitive and real with the conductors $d>1$, $d_1>1$ and $d_2>1$, respectively. Moreover $\chi_1\chi_2$ is not trivial. These functions are the analogue of classical auxiliary functions in the theory of the zeros of Dirichlet L-functions. A novelty of the present paper is to introduce the new auxiliary function

$$\psi(s,\chi) = \zeta(s)\phi(s,\chi),$$

where $\zeta(s)$ is the Riemann zeta function. This function is helpful for the proof of the main theorem when there is no real zero of $L_1(s,\chi)$ near $s=1$. The functions $\phi(s,\chi)$, $\phi(s,\chi_1,\chi_2)$ and $\psi(s,\chi)$ are convergent absolutely for

$\sigma > 1$. We write the Dirichlet series expansion of them as follows;

$$\phi(s,\chi) = \sum_{n=1}^{\infty} \delta(n) n^{-s},$$
$$\phi(s,\chi_1,\chi_2) = \sum_{n=1}^{\infty} \lambda(n) n^{-s},$$
$$\psi(s,\chi) = \sum_{n=1}^{\infty} \omega(n) n^{-s}.$$

From (1) we have

$$\log \phi(s,\chi) = \sum_{p:\text{prime}} \sum_{h=1}^{\infty} \frac{b(h,p)}{h} \left(1+\chi^h(p)\right) p^{-hs} \tag{5}$$

and

$$\log \phi(s,\chi_1,\chi_2) = \sum_{p:\text{prime}} \sum_{h=1}^{\infty} \frac{b(h,p)}{h} \left(1+\chi_1^h(p)\right)\left(1+\chi_2^h(p)\right) p^{-hs}, \tag{6}$$

hence hypothesis (H1) implies that $\delta(1) = \lambda(1) = \omega(1) = 1$ and $\delta(n)$, $\lambda(n)$ and $\omega(n)$ are non-negative real numbers for $n > 1$. We prove the following four lemmas on these functions. The residue of the function $f(s)$ at $s = a$ is denoted by $\mathrm{Res}_{s=a}(f(s))$.

Lemma 1. *Let β_0 be a real number with $3/4 < \beta_0 < 1$ and ℓ be a sufficiently large natural number. Then there exists an effective constant $c_1 > 0$, independent of β_0, such that*

$$\frac{\phi(\beta_0,\chi)}{\ell!} + \operatorname*{Res}_{s=1-\beta_0}\left(\frac{\phi(s+\beta_0,\chi)d^{c_1 s}}{s(s+1)\cdots(s+\ell)}\right) \gg 1.$$

Lemma 2. *The definition of β_0 and ℓ are the same as in Lemma 1. Then there exists an effective constant $c_2 > 0$, independent of β_0, such that*

$$\frac{\phi(\beta_0,\chi_1,\chi_2)}{\ell!} + \operatorname*{Res}_{s=1-\beta_0}\left(\frac{\phi(s+\beta_0,\chi_1,\chi_2)(d_1 d_2)^{c_2 s}}{s(s+1)\cdots(s+\ell)}\right) \gg 1.$$

Lemma 3. *The definition of β_0 and ℓ are the same as in Lemma 1. Then there exists an effective constant $c_3 > 0$, independent of β_0, such that*

$$\frac{\psi(\beta_0, \chi)}{\ell!} + \operatorname*{Res}_{s=1-\beta_0} \left(\frac{\psi(s+\beta_0, \chi) d^{c_3 s}}{s(s+1)\cdots(s+\ell)} \right) \gg 1.$$

Lemma 4. *There exists an effective positive constant c_4 such that $\phi(s, \chi_1, \chi_2)$ has at most r real zeros (counted with multiplicity) in the range*

$$1 - \frac{c_4}{\log d_1 d_2} < \sigma < 1.$$

Lemma 1, Lemma 2 and Lemma 3 are analogues of Proposition 1.1 in Hoffstein and Lockhart [8]. We only show the proofs of Lemma 1 and Lemma 4, because the proofs of Lemma 2 and Lemma 3 are similar to the proof of Lemma 1.

Proof of Lemma 1. Let $x > 2$ be not an integer. It is known that

$$\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{x^s ds}{s(s+1)(s+2)\cdots(s+\ell)} = \begin{cases} \frac{1}{\ell!}\left(1 - \frac{1}{x}\right)^{\ell} & \text{if } x > 1, \\ 0 & \text{if } 0 < x \leq 1, \end{cases}$$

for any integer ℓ. By using the above formula, we have

$$\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{\phi(s+\beta_0, \chi) x^s}{s(s+1)(s+2)\cdots(s+\ell)} ds = \sum_{n<x} \frac{\delta(n)}{n^{\beta_0}} \frac{1}{\ell!} \left(1 - \frac{n}{x}\right)^{\ell} > \frac{1}{\ell! 2^{\ell}} \quad (7)$$

because $\delta(n)$ are non-negative and $\delta(1) = 1$. We shift the path of integration on the left hand side of (7) to $\sigma = 1/2 - \beta_0$. From assumptions (A2), (A3) and the Phragmén-Lindelöf theorem, we obtain

$$L_k(\sigma + it, \chi) \ll d^{\gamma(k)(1+\varepsilon-\sigma)} (1 + |t|)^{A(k)(1+\varepsilon-\sigma)}$$

and

$$L_k(\sigma + it) \ll (1 + |t|)^{A(k)(1+\varepsilon-\sigma)} \qquad (t \neq 0)$$

for $-\varepsilon < \sigma < 1 + \varepsilon$, where $A(k) = \sum_{\nu=1}^{N(k)} \alpha_\nu(k)$. Hence

$$\phi(s, \chi) \ll d^{\gamma(K)(1+\varepsilon-\sigma)} (1 + |t|)^{2A(K)(1+\varepsilon-\sigma)},$$

where $\gamma(K) = \sum_{k=1}^{K} \gamma(k) e_k$ and $A(K) = \sum_{k=1}^{K} A(k) e_k$. By using this estimate, we obtain that there exists a sufficiently large $\ell_0 = \ell_0(A(K))$ such that

$$\begin{aligned}\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} &\frac{\phi(s+\beta_0,\chi) x^s}{s(s+1)(s+2)\cdots(s+\ell)} ds \\ &= \frac{\phi(\beta_0,\chi)}{\ell!} + \operatorname*{Res}_{s=1-\beta_0} \left(\frac{\phi(s+\beta_0,\chi)x^s}{s(s+1)(s+2)\cdots(s+\ell)} \right) \\ &\quad + O\left(d^{\gamma(K)(1/2+\varepsilon)} x^{1/2-\beta_0}\right) \end{aligned} \tag{8}$$

for any $\ell \geq \ell_0$. Then results (7) and (8) imply that

$$\frac{\phi(\beta_0,\chi)}{\ell!} + \operatorname*{Res}_{s=1-\beta_0} \left(\frac{\phi(s+\beta_0,\chi)x^s}{s(s+1)\cdots(s+\ell)} \right) + O\left(d^{\gamma(K)(1/2+\varepsilon)} x^{1/2-\beta_0}\right) \geq \frac{1}{\ell! 2^\ell}. \tag{9}$$

We put $x = d^{c_1}$. It is easy to find an effective constant $c_1 > 0$ such that the error term on the left hand side of (9) is smaller than $(\ell! 2^{\ell+1})^{-1}$. We complete the proof of Lemma 1. □

Proof of Lemma 4. From (4) we have

$$-\frac{\phi'}{\phi}(s,\chi_1,\chi_2) = \frac{r}{s-1} - \sum_{|t-\gamma|\leq 1} \frac{1}{s-\rho} + O\left(\log(d_1 d_2(|t|+2))\right) \tag{10}$$

for $-1 \leq \sigma \leq 2$, where $\rho = \beta + i\gamma$ runs over the zeros of $\phi(s,\chi_1,\chi_2)$. We know

$$-\frac{\phi'}{\phi}(\sigma,\chi_1,\chi_2) \geq 0 \tag{11}$$

for $\sigma > 1$ from (6). We assume that there exist $r+1$ real zeros $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_{r+1}$ of $\phi(s,\chi_1,\chi_2)$. We obtain that

$$0 \leq \frac{r}{\sigma-1} - \sum_{j=1}^{r+1} \frac{1}{\sigma-\rho_j} + A_1 \log d_1 d_2 \leq \frac{r}{\sigma-1} - \frac{r+1}{\sigma-\rho_1} + A_1 \log d_1 d_2$$

for $\sigma > 1$ by using (10) and (11), where A_1 is an effective positive constant. We put $0 < A_2 < 1/A_1$ and $\sigma = 1 + A_2/\log d_1 d_2$. Then it follows that there exists an effective positive constant c_4 such that

$$\rho_1 \leq 1 - \frac{c_4}{\log d_1 d_2}.$$

□

Lastly in this section we explain how to show the claim of Remark 2. This can be done quite similarly to the argument in Section 14 of [4]. In fact, if $\rho = \beta + i\gamma$ is a zero of $L_1(s,\chi)$ with $|\gamma| \geq 1/2$, then, using (4), we can show

$$\frac{4e_1}{\sigma - \beta} \leq \frac{3r}{\sigma - 1} + \Re\left(\frac{r}{\sigma - 1 + 2i\gamma}\right) + O\left(\log(d(|\gamma| + 2))\right)$$

for $\sigma > 1$, from which

$$\beta \leq 1 - \frac{C_1}{\log(d(|\gamma| + 2))}$$

follows under hypothesis (H2). If $0 < |\gamma| < 1/2$, we use the fact that not only ρ but also $\overline{\rho} = \beta - i\gamma$ are roots of $\Lambda(s,\chi) = 0$. This follows from the assumption of Remark 1. By using this fact we obtain

$$\begin{aligned}\frac{8e_1}{\sigma - \beta} &\leq \frac{3r}{\sigma - 1} + \Re\left(\frac{r}{\sigma - 1 + 2i\gamma}\right) + O\left(\log(d(|\gamma| + 2))\right) \\ &\leq \frac{4r}{\sigma - 1} + O\left(\log(d(|\gamma| + 2))\right),\end{aligned}$$

which implies the desired result.

4 The proof of the main theorem

We fix a sufficiently small real number ε_1 with $0 < \varepsilon_1 < 1/4$. We divide the argument into three cases.

Case 1. If r is an odd (resp. even) number, we consider the case that $\phi(s,\chi)$ has even (resp. odd) number of real zeros in the range $1 - \varepsilon_1 \leq \sigma \leq 1$. We see that $\phi(s,\chi)$ tends to $-\infty$ (resp. $+\infty$) as $s \to 1 - 0$ if r is odd (resp. even) number. Therefore we have $\phi(1 - \varepsilon_1, \chi) \leq 0$. We put $\beta_0 = 1 - \varepsilon_1$. By using Lemma 1, we obtain

$$\operatorname*{Res}_{s=\varepsilon_1}\left(\frac{\phi(s + 1 - \varepsilon_1)d^{c_1 s}}{s(s+1)\cdots(s+\ell)}\right) \gg 1. \tag{12}$$

We write the Laurant expansion of $\phi(s,\chi)$ at $s = 1$ as

$$\phi(s,\chi) = \sum_{j=-r}^{\infty} \alpha_j (s-1)^j,$$

where $\alpha_{-r} \neq 0$. Then we see that

$$\operatorname*{Res}_{s=\varepsilon_1}\left(\frac{\phi(s+1-\varepsilon_1)d^{c_1 s}}{s(s+1)\cdots(s+\ell)}\right) = \sum_{\substack{-r\le j,\ 0\le m,n_0,\dots n_\ell \\ j+m+n_0+\cdots+n_\ell=-1}} \frac{(-1)^{n_0+\cdots+n_\ell}\alpha_j d^{c_1\varepsilon_1}(c_1\log d)^m}{m!{\varepsilon_1}^{n_0+1}(\varepsilon_1+1)^{n_1+1}\cdots(\varepsilon_1+\ell)^{n_\ell+1}}. \quad (13)$$

From (12) and (13) we obtain

$$d^{c_1\varepsilon_1} \sum_{\substack{-r\le j,\ 0\le m,n_0,\dots n_\ell \\ j+m+n_0+\cdots+n_\ell=-1}} \alpha_j(\log d)^m \gg_{\varepsilon_1} 1. \quad (14)$$

Next we consider the coefficients α_j for $-r \le j \le -1$. We write the Laurant expansions of $\Lambda(s)$ and $L_k(s,\chi)$ as

$$\Lambda(s) = \sum_{j=-r}^{\infty} \beta_j(s-1)^j$$

and

$$L_k(s,\chi) = \sum_{m_k=0}^{\infty} \frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k},$$

respectively. We have

$$\begin{aligned}\phi(s,\chi) = &\left(\sum_{j=-r}^{\infty}\beta_j(s-1)^j\right)\\ &\times\left\{\sum_{l_1=0}^{e_1}\binom{e_1}{e_1-l_1}L_1(1,\chi)^{e_1-l_1}\left(\sum_{m_1=1}^{\infty}\frac{L_1^{(m_k)}(1,\chi)}{m!}(s-1)^{m_1}\right)^{l_1}\right\}\\ &\times\prod_{k=2}^{K}\left(\sum_{m_k=0}^{\infty}\frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k}\right)^{e_k}. \quad (15)\end{aligned}$$

Therefore we see that the terms $\alpha_j(s-1)^j$, $-r \le j \le -1$, appear in the expansion of the following part of the right hand side of (15);

$$\left(\sum_{j=-r}^{-1}\beta_j(s-1)^j\right)$$

$$
\begin{aligned}
&\times \left\{ \sum_{l_1=0}^{r-1} \binom{e_1}{e_1-l_1} L_1(1,\chi)^{e_1-l_1} \left(\sum_{m_1=1}^{r-1} \frac{L_1^{(m_k)}(1,\chi)}{m!}(s-1)^{m_1} \right)^{l_1} \right\} \\
&\times \prod_{k=2}^{K} \left(\sum_{m_k=0}^{r-1} \frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k} \right)^{e_k} \\
=&L_1(1,\chi) \left(\sum_{j=-r}^{-1} \beta_j (s-1)^j \right) \\
&\times \left\{ \sum_{l_1=0}^{r-1} \binom{e_1}{e_1-l_1} L_1(1,\chi)^{e_1-l_1-1} \left(\sum_{m_1=1}^{r-1} \frac{L_1^{(m_k)}(1,\chi)}{m!}(s-1)^{m_1} \right)^{l_1} \right\} \\
&\times \prod_{k=2}^{K} \left(\sum_{m_k=0}^{r-1} \frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k} \right)^{e_k}. \qquad (16)
\end{aligned}
$$

Recalling hypothesis (H2), we see that $e_1 - l_1 - 1 \geq 0$. By using (3) and (16), we obtain that $\alpha_j \ll_{\varepsilon_1} |L_1(1,\chi)| d^{c\varepsilon_1}$ for $-r \leq j \leq -1$. Here, and in what follows, c is an effective positive constant, not necessarily the same at each occurrence. This estimate and (14) imply (2).

Case 2. If r is an even (resp. odd) number, we consider the case that $\phi(s,\chi)$ has even (resp. odd) number of real zeros in the range $1-\varepsilon_1 \leq \sigma \leq 1$ and $L_1(s,\chi)$ has no real zeros in the same range. Then $\psi(s,\chi)$ also has even (resp. odd) number of real zeros if r is even (resp. odd) number. The order of the pole of $\psi(s,\chi)$ at $s=1$ is $r+1$, hence $\psi(s,\chi)$ tends to $-\infty$ (resp. $+\infty$) as $s \to 1-0$ if r is even (resp. odd) number. Therefore we have $\psi(1-\varepsilon_1,\chi) \leq 0$. We put $\beta_0 = 1-\varepsilon_1$. By using Lemma 3, we obtain

$$
\operatorname*{Res}_{s=1-\beta} \left(\frac{\psi(s+\beta,\chi) d^{c_3 s}}{s(s+1)\cdots(s+\ell)} \right) \gg 1.
$$

We write the Laurant expansion of $\psi(s,\chi)$ at $s=1$ as

$$
\psi(s,\chi) = \sum_{j=-r-1}^{\infty} \alpha'_j (s-1)^j,
$$

where $\alpha'_{-r-1} \neq 0$. We obtain that

$$
d^{c_3\varepsilon_1} \sum_{\substack{-r-1 \leq j,\ 0 \leq m, n_0, \ldots n_\ell \\ j+m+n_0+\cdots+n_\ell=-1}} \alpha'_j (\log d)^m \gg_{\varepsilon_1} 1, \qquad (17)
$$

by using the same argument as in the proof of (14). We put

$$\zeta(s)\Lambda(s)=\sum_{j=-r-1}^{\infty}{\beta'}_j(s-1)^j.$$

It is clear that $\beta'_{-r-1}\neq 0$. We have

$$\begin{aligned}\psi(s,\chi)=&\left(\sum_{j=-r-1}^{\infty}{\beta'}_j(s-1)^j\right)\\&\times\left\{\sum_{l_1=0}^{e_1}\binom{e_1}{e_1-l_1}L_1(1,\chi)^{e_1-l_1}\left(\sum_{m_1=1}^{\infty}\frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1}\right)^{l_1}\right\}\\&\times\prod_{k=2}^{K}\left(\sum_{m_k=0}^{\infty}\frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k}\right)^{e_k}.\end{aligned}\tag{18}$$

We rearrange the above formula as follows;

$$\begin{aligned}\psi(s,\chi)=&\left(\sum_{j=-r-1}^{\infty}{\beta'}_j(s-1)^j\right)\left\{\left(\sum_{m_1=1}^{\infty}\frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1}\right)^{e_1}\right.\\&\left.+\sum_{l_1=0}^{e_1-1}\binom{e_1}{e_1-l_1}L_1(1,\chi)^{e_1-l_1}\left(\sum_{m_1=1}^{\infty}\frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1}\right)^{l_1}\right\}\\&\times\prod_{k=2}^{K}\left(\sum_{m_k=0}^{\infty}\frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k}\right)^{e_k}\\=&\left(\sum_{j=e_1-r-1}^{\infty}{\beta'}_{j-e_1}(s-1)^j\right)\left(\sum_{m_1=1}^{\infty}\frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1-1}\right)^{e_1}\\&\times\prod_{k=2}^{K}\left(\sum_{m_k=0}^{\infty}\frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k}\right)^{e_k}\\&+\left(\sum_{j=-r-1}^{\infty}{\beta'}_j(s-1)^j\right)L_1(1,\chi)\\&\times\left\{\sum_{l_1=0}^{e_1-1}\binom{e_1}{e_1-l_1}L_1(1,\chi)^{e_1-l_1-1}\left(\sum_{m_1=1}^{\infty}\frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1}\right)^{l_1}\right\}\end{aligned}$$

$$\times \prod_{k=2}^{K} \left(\sum_{m_k=0}^{\infty} \frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k} \right)^{e_k}. \tag{19}$$

Therefore we see that the terms $\alpha'_j(s-1)^m$, $-r-1 \le m \le -2$, appear in the expansion of the following part of the right hand side of (18);

$$\left(\sum_{j=-r-1}^{-2} \beta'_j (s-1)^j \right) L_1(1,\chi)$$
$$\times \left\{ \sum_{l_1=0}^{r-1} \binom{e_1}{e_1-l_1} L_1(1,\chi)^{e_1-l_1-1} \left(\sum_{m=1}^{r-1} \frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1} \right)^{l_1} \right\}$$
$$\times \prod_{k=2}^{K} \left(\sum_{m_k=0}^{r-1} \frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k} \right)^{e_k}. \tag{20}$$

Recalling hypothesis (H2), we see that $e_1 - l_1 - 1 \ge 0$. By using (3) and (20) we obtain that $\alpha'_j \ll_{\varepsilon_1} |L_1(1,\chi)| d^{c\varepsilon_1}$ for $-r-1 \le j \le -2$. We consider the term of $\alpha'_{-1}(s-1)^{-1}$. We see that it appears in the expansion of the following part of the right hand side of (19);

$$\beta'_{-r-1}(s-1)^{e_1-r-1} L_1'(1,\chi)^{e_1} \prod_{k=2}^{K} L_k(1,\chi)^{e_k} + \left(\sum_{j=-r-1}^{-1} \beta'_j (s-1)^j \right)$$
$$\times L_1(1,\chi) \left\{ \sum_{l_1=0}^{r} \binom{e_1}{e_1-l_1} L_1(1,\chi)^{e_1-l_1-1} \left(\sum_{m_1=1}^{r} \frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1} \right)^{l_1} \right\}$$
$$\times \prod_{k=2}^{K} \left(\sum_{m_k=0}^{r} \frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k} \right)^{e_k} \tag{21}$$

because we have that $e_1 - r - 1 \ge -1$ from hypothesis (H2). By using (3) and (21), we see that $\alpha'_{-1} \ll |L_1(1,\chi)| d^{c\varepsilon_1}$, if $L_1'(1,\chi) \ll_{\varepsilon_1} |L_1(1,\chi)| d^{c\varepsilon_1}$ is true. The proof of the latter estimate is as follows. We have

$$\frac{L_1'}{L_1}(1,\chi) = \sum_{\rho:\text{real}} \frac{1}{1-\rho} + \sum_{0<|\Im(\rho)|<1} \frac{1}{1-\rho} + O(\log d) \tag{22}$$

from (4), were ρ runs over the zeros of $L_1(s,\chi)$. We recall that there is no real zero of $L_1(s,\chi)$ in the range $1-\varepsilon_1 \le \sigma \le 1$ by the assumption of Case 2.

This implies that

$$\sum_{\rho:\text{real}} \frac{1}{1-\rho} \le \sum_{\rho:\text{real}} \frac{1}{\varepsilon_1} \ll_{\varepsilon_1} \log d \tag{23}$$

by using Theorem 1 in Perelli [13]. Also we have

$$\sum_{0<|\Im(\rho)|<1} \frac{1}{1-\rho} \ll \sum_{0<|\Im(\rho)|<1} \frac{\log d}{C_1} \ll (\log d)^2 \tag{24}$$

by using hypothesis (H3) and Theorem 1 in [13]. Consequently we obtain that $L_1'(1,\chi) \ll_{\varepsilon_1} |L_1(1,\chi)| d^{c\varepsilon_1}$ from (22), (23) and (24). Hence we obtain that $\alpha_j' \ll_{\varepsilon_1} |L_1(1,\chi)| d^{c\varepsilon_1}$ for $-r-1 \le j \le -1$. This estimate and (17) imply (2).

Case 3. Finally we consider the case when r is an even (resp. odd) number, $\phi(s,\chi)$ has even (resp. odd) number of real zeros in the range $1-\varepsilon_1 \le \sigma \le 1$, and $L_1(s,\chi)$ has some real zeros in the same range. We denote by X^* the set of all real Dirichlet characters satisfying the assumptions of Case 3. There is a minimal conductor of the character in X^* and we denote it by d_2. We fix a character χ_2 with modulus d_2 in X^*. We shall show the theorem for $L_1(s,\chi_1)$ with any character $\chi_1 \in X^*$ which is not equal to χ_2. Let d_1 be the modulus of χ_1. Now we know that $\Lambda(s,\chi_1,\chi_2)$ has at least two real zeros ρ_1, ρ_2 in the range $1-\varepsilon_1 \le \sigma \le 1$. We may assume $\rho_1 \le \rho_2$. Then $\phi(s,\chi_1,\chi_2)$ has at least $2e_1$ $(>r)$ real zeros. We put $\beta_0 = \rho_1$. By using Lemma 2, we obtain

$$\operatorname*{Res}_{s=1-\rho_1} \left(\frac{\phi(s+\rho_1,\chi_1,\chi_2)(d_1 d_2)^{c_2 s}}{s(s+1)\cdots(s+\ell)} \right) \gg 1.$$

We write the Laurant expansion of $\phi(s,\chi_1,\chi_2)$ at $s=1$ as

$$\phi(s,\chi_1,\chi_2) = \sum_{j=-r}^{\infty} \alpha_j''(s-1)^j.$$

We have that $\varepsilon_1 \ge 1-\rho_1 \ge c_4/(\log d_1 d_2)$ from Lemma 4 and the assumption of Case 3. By using these inequalities and the same argument as in the proof of (14), we obtain that

$$(d_1 d_2)^{c_2\varepsilon_1} \sum_{\substack{-r\le j,\ 0\le m,n_0,\dots n_\ell \\ j+m+n_0+\cdots+n_\ell=-1}} \alpha_j''(\log d_1 d_2)^{m+n_0+1} \gg_{\varepsilon_1} 1. \tag{25}$$

We have

$$
\begin{aligned}
&\phi(s,\chi_1,\chi_2)\\
&=\left(\sum_{j=-r}^{\infty}\beta_j(s-1)^j\right)\prod_{k_1=1}^{K}\left(\sum_{m_{k_1}=0}^{\infty}\frac{L_{k_1}^{(m_{k_1})}(1,\chi_1)}{m_{k_1}!}(s-1)^{m_{k_1}}\right)^{e_{k_1}}\\
&\quad\times\prod_{k_2=1}^{K}\left(\sum_{m_{k_2}=0}^{\infty}\frac{L_{k_2}^{(m_{k_2})}(1,\chi_2)}{m_{k_2}!}(s-1)^{m_{k_2}}\right)^{e_{k_2}}\\
&\quad\times\prod_{k_3=1}^{K}\left(\sum_{m_{k_3}=0}^{\infty}\frac{L_{k_3}^{(m_{k_3})}(1,\chi_1\chi_2)}{m_{k_3}!}(s-1)^{m_{k_3}}\right)^{e_{k_3}}\\
&=\left(\sum_{j=-r}^{\infty}\beta_j(s-1)^j\right)\\
&\quad\times\left\{\sum_{l_1=0}^{e_1}\binom{e_1}{e_1-l_1}L_1(1,\chi)^{e_1-l_1}\left(\sum_{m_1=1}^{\infty}\frac{L_1^{(m_1)}(1,\chi_1)}{m_1!}(s-1)^{m_1}\right)^{l_1}\right\}\\
&\quad\times\prod_{k_1=2}^{K}\left(\sum_{m_{k_1}=0}^{\infty}\frac{L_{k_1}^{(m_{k_1})}(1,\chi_1)}{m_{k_1}!}(s-1)^{m_{k_1}}\right)^{e_{k_1}}\\
&\quad\times\prod_{k_2=1}^{K}\left(\sum_{m_{k_2}=0}^{\infty}\frac{L_{k_2}^{(m_{k_2})}(1,\chi_2)}{m_{k_2}!}(s-1)^{m_{k_2}}\right)^{e_{k_2}}\\
&\quad\times\prod_{k_3=1}^{K}\left(\sum_{m_{k_3}=0}^{\infty}\frac{L_{k_3}^{(m_{k_3})}(1,\chi_1\chi_2)}{m_{k_3}!}(s-1)^{m_{k_3}}\right)^{e_{k_3}}.
\end{aligned}
$$

Therefore we see that the terms $\alpha''_j(s-1)^j$, $-r\le j\le -1$, appear in the expansion of the following part of the right hand side of the above formula;

$$
\begin{aligned}
&\left(\sum_{j=-r}^{-1}\beta_j(s-1)^j\right)L_1(1,\chi_1)\\
&\times\left(\sum_{l_1=0}^{r-1}\binom{e_1}{e_1-l_1}L_1(1,\chi)^{e_1-l_1-1}\left(\sum_{m_1=1}^{r-1}\frac{L_1^{(m_1)}(1,\chi_1)}{m_1!}(s-1)^{m_1}\right)^{l_1}\right)
\end{aligned}
$$

$$\times \prod_{k_1=2}^{K} \left(\sum_{m_{k_1}=0}^{r-1} \frac{L_{k_1}^{(m_{k_1})}(1,\chi_1)}{m_{k_1}!}(s-1)^{m_{k_1}} \right)^{e_{k_1}}$$
$$\times \prod_{k_2=1}^{K} \left(\sum_{m_{k_2}=0}^{r-1} \frac{L_{k_2}^{(m_{k_2})}(1,\chi_2)}{m_{k_2}!}(s-1)^{m_{k_2}} \right)^{e_{k_2}}$$
$$\times \prod_{k_3=1}^{K} \left(\sum_{m_{k_3}=0}^{r-1} \frac{L_{k_3}^{(m_{k_3})}(1,\chi_1\chi_2)}{m_{k_3}!}(s-1)^{m_{k_3}} \right)^{e_{k_3}} .$$

We see that $e_1 - l_1 - 1 \geq 0$ from hypothesis (H2), hence we have

$$\alpha_j'' \ll_{\varepsilon_1} |L_1(1,\chi_1)|(d_1 d_2)^{c\varepsilon_1} \ll_{\varepsilon_1} |L_1(1,\chi_1)|(d_1)^{c\varepsilon_1}$$

by using (3). This estimate and (25) imply (2). The proof of Theorem 1 is now complete.

Here we mention how to prove the claim stated in Remark 3. We put $F(s) = \log \phi(s,\chi)$. We can write

$$F(s) = \sum_{n=1}^{\infty} \delta'(n) n^{-s} \tag{26}$$

in $\sigma > 1$ from (5), where $\delta'(n) \geq 0$. We prove that the Dirichlet series (26) has the finite abscissa of convergence σ_c. Suppose the contrary. Then the Dirichlet series (26) is convergent in the whole plane. This implies that $F(\sigma + it) = O(1)$ as $|t| \to \infty$. However, we have

$$\begin{aligned}\log \Lambda(s,\chi) = \sum_{k=1}^{K} e_k \Big[& \log W_{k,\chi} + (1-2s)\log Q_{k,\chi} \\ & + \sum_{\nu=1}^{N(k)} \Big\{ \log \Gamma(\alpha_\nu(k)(1-s) + \beta_\nu(k,\chi)) \\ & - \log \Gamma(\alpha_\nu(k)s + \beta_\nu(k,\chi)) \Big\} + \log L_k(1-s,\chi) \Big]\end{aligned}$$

by using the functional equation for $L_k(s,\chi)$. If $t \geq 2$ and $\sigma < 0$ we have

$$\begin{aligned}&\log \Gamma(\alpha_\nu(k)(1-s) + \beta_\nu(k,\chi)) - \log \Gamma(\alpha_\nu(k)s + \beta_\nu(k,\chi)) \\ &= \log \Gamma(\alpha_\nu(k)(1-s) + \beta_\nu(k,\chi)) + \log \Gamma(1 - \alpha_\nu(k)s - \beta_\nu(k,\chi))\end{aligned}$$

$$
\begin{aligned}
& - \log \pi + \log \sin(\pi(\alpha_\nu(k)s + \beta_\nu(k,\chi))) \\
\sim & - 2i\alpha_\nu(k)t \log t
\end{aligned}
$$

by using Stirling's formula. Hence $\log \Lambda(s,\chi) \sim -2iA(K)t \log t$. The same conclusion holds for $\log \Lambda(s)$, hence $F(s) \sim -4iA(K)t \log t$ for $t \geq 2$ and $\sigma < 0$. This is a contradiction to $F(\sigma+it) = O(1)$. Hence $F(s) = \sum_{n+1}^{\infty} \delta'(n) n^{-s}$ has the finite abscissa of convergence σ_c and hence has a singularity at $s = \sigma_c$ by Theorem 11.13 in [1]. Now suppose that $L_1(1,\chi) = 0$. Then the function $\phi(s,\chi)$ is entire because of (H2). Therefore the singularity at $s = \sigma_c$ implies that $\phi(\sigma_c,\chi) = 0$. However we have $\phi(\sigma,\chi) \geq 1$ for $\sigma > \sigma_c$ since $\delta'(n) \geq 0$. We get a contradiction by letting $\sigma \to \sigma_c$. Hence we see that $L_1(1,\chi) \neq 0$. From this fact and Theorem 1, Siegel's theorem for $L_1(s,\chi)$ follows immediately.

5 The symmetric power L-functions

In this final section we apply our main theorem to the case of symmetric power L-functions. Let

$$
f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}
$$

be a holomorphic cusp form, which is a newform of weight k and level N. Let us write for each prime p, which does not divide N,

$$
a(p) = 2p^{\frac{k-1}{2}} \cos \theta_p.
$$

For each integer $n \geq 0$, let

$$
L_{sym,n}(s,\chi) = \prod_{p \nmid N} \prod_{j=0}^{n} \left(1 - \frac{\chi(p) e^{i\theta_p (n-2j)}}{p^s}\right)^{-1}.
$$

Clearly, $L_{sym,n}(s,\chi)$ converges absolutely for $\sigma > 1$. It is in fact conjectured that each $L_{sym,n}(s,\chi)$ can be extended to an entire function for any $n \geq 1$ and χ. This is stronger than (A1). Here we suppose that this conjecture is true and the L-functions satisfy (A2) and (A3). In view of Serre [15], these assumptions are very natural. In the case $n = 0$, the L-function is equal to the Dirichlet L-function, hence the Siegel-Tatuzawa theorem in this case is classical. When $n \geq 1$, we divide the situation into two cases; n is even, or n is odd. Put

$$\Lambda_{even}(s,\chi) = L_{sym,0}(s,\chi)L_{sym,2}(s,\chi)\cdots L_{sym,2n}(s,\chi)$$

and

$$\begin{aligned}\Lambda_{odd}(s,\chi) =& \left(L_{sym,0}(s,\chi)L_{sym,1}(s,\chi)L_{sym,2}(s,\chi)\cdots L_{sym,2n+1}(s,\chi)\right)^2 \\ &\times L_{sym,2n+2}(s,\chi).\end{aligned}$$

Note that the case $\chi = \chi_0$ of these functions was introduced by Ram Murty in [12]. These functions satisfy (H1), (H2) and (H3). In fact, as in the proof of Theorem 3 in [12], we can show

$$\log \Lambda_{even}(s,\chi) = \sum_{p:prime}\sum_{h=1}^{\infty} \frac{\chi^h(p)}{hp^{hs}} \left(\frac{\sin((n+1)\theta_p h)}{\sin(\theta_p h)}\right)^2$$

and

$$\log \Lambda_{odd}(s,\chi) = \sum_{p:prime}\sum_{h=1}^{\infty} \frac{\chi^h(p)}{hp^{hs}} \left(\frac{\sin((n+3/2)\theta_p h)}{\sin(\theta_p h/2)}\right)^2,$$

hence (H1) follows. Hypothesis (H2) trivially holds because we assume $L_{sym,n}(s,\chi_0)$, $L_{sym,n}(s,\chi)$ are entire for $n \geq 1$, and (H3) follows from Remark 1. Therefore the Siegel-Tatuzawa theorem for $L_{syn,2n}(s,\chi)$ and $L_{sym,2n+1}(s,\chi)$ follows from Theorem 1 under (A1)-(A3).

Theorem 2. *Let X be the same as in Theorem 1. If the n-th symmetric power L-function $L_{sym,n}(s,\chi)$ can be extended to an entire function (for any $n \geq 1$ and any χ), and satisfies (A2) and (A3), then there exists an effective constant $C(\varepsilon)$ such that*

$$|L_{sym,n}(1,\chi)| > \frac{C(\varepsilon)}{d^{\varepsilon}}$$

for any $\varepsilon > 0$, except for at most one possible element of X. Here d is the conductor of χ.

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Yumiko Ichihara:
Graduate School of Information Engineering,
Hiroshima University,
1-4-1 Kagamiyama, Higashi-hiroshima, 739-8527 Japan
E-mail: ichihara@amath.hiroshima-u.ac.jp

Kohji Matsumoto:
Graduate School of Mathematics,
Nagoya University,
Chikusa-ku, Nagoya, 464-8602 Japan
E-mail: kohjimat@math.nagoya-u.ac.jp