The analytic continuation and the asymptotic behaviour of multiple zeta-functions II

Kohji Matsumoto

Abstract

The meromorphic continuation of generalized multiple zeta-functions, which has been shown under certain restrictions in the author's former paper, is proved in a fairly general situation.

1 Introduction

The definition of generalized multiple zeta-functions is as follows:

$$\begin{aligned}&\zeta_r((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_r),(w_1,\ldots,w_r))\\&\quad=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha_1+m_1w_1)^{-s_1}(\alpha_2+m_1w_1+m_2w_2)^{-s_2}\\&\qquad\times\cdots\times(\alpha_r+m_1w_1+\cdots+m_rw_r)^{-s_r},\end{aligned} \tag{1.1}$$

where r be a positive integer, $s_1,\ldots,s_r$ be complex variables, $\alpha_1,\ldots,\alpha_r$, $w_1,\ldots,$ w_r be complex parameters. Let ℓ be a fixed line on the complex plane $\mathbf{C}$ crossing the origin. Then ℓ divides $\mathbf{C}$ into two open half-planes and ℓ itself, and denote by $H(\ell)$ one of those half-planes. We can write

$$H(\ell)=\left\{w\in\mathbf{C}\setminus\{0\}\;\middle|\;\theta-\frac{\pi}{2}<\arg w<\theta+\frac{\pi}{2}\right\},$$

with $-\pi<\theta\leq\pi$. To assure the convergence of (1.1), we assume

$$w_j\in H(\ell)\qquad(1\leq j\leq r). \tag{1.2}$$

It might happen that $\alpha_j+m_1w_1+\cdots+m_jw_j=0$ holds for some j and some $(m_1,\ldots,m_j)$, but only finitely many times under the assumption (1.2). We adopt the convention that the terms corresponding to such $(m_1,\ldots,m_j)$'s are removed from (1.1). For any j, $\alpha_j+m_1w_1+\cdots+m_jw_j\in H(\ell)$ except for finitely many $(m_1,\ldots,m_j)$'s. If $\alpha_j+m_1w_1+\cdots+m_jw_j\in H(\ell)$, then the branch of the logarithm in the factor

$$(\alpha_j+m_1w_1+\cdots+m_jw_j)^{-s_j}=\exp(-s_j\log(\alpha_j+m_1w_1+\cdots+m_jw_j))$$

is chosen as

$$\theta - \pi/2 < \arg(\alpha_j + m_1 w_1 + \cdots + m_j w_j) < \theta + \pi/2.$$

In [4] we have shown that, under the above convention and the assumption (1.2), the series (1.1) converges absolutely in the region

$$\mathcal{A}_r = \{(s_1, \ldots, s_r) \in \mathbf{C}^r \mid \Re(s_{r-k+1} + \cdots + s_r) > k \quad (1 \leq k \leq r)\},$$

uniformly in any compact subset of $\mathcal{A}_r$.

The purpose of the present paper is to complete the proof of the following

Theorem *Under the above convention and the assumption* (1.2), *the function defined by* (1.1) *can be continued meromorphically to the whole* $\mathbf{C}^r$ *space.*

This result has been proved in [5] under the additional assumptions

$$\alpha_j \in H(\ell) \qquad (1 \leq j \leq r) \tag{1.3}$$

and

$$\alpha_{j+1} - \alpha_j \in H(\ell) \qquad (1 \leq j \leq r-1). \tag{1.4}$$

Therefore, our remaining task is to remove these two assumptions. However, the proof given in the following sections does not depend on the results proved in [5], except two lemmas on Hurwitz zeta-functions given in Section 2 of [5].

The previous history on the analytic continuation of various special cases of (1.1) is mentioned in [4] [5].

2 The case $r = 1$

First we consider the case $r = 1$, that is

$$\zeta_1(s_1; \alpha_1, w_1) = \sum_{m_1=0}^{\infty} (\alpha_1 + m_1 w_1)^{-s_1}. \tag{2.1}$$

We prove that, if $w_1 \in H(\ell)$, then (2.1) can be continued meromorphically to the whole plane. Since $w_1 \in H(\ell)$, we find a positive integer μ_1 such that $\alpha_1 + m_1 w_1 \in H(\ell)$ for any $m_1 \geq \mu_1$. If we choose μ_1 sufficiently large, then $\arg(m_1 + \alpha_1 w_1^{-1})$ is small for $m_1 \geq \mu_1$, and so

$$\arg(\alpha_1 + m_1 w_1) = \arg w_1 + \arg(m_1 + \alpha_1 w_1^{-1}).$$

Hence we can write

$$\zeta_1(s_1; \alpha_1, w_1) = \sum_{m_1=0}^{\mu_1 - 1} (\alpha_1 + m_1 w_1)^{-s_1} + w_1^{-s_1} \sum_{m_1=\mu_1}^{\infty} \left(m_1 + \frac{\alpha_1}{w_1} \right)^{-s_1}. \tag{2.2}$$

Under our convention we may assume that $\alpha_1 w_1^{-1} \notin \{0, -1, -2, \ldots\}$. Hence the second term on the right-hand side of (2.2) can be continued to $\mathbf{C}$ by Lemma 1 of [5]. The first term is clearly continuable. We have proved our claim, which implies that our theorem is true for $r = 1$.

Hence now we can apply the induction argument. In the following sections we assume the validity of the theorem for ζ_{r-1}, and prove the theorem for ζ_r.

3 Removing the condition (1.4)

Now we assume that the theorem is true for ζ_{r-1} under the conditions (1.2) and (1.3) (but without (1.4)), and prove the theorem for ζ_r under the same conditions.

First of all we note that, under the conditions (1.2) and (1.3), we may assume that ℓ is the imaginary axis, and $H(\ell)$ is the half-plane H_+ which consists of all complex numbers with positive real part. In fact, putting $\tilde{\alpha}_j = \alpha_j e^{-i\theta}$ and $\tilde{w}_j = w_j e^{-i\theta}$ $(1 \le j \le r)$, we find easily (as in Section 6 of [5])

$$\begin{aligned}&\zeta_r((s_1, \ldots, s_r); (\alpha_1, \ldots, \alpha_r), (w_1, \ldots, w_r)) \\ &\quad = \exp(-i\theta(s_1 + \cdots + s_r)) \\ &\qquad \times \zeta_r((s_1, \ldots, s_r); (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r), (\tilde{w}_1, \ldots, \tilde{w}_r)),\end{aligned}$$

hence our problem is reduced to the continuation of

$$\zeta_r((s_1, \ldots, s_r); (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r), (\tilde{w}_1, \ldots, \tilde{w}_r)).$$

Therefore in this section we assume $H(\ell) = H_+$, and replace the conditions (1.2) and (1.3) by

$$w_j \in H_+ \qquad (1 \le j \le r), \tag{3.1}$$

and

$$\alpha_j \in H_+ \qquad (1 \le j \le r), \tag{3.2}$$

respectively.

At first we assume $\Re s_j > 1$ $(1 \le j \le r)$. Since $w_r \in H_+$, we can find a positive integer μ_r for which $\alpha_r - \alpha_{r-1} + m_r w_r \in H_+$ holds for any $m_r \ge \mu_r$. We divide the definition (1.1) of ζ_r as

$$\begin{aligned}&\zeta_r((s_1, \ldots, s_r); (\alpha_1, \ldots, \alpha_r), (w_1, \ldots, w_r)) \\ &\quad = \sum_{m_1=0}^{\infty} \cdots \sum_{m_{r-1}=0}^{\infty} \sum_{m_r=0}^{\mu_r - 1} (\alpha_1 + m_1 w_1)^{-s_1} \\ &\qquad\quad \times \cdots \times (\alpha_r + m_1 w_1 + \cdots + m_r w_r)^{-s_r} \\ &\qquad + \sum_{m_1=0}^{\infty} \cdots \sum_{m_{r-1}=0}^{\infty} \sum_{m_r=\mu_r}^{\infty} (\alpha_1 + m_1 w_1)^{-s_1} \\ &\qquad\quad \times \cdots \times (\alpha_r + m_1 w_1 + \cdots + m_r w_r)^{-s_r}.\end{aligned} \tag{3.3}$$

Putting $\alpha'_r = \alpha_r + \mu_r w_r$, we can see that the second sum on the right-hand side is equal to

$$\zeta_r((s_1, \ldots, s_r); (\alpha_1, \ldots, \alpha_{r-1}, \alpha'_r), (w_1, \ldots, w_r)). \tag{3.4}$$

On the other hand, the first sum can be written as

$$\sum_{m_r=0}^{\mu_r-1} \xi_{r-1}((s_1, \ldots, s_r); (\alpha_1, \ldots, \alpha_{r-1}, \alpha_r + m_r w_r), (w_1, \ldots, w_{r-1})),$$

where

$$\begin{aligned}
&\xi_{r-1}((s_1, \ldots, s_r); (\alpha_1, \ldots, \alpha_{r-1}, \beta), (w_1, \ldots, w_{r-1})) \\
&\quad = \sum_{m_1=0}^{\infty} \cdots \sum_{m_{r-1}=0}^{\infty} (\alpha_1 + m_1 w_1)^{-s_1} \cdots (\alpha_{r-2} + m_1 w_1 + \cdots + m_{r-2} w_{r-2})^{-s_{r-2}} \\
&\qquad\qquad \times (\alpha_{r-1} + m_1 w_1 + \cdots + m_{r-1} w_{r-1})^{-s_{r-1}} \\
&\qquad\qquad \times (\beta + m_1 w_1 + \cdots + m_{r-1} w_{r-1})^{-s_r}.
\end{aligned} \tag{3.5}$$

Therefore the problem is reduced to the continuation of (3.4) and (3.5).

We first treat (3.4) by using the formula

$$\Gamma(s)(1+\lambda)^{-s} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s+z)\Gamma(-z)\lambda^z dz \tag{3.6}$$

where $\Re s > 0$, $|\arg \lambda| < \pi$, $\lambda \neq 0$, $-\Re s < c < 0$, and the path of integration is the vertical line $\Re z = c$. This is the classical Mellin-Barnes formula, and a simple proof is given in Section 4 of [4]. We apply (3.6) with $s = s_r$ and

$$\lambda = \frac{\alpha'_r - \alpha_{r-1} + m'_r w_r}{\alpha_{r-1} + m_1 w_1 + \cdots + m_{r-1} w_{r-1}},$$

where $m'_r = m_r - \mu_r (\geq 0)$. Both the denominator and the numerator of λ are belonging to H_+, because $\alpha_{r-1} \in H_+$ by (3.2) while $\alpha'_r - \alpha_{r-1} \in H_+$ is implied by the definition of α'_r. Hence $|\arg \lambda| < \pi$ and $\lambda \neq 0$. Moreover, since $\Re s_r > 1$, we can choose c satisfying $-\Re s_r < c < -1$. From (3.6) we have

$$\begin{aligned}
&(\alpha_{r-1} + m_1 w_1 + \cdots + m_{r-1} w_{r-1})^{s_r} (\alpha'_r + m_1 w_1 + \cdots + m_{r-1} w_{r-1} + m'_r w_r)^{-s_r} \\
&\qquad = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \left(\frac{\alpha'_r - \alpha_{r-1} + m'_r w_r}{\alpha_{r-1} + m_1 w_1 + \cdots + m_{r-1} w_{r-1}} \right)^z dz.
\end{aligned}$$

Multiply the both sides by

$$\begin{aligned}
&(\alpha_1 + m_1 w_1)^{-s_1} \cdots (\alpha_{r-2} + m_1 w_1 + \cdots + m_{r-2} w_{r-2})^{-s_{r-2}} \\
&\qquad \times (\alpha_{r-1} + m_1 w_1 + \cdots + m_{r-1} w_{r-1})^{-s_{r-1} - s_r}
\end{aligned}$$

and summing up with respect to $m_1, \ldots, m_{r-1}, m'_r$, we obtain

$$\begin{aligned}
&\zeta_r((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_{r-1},\alpha'_r),(w_1,\ldots,w_r)) \\
&\quad = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+z); \\
&\qquad (\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))\sum_{m'_r=0}^{\infty}(\alpha'_r-\alpha_{r-1}+m'_r w_r)^z dz. \qquad (3.7)
\end{aligned}$$

If we choose μ_r sufficiently large, then

$$\arg\left(m'_r + \frac{\alpha'_r - \alpha_{r-1}}{w_r}\right)$$

is small for any $m'_r \geq 0$. Hence, as in Section 2, we can verify

$$\begin{aligned}
\sum_{m'_r=0}^{\infty}(\alpha'_r-\alpha_{r-1}+m'_r w_r)^z &= w_r^z \sum_{m'_r=0}^{\infty}\left(m'_r + \frac{\alpha'_r-\alpha_{r-1}}{w_r}\right)^z \\
&= w_r^z \zeta\left(-z, \frac{\alpha'_r-\alpha_{r-1}}{w_r}\right).
\end{aligned}$$

The right-hand side is, by Lemma 2 of [5], estimated as

$$O\Big((|y|+1)^{\max\{0,1+x\}+\varepsilon}\exp(|y|\rho)\Big) \qquad (3.8)$$

for any $\varepsilon > 0$, where $x = \Re z$, $y = \Im z$, and

$$\rho = \max\left\{|\arg(\alpha'_r - \alpha_{r-1})|, |\arg w_r|\right\},$$

hence $|\rho| < \pi/2$. The factor ζ_{r-1} in the integrand on the right-hand side of (3.7) is convergent absolutely if $\Re z \geq c$, hence this factor is estimated as $O(\exp(|y|\theta_0))$, where

$$\theta_0 = \sup_{m_1,\ldots,m_{r-1}} |\arg(\alpha_{r-1} + m_1 w_1 + \cdots + m_{r-1}w_{r-1})|$$

so $|\theta_0| < \pi/2$. (The implied constant depends on $\sigma_1, \ldots, \sigma_r$, $t_1, \ldots, t_r$, x etc. but does not depend on y.) Combining this estimate, (3.8), and Stirling's formula, we find that the integrand on the right-hand side of (3.7) tends to 0 when $|y| \to \infty$ in the region $\Re z \geq c$. Therefore we can shift the path of integration to the line $\Re z = M - \varepsilon$, where M is a positive integer. The relevant poles are at $z = -1, 0, 1, 2, \ldots, M-1$, and counting the residues of those poles we obtain

$$\begin{aligned}
&\zeta_r((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_{r-1},\alpha'_r),(w_1,\ldots,w_r)) \\
&\quad = \frac{1}{s_r-1}\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r-1); \\
&\qquad\qquad (\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))w_r^{-1}
\end{aligned}$$

$$+\sum_{k=0}^{M-1}\binom{-s_r}{k}\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+k);$$
$$(\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))\zeta\left(-k,\frac{\alpha'_r-\alpha_{r-1}}{w_r}\right)w_r^k$$
$$+\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+z);$$
$$(\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))\zeta\left(-z,\frac{\alpha'_r-\alpha_{r-1}}{w_r}\right)w_r^z dz. \quad (3.9)$$

In the integrand of the above integral term, the factor $\Gamma(s_r+z)$ is holomorphic if $\Re(s_r+z)>0$, and the factor ζ_{r-1} is convergent absolutely if

$$\Re(s_{r-j}+\cdots+s_r+z)>j \qquad (1\leq j\leq r-1).$$

Hence the integral term on the right-hand side of (3.9) is holomorphic in the region

$$\mathcal{F}_r(M;\varepsilon)=\left\{(s_1,\ldots,s_r)\in\mathbf{C}^r \;\middle|\; \begin{array}{r}\Re(s_{r-j}+\cdots+s_r)>j-M+\varepsilon\\ (0\leq j\leq r-1)\end{array}\right\},$$

while the other terms can be continued meromorphically by the induction assumption. Since M is arbitrary, this implies the meromorphic continuation of (3.4) to the whole $\mathbf{C}^r$ space.

The idea of using the Mellin-Barnes formula (3.6) to this type of problems goes back to Katsurada's papers [1] [2]. Then, inspired by Katsurada's works, the author wrote [3] [4] [5]. The above treatment of (3.4) is similar to the argument in Sections 3 and 4 of [5], but we repeat the details for the convenience of readers. (The method of estimating the factor ζ_{r-1} is different from that in [5].)

Next we prove the analytic continuation of (3.5). Since either $\beta-\alpha_{r-1}$ or $\alpha_{r-1}-\beta$ clearly has the non-negative real part, we may assume $\Re(\beta-\alpha_{r-1})\geq 0$ without loss of generality. Moreover, if $\beta=\alpha_{r-1}$ then

$$\xi_{r-1}((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_{r-1},\beta),(w_1,\ldots,w_{r-1}))$$
$$=\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r);(\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))$$

which can be continued by the induction assumption. Hence we may assume $\beta\neq\alpha_{r-1}$. Now we apply (3.6) with $s=s_r$ and

$$\lambda=\frac{\beta-\alpha_{r-1}}{\alpha_{r-1}+m_1w_1+\cdots+m_{r-1}w_{r-1}}.$$

The above assumptions imply $|\arg\lambda|<\pi$ and $\lambda\neq 0$, hence we can use (3.6). As before, we obtain

$$\xi_{r-1}((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_{r-1},\beta),(w_1,\ldots,w_{r-1}))$$

$$= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+z);$$
$$(\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))(\beta-\alpha_{r-1})^z dz, \tag{3.10}$$

and by shifting the path we find that the right-hand side is equal to

$$\sum_{k=0}^{M-1}\binom{-s_r}{k}\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+k);$$
$$(\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))(\beta-\alpha_{r-1})^k$$
$$+\frac{1}{2\pi i}\int_{(M-\varepsilon)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+z);$$
$$(\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))(\beta-\alpha_{r-1})^z dz. \tag{3.11}$$

This last integral is holomorphic in $\mathcal{F}_r(M;\varepsilon)$, hence we obtain the meromorphic continuation of (3.5). Therefore we now obtain the proof of meromorphic continuation of ζ_r without the condition (1.4).

4 Removing the condition (1.3)

Finally we remove the condition (1.3). Assume that the theorem is true for ζ_{r-1} under the only condition (1.2). Write $\alpha_j=\alpha_j^{(1)}+\alpha_j^{(2)}$ with $\arg\alpha_j^{(1)}=\theta-\pi/2$ or $\theta+\pi/2$ (or $\alpha_j^{(1)}=0$) and $\arg\alpha_j^{(2)}=\theta$ or $-\theta$ (or $\alpha_j^{(2)}=0$). Consider the set of all $\alpha_j^{(2)}$ whose argument is not θ, and denote by $\tilde{\alpha}$ (one of) the element(s) of this set whose absolute value is the largest. Choose a positive integer μ such that $\tilde{\alpha}+m_1w_1\in H(\ell)$ for any $m_1\geq\mu$. Divide the series (1.1) as

$$\zeta_r((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_r),(w_1,\ldots,w_r))$$
$$=\sum_{m_1=0}^{\mu-1}\sum_{m_2=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}+\sum_{m_1=\mu}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}=T_1+T_2, \tag{4.1}$$

say. This idea of dividing the series (1.1) as above has already appeared in Section 6 of [4], in the proof of absolute convergence of (1.1).

Putting $\alpha_j'(m_1)=\alpha_j+m_1w_1$ for $0\leq m_1\leq\mu-1$, we find that

$$T_1 = \sum_{m_1=0}^{\mu-1}\alpha_1'(m_1)^{-s_1}$$
$$\times\zeta_{r-1}((s_2,\ldots,s_r);(\alpha_2'(m_1),\ldots,\alpha_r'(m_1)),(w_2,\ldots,w_r)),$$

which can be continued by the induction assumption. As for T_2, writing $m_1'=m_1-\mu$ and $\alpha_j'=\alpha_j+\mu w_1$ $(1\leq j\leq r)$, we have

$$T_2 = \sum_{m_1'=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha_1'+m_1'w_1)^{-s_1}(\alpha_2'+m_1'w_1+m_2w_2)^{-s_2}$$

$$
\begin{aligned}
&\times \cdots \times (\alpha'_r + m'_1 w_1 + m_2 w_2 + \cdots + m_r w_r)^{-s_r} \\
= \quad & \zeta_r((s_1, \ldots, s_r); (\alpha'_1, \ldots, \alpha'_r), (w_1, \ldots, w_r)).
\end{aligned}
$$

Since $\alpha'_j \in H(\ell)$ $(1 \leq j \leq r)$, the right-hand side can be continued meromorphically by the fact already shown in Section 3. Therefore now (4.1) is continued to the whole $\mathbf{C}^r$ space, and our theorem is proved completely.

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Graduate Schoolof Mathematics
Nagoya University
Chikusa-ku, Nagoya 464-8602
Japan