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# An explicit formula of Atkinson type for the product of the Riemann zeta-function and a Dirichlet polynomial

**Research Article**

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**Abstract:** We prove an explicit formula of Atkinson type for the error term in the asymptotic formula for the mean square of the product of the Riemann zeta-function and a Dirichlet polynomial. To deal with the case when coefficients of the Dirichlet polynomial are complex, we apply the idea of the first author in his study on mean values of Dirichlet $L$-functions.

**MSC:** 11M06, 11M41

**Keywords:** Riemann zeta-function • Dirichlet polynomial • Atkinson formula



## 1. Introduction and statement of results

Let $s = \sigma + it$ be a complex variable, $\zeta(s)$ denote the Riemann zeta-function, and

$$A(s) = \sum_{m \le M} a(m)\, m^{-s}$$

be a Dirichlet polynomial, where $M \ge 1$ and $a(m) \in \mathbb{C}$. Let us assume

$$a(m) = O(m^{\varepsilon}) \tag{1}$$

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$\varepsilon > 0$, where and in what follows $\varepsilon$ always denotes a small positive number, not necessarily the same at each occurrence. The mean value

$$I(T,A) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right) A\left(\frac{1}{2}+it\right)\right|^2 dt, \qquad T \geq 2,$$

is an interesting object in number theory in connection with power moments and the distribution of zeros of $\zeta(s)$. The asymptotic formula for $I(T,A)$ was first established by Balasubramanian, Conrey and Heath-Brown [2]. Denote by $(k,l)$ the greatest common divisor of $k$ and $l$, and by $[k,l] = kl/(k,l)$ their least common multiple. Let

$$\mathcal{M}(T,A) = \sum_{k\leq M}\sum_{l\leq M} \frac{a(k)\,\overline{a(l)}}{[k,l]}\left(\log\frac{(k,l)^2 T}{2\pi kl} + 2\gamma - 1\right) T,$$

where $\overline{a(l)}$ is the complex conjugate of $a(l)$ and $\gamma$ is Euler's constant. They proved

$$I(T,A) = \mathcal{M}(T,A) + E(T,A)$$

under the condition $\log M \ll \log T$ (the symbol $f \ll g$ means $f = O(g)$), where $E(T,A)$ is the error term. The error estimate proved in [2] is

$$E(T,A) \ll M^2 T^{\varepsilon} + T\log^{-B} T$$

for any $B > 0$.

The second author [14] proved that if $C_0 \log^{1/2} T \leq M^{\mu} T^{\rho} \leq T/C_0^2 \log^{1/2} T$ holds where $\mu \geq 0$, $0 \leq \mu < 1$, $\mu + \rho > 0$, and $C_0$ is sufficiently large, then

$$E(T,A) \ll M^{2-\mu/2}\, T^{1/2-\rho/2+\varepsilon} + M^{\mu}\, T^{\rho+\varepsilon}. \tag{2}$$

The aim of this article is to study $E(T,A)$ in more detail. For that we will prove an analogue of Atkinson's formula for $E(T,A)$.

Atkinson's formula was originally obtained [1] for the mean square of $\zeta(s)$. It gives an explicit formula for $E(T)$ defined by

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt = T\log T + (2\gamma - 1 - \log 2\pi)\,T + E(T).$$

It is known that Atkinson's formula is very useful in the mean square theory of $\zeta(s)$ (see [7, 8, 13]). Therefore it is natural to search for Atkinson-type formulas for other zeta and $L$-functions. In [16], Motohashi made a first attempt and applied Atkinson's idea [1] to the study of $E(T,A)$. He did not obtain an explicit formula of Atkinson type. His idea was to consider a certain exponential integral including $E(T,A)$ to obtain a sharp estimate. His estimate is $O(M^{4/3}T^{1/3+\varepsilon})$ for $M \ll T^{1/2}\log^{-3/4} T$ (actually for the integral from $-T$ to $T$). Steuding made further use of Motohashi's idea and found an interesting result on the distribution of the zeros of $\zeta(s)$ in short intervals. The proof of (2) in [14] is also based on a variant of Atkinson's idea (cf. Section 2.7 of Ivić [8]).

In the case of $\zeta(s)$, it is clear that

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt = \frac{1}{2}\int_{-T}^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt. \tag{3}$$

This property was used by Atkinson in the proof of his explicit formula. However, since the $a(m)$ are complex, $I(T,A)$ is not necessarily equal to

$$\frac{1}{2}\int_{-T}^T \left|\zeta\left(\frac{1}{2}+it\right) A\left(\frac{1}{2}+it\right)\right|^2 dt.$$

This causes some trouble when one tries to prove an Atkinson-type formula for $E(T,A)$. The same happens in the case of Dirichlet $L$-function $L(s,\chi)$ associated with a complex character $\chi$ (see [12]). In [5], the first author introduced a new idea to prove an explicit formula of Atkinson type for

$$I(T,\chi)=\int_0^T\left|L\left(\frac{1}{2}+it,\chi\right)\right|^2dt,$$

and showed that if $\chi$ is a complex primitive character, then $I(T,\chi)$ and the integral of $|L(1/2+it,\chi)|^2$ from $-T$ to 0 do not coincide. (The difference of these two quantities is $\Omega\big(T^{1/4}\big)$.)

The fundamental idea in [5], inspired by Hafner and Ivić [3], is to consider the integral over the interval $[T,2T]$ instead of $[-T,T]$ to avoid the trouble mentioned above. Then, in the course of the proof, there appear some integrals which cannot be treated by Atkinson's original argument. How to deal with those integrals is the most novel point of [5]. In the present paper we apply the idea from [5] to $E(T,A)$ to obtain an analogue of Atkinson's formula.

Throughout the paper we will use the following notation:

$$\kappa=\frac{k}{(k,l)},\qquad \lambda=\frac{l}{(k,l)},\qquad \operatorname{arcsinh}x=\log\left(x+\sqrt{x^2+1}\right),\qquad \xi(T,u)=\frac{T}{2\pi}+\frac{u}{2}-\sqrt{\frac{u^2}{4}+\frac{uT}{2\pi}},$$

$$f(T,u)=2T\operatorname{arcsinh}\sqrt{\frac{\pi u}{2T}}+\sqrt{2\pi uT+\pi^2u^2}-\frac{\pi}{4},\qquad g(T,u)=T\log\frac{T}{2\pi u}-T+2\pi u+\frac{\pi}{4}.$$

Let us define

$$\Sigma_1(T,Y)=\sum_{k,l\le M}\sum_{n\le\kappa\lambda Y}\Im\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}(\kappa\lambda)^{1/2}\frac{d(n)}{n^{1/2}}e^{2\pi in\overline{\kappa}/\lambda}\left(\operatorname{arcsinh}\sqrt{\frac{\pi n}{2T\kappa\lambda}}\right)^{-1}\right.$$
$$\left.\times\left(1+\frac{2T\kappa\lambda}{\pi n}\right)^{-1/4}\exp\left(i\left(f\left(T,\frac{n}{\kappa\lambda}\right)-\frac{\pi n}{\kappa\lambda}+\frac{\pi}{2}\right)\right)\right\},$$

where $d(n)$ is the number of positive divisors of $n$, $\overline{\kappa}$ is defined by $\kappa\overline{\kappa}\equiv 1\,(\mathrm{mod}\,\lambda)$, and

$$\Sigma_2(T,Y)=-2\sum_{k,l\le M}\sum_{n\le(\lambda/\kappa)Y}\Re\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}(\kappa\lambda)^{1/2}\frac{d(n)}{n^{1/2}}e^{-2\pi in\kappa/\lambda}\left(\log\frac{T\lambda}{2\pi n\kappa}\right)^{-1}\exp\left(i\,g\left(T,\frac{\kappa n}{\lambda}\right)\right)\right\}.$$

The main results in this paper are the following three theorems.

### Theorem 1.1.

*Let $T$, $Y$ be positive numbers satisfying $C_1T<Y<C_2T$ and $T\ge C^*=\max\{e,C_1^{-1}\}$ (where $C_1$, $C_2$ are fixed constants with $0<C_1<C_2$ and $e=2.71828\ldots$). Then we have*

$$\int_T^{2T}\left|\zeta\left(\frac{1}{2}+it\right)A\left(\frac{1}{2}+it\right)\right|^2dt=\mathcal{M}(2T,A)+\Sigma_1(2T,2Y)+\Sigma_2\big(2T,\xi(2T,2Y)\big)$$
$$-\mathcal{M}(T,A)-\Sigma_1(T,Y)-\Sigma_2\big(T,\xi(T,Y)\big)+R(T,2T,A),\qquad(4)$$

*where $R(T,2T,A)$ is the error term satisfying*

$$R(T,2T,A)\ll M^{1+\varepsilon}\log^2T+M^{5/2+\varepsilon}\,T^{-1/4}.$$

From Theorem 1.1 it is easy to deduce:

**Theorem 1.2.**
*Under the same assumptions as in Theorem 1.1, we have*

$$E(T,A) = \Sigma_1(T,Y) + \Sigma_2\big(T,\xi(T,Y)\big) + R(T,A) \tag{5}$$

*with*

$$R(T,A) \ll M^{1+\varepsilon}\log^3 T + M^{2+\varepsilon}\log^{3/2-2\delta} T\cdot(\log\log T)^2 + M^{5/2+\varepsilon}\log^{-3/4+\delta} T$$

*for any* $\delta > 0$.

When the coefficients $a(m)$ are real, we do not encounter the problem mentioned above. Therefore in this case the treatment is simpler, and we obtain the following sharper result:

**Theorem 1.3.**
*When the coefficients $a(m)$ are real, we have (5) with*

$$R(T,A) \ll M^{1+\varepsilon}\log^2 T + M^{5/2+\varepsilon}\,T^{-1/4}.$$

Theorems 1.2 and 1.3 are analogues of Atkinson's formula [1] for $\zeta(s)$. In fact, the case $M=1$, $a(1)=1$ of Theorem 1.3 gives exactly Atkinson's formula.

Our results give an explicit formula for $E(T,A)$, by which we can study the behavior of $E(T,A)$, especially the oscillation property, quite accurately. In a forthcoming paper we will apply our formula to discuss the difference between $I(T,A)$ and

$$\int_{-T}^{0}\left|\zeta\left(\frac{1}{2}+it\right)A\left(\frac{1}{2}+it\right)\right|^2 dt$$

(analogously to the work of the first author [6]), and also power moments of $E(T,A)$.



## 2. The fundamental decomposition

In this and the next section, we follow Motohashi [16]. Only a brief sketch of the proof is given in [16]. In [17], Steuding explained Motohashi's method, including a treatment of derivatives of the zeta-function, for some special $a(m)$. Here we will supply more details of Motohashi's method.

Let $u$, $v$ be two complex variables, and we first assume $\Re u > 1, \Re v > 1$. Since

$$\zeta(u)\,A(u) = \sum_{m=1}^{\infty}\frac{1}{m^u}\sum_{k\le M}\frac{a(k)}{k^u} = \sum_{q=1}^{\infty}\left(\sum_{k\le M,\,k|q} a(k)\right)q^{-u},$$

we have

$$\zeta(u)\,\zeta(v)\,A(u)\,\overline{A(\overline{v})} = \sum_{q=1}^{\infty}\left(\sum_{k\le M,\,k|q}a(k)\right)q^{-u}\sum_{r=1}^{\infty}\left(\sum_{l\le M,\,l|r}\overline{a(l)}\right)r^{-v} = \sum_{q=r}+\sum_{q<r}+\sum_{q>r} = B_0 + B(u,v) + \overline{B(\overline{v},\overline{u})}. \tag{6}$$

The term $B_0$ can be written as

$$B_0 = \sum_{k\le M}\sum_{l\le M}a(k)\,\overline{a(l)}\sum_{[k,l]|r}r^{-u-v} = \zeta(u+v)\sum_{k\le M}\sum_{l\le M}\frac{a(k)\,\overline{a(l)}}{[k,l]^{u+v}}. \tag{7}$$

Let us now consider $B(u,v)$. Set

$$b(m) = \sum_{k\le M,\, k|m} a(k).$$

Then,

$$B(u,v) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b(m)\,\overline{b(m+n)}\,m^{-u}(m+n)^{-v}.$$

We show that $B(u,v)$ can be extended meromorphically to the region $\Re u > 0, \Re v > 1, \Re(u+v) > 2$, with the formula

$$B(u,v) = \frac{1}{\Gamma(u)\Gamma(v)} \sum_{k\le M}\sum_{l\le M} a(k)\,\overline{a(l)}\,l^{-1}\sum_{f=1}^{l}\int_0^\infty \frac{y^{v-1}}{e^{y-2\pi i f/l}-1}\int_0^\infty \frac{x^{u-1}}{e^{k(x+y)-2\pi i f k/l}-1}\,dx\,dy. \tag{8}$$

In fact, for $y > 0, \Re u > 0$, the inner integral on the right-hand side equals

$$\int_0^\infty \sum_{h=1}^{\infty} e^{-hky+2\pi i f h k/l} e^{-hkx} x^{u-1} dx = k^{-u}\Gamma(u)\sum_{h=1}^{\infty} e^{-hky+2\pi i f h k/l} h^{-u}. \tag{9}$$

Hence, assuming further $\Re v > 1$ and $\Re(u+v) > 2$, we see that the double integral on the right-hand side of (8) equals

$$\frac{\Gamma(u)}{k^u}\sum_{h=1}^{\infty}\frac{e^{2\pi i f h k/l}}{h^u}\int_0^\infty \frac{y^{v-1}e^{-hky}}{e^{y-2\pi i f/l}-1}\,dy,$$

as the summation and the integration can be interchanged in view of the absolute convergence. The integral on the right-hand side of the above can be expanded as

$$\sum_{n=1}^{\infty} e^{2\pi i f n/l}\int_0^\infty y^{v-1}e^{-(hk+n)y}\,dy = \Gamma(v)\sum_{n=1}^{\infty} e^{2\pi i f n/l}(hk+n)^{-v}.$$

Therefore the right-hand side of (8) equals

$$\sum_{k\le M}\sum_{l\le M}\frac{a(k)\,\overline{a(l)}}{k^u l}\sum_{h=1}^{\infty}\frac{1}{h^u}\sum_{n=1}^{\infty}\frac{1}{(hk+n)^v}\sum_{f=1}^{l} e^{2\pi i (hk+n) f/l} = \sum_{k\le M}\sum_{l\le M}\frac{a(k)\,\overline{a(l)}}{k^u}\sum_{\substack{h,n\ge 1\\ l|(hk+n)}}\frac{1}{h^u(hk+n)^v},$$

which, letting $hk = m$, equals $B(u,v)$. Thus (8) follows.

Next, define

$$H(z;k,l,f) = \frac{1}{e^{kz-2\pi i f k/l}-1} - \frac{\delta(f)}{kz},$$

where $\delta(f) = 1$ or $0$ depending on whether $l$ divides $kf$ or not. The function $H(z;k,l,f)$ is holomorphic at $z = 0$ and is $O(\min\{z^{-1},1\})$ for $z \ge 0$. Dividing the inner integral on the right-hand side of (8) as

$$\frac{\delta(f)}{k}\int_0^\infty \frac{x^{u-1}}{x+y}\,dx + \int_0^\infty x^{u-1}H(x+y;k,l,f)\,dx$$

and noting that

$$\int_0^\infty \frac{x^{u-1}}{x+y}\,dx = y^{u-1}\Gamma(u)\Gamma(1-u), \qquad 0 < \Re u < 1,$$

we find that the right-hand side of (8) equals

$$\frac{\Gamma(1-u)}{\Gamma(v)} \sum_{k\le M} \sum_{l\le M} \frac{a(k)\,\overline{a(l)}}{kl} \sum_{f=1}^{l} \delta(f) \int_0^\infty \frac{y^{u+v-2}}{e^{y-2\pi i f/l}-1}\,dy + g(u,v;A) \tag{10}$$

in the region $0<\Re u<1$, $\Re(u+v)>2$, where

$$g(u,v;A) = \frac{1}{\Gamma(u)\,\Gamma(v)} \sum_{k\le M} \sum_{l\le M} a(k)\,\overline{a(l)}\, l^{-1} \sum_{f=1}^{l} \int_0^\infty \frac{y^{v-1}}{e^{y-2\pi i f/l}-1} \int_0^\infty x^{u-1} H(x+y;k,l,f)\,dx\,dy.$$

Similarly to (9), we see that the integral in the first term of (10) equals

$$\Gamma(u+v-1) \sum_{m=1}^{\infty} e^{2\pi i m f/l} m^{-u-v+1}.$$

Therefore from (8) and (10) we obtain

$$B(u,v) = \frac{\Gamma(1-u)}{\Gamma(v)}\,\Gamma(u+v-1) \sum_{k\le M} \sum_{l\le M} \frac{a(k)\,\overline{a(l)}}{kl} \sum_{f=1}^{l} \delta(f)\, \phi\left(u+v-1, \frac{f}{l}\right) + g(u,v;A),$$

where $\phi(s,\alpha) = \sum_{m\ge1} e^{2\pi i m\alpha} m^{-s}$ is the Lerch zeta-function. Since

$$\sum_{f=1}^{l} \delta(f)\, \phi\left(u+v-1, \frac{f}{l}\right) = \sum_{j=1}^{(k,l)} \phi\left(u+v-1, \frac{j}{(k,l)}\right) = \frac{\zeta(u+v-1)}{(k,l)^{u+v-2}},$$

we obtain

$$B(u,v) = \frac{\Gamma(1-u)}{\Gamma(v)}\,\Gamma(u+v-1)\,\zeta(u+v-1) \sum_{k\le M} \sum_{l\le M} \frac{a(k)\,\overline{a(l)}}{(k,l)^{u+v-1}[k,l]} + g(u,v;A) \tag{11}$$

in the region $0<\Re u<1$, $\Re(u+v)>2$.

## 3. The contour integral expression

Let $C$ be the contour which comes from $+\infty$ along the positive real axis, rounds the origin counterclockwise, and goes back to $+\infty$ again along the positive real axis. Then it is easy to see that, under the assumptions $0<\Re u<1$, $\Re(u+v)>2$, $g(u,v;A)$ can be rewritten as

$$g(u,v;A) = \frac{1}{\Gamma(u)\,\Gamma(v)\,(e^{2\pi i u}-1)(e^{2\pi i v}-1)} \sum_{k\le M} \sum_{l\le M} a(k)\,\overline{a(l)}\, l^{-1} \sum_{f=1}^{l} \int_C \frac{y^{v-1}}{e^{y-2\pi i f/l}-1} \int_C x^{u-1} H(x+y;k,l,f)\,dx\,dy. \tag{12}$$

However, the right-hand side of (12) is convergent for $\Re u<1$ and any $v\in\mathbb{C}$. Hence (12) gives the meromorphic extension of $g(u,v;A)$ to that region, and therefore (11) is also valid in that region. Combining (6), (7) and (11), we obtain

$$\begin{aligned}\zeta(u)\,\zeta(v)\,A(u)\,\overline{A(\overline{v})} &= \zeta(u+v) \sum_{k\le M} \sum_{l\le M} \frac{a(k)\,\overline{a(l)}}{[k,l]^{u+v}} + \frac{\Gamma(1-u)}{\Gamma(v)}\,\Gamma(u+v-1)\,\zeta(u+v-1) \sum_{k\le M} \sum_{l\le M} \frac{a(k)\,\overline{a(l)}}{(k,l)^{u+v-1}[k,l]} \\ &\quad + \frac{\Gamma(1-v)}{\Gamma(u)}\,\Gamma(u+v-1)\,\zeta(u+v-1) \sum_{k\le M} \sum_{l\le M} \frac{\overline{a(k)}\,a(l)}{(k,l)^{u+v-1}[k,l]} + g(u,v;A) + \overline{g(\overline{v},\overline{u};A)}\end{aligned}$$

in the region $\Re u < 1, \Re v < 1$. Note that, changing $k$ and $l$ in the third double sum, we can combine the second and the third term on the right-hand side as

$$\left(\frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)}\right) \Gamma(u+v-1)\, \zeta(u+v-1) \sum_{k \le M} \sum_{l \le M} \frac{a(k)\, \overline{a(l)}}{(k,l)^{u+v-1}[k,l]}.$$

Now assume $0 < \Re u < 1$, and take the limit $v \to 1-u$. The singularities coming from $\zeta(u+v)$ and $\Gamma(u+v-1)$ cancel with each other, and the result is

$$\zeta(u)\, \zeta(1-u)\, A(u)\, \overline{A(1-\overline{u})} = \sum_{k \le M} \sum_{l \le M} \frac{a(k)\, \overline{a(l)}}{[k,l]} \left\{ \frac{1}{2} \left( \frac{\Gamma'}{\Gamma}(u) + \frac{\Gamma'}{\Gamma}(1-u) \right) + \log \frac{(k,l)^2}{kl} + 2\gamma - \log 2\pi \right\} + g(u, 1-u; A) + \overline{g(1-\overline{u}, \overline{u}; A)} \tag{13}$$

for $0 < \Re u < 1$.

Next we prove another expression of $g(u, 1-u; A)$ in the region $\Re u < 0$. Let $R$ be a large positive integer and

$$\mathcal{C}_R = \mathcal{C}_R(k,l,f) = \left\{ x = -y + \frac{2\pi i f}{l} + \frac{2\pi}{k}\left(R + \frac{1}{2}\right) e^{i\theta} \,\middle|\, 0 \le \theta < 2\pi \right\}.$$

Then we can easily see that $H(x+y; k, l, f) \ll 1$ if $x \in \mathcal{C}_R(k,l,f)$. Hence

$$\int_{\mathcal{C}_R} H(x+y; k,l,f)\, x^{u-1}\, dx \ll \int_0^{2\pi} R^{\Re u - 1} R\, d\theta \ll R^{\Re u},$$

which tends to 0 as $R \to \infty$ if $\Re u < 0$. Hence

$$\int_{\mathcal{C}} H(x+y; k,l,f)\, x^{u-1}\, dx = -2\pi i {\sum_n}^{*} \mathrm{Res}_n \left( H(x+y; k,l,f)\, x^{u-1} \right) \tag{14}$$

if $\Re u < 0$, where $\mathrm{Res}_n(\cdot)$ denotes the residue of the function at $x = -y + 2\pi i (l^{-1} f + k^{-1} n)$ and ${\sum_n}^{*}$ means the summation running over all integers $n \ne -kf/l$. Since

$$\mathrm{Res}_n \left( H(x+y; k,l,f)\, x^{u-1} \right) = \frac{1}{k} \left( -y + 2\pi i \left( \frac{f}{l} + \frac{n}{k} \right) \right)^{u-1},$$

if we can interchange the summation and integration, from (12) and (14) we obtain

$$g(u, 1-u; A) = \frac{-2\pi i}{\Gamma(u)\, \Gamma(1-u)\, (e^{2\pi i u} - 1)(e^{-2\pi i u} - 1)} \sum_{k \le M} \sum_{l \le M} \frac{a(k)\, \overline{a(l)}}{kl} \sum_{f=1}^{l} {\sum_n}^{*} I(n; k, l, f), \tag{15}$$

where

$$I(n; k,l,f) = \int_{\mathcal{C}} \frac{y^{-u}}{e^{y - 2\pi i f/l} - 1} \left( -y + 2\pi i \left( \frac{f}{l} + \frac{n}{k} \right) \right)^{u-1} dy = (e^{-2\pi i u} - 1) \int_0^{\infty} \frac{y^{-u}}{e^{y - 2\pi i f/l} - 1} \left( -y + 2\pi i \left( \frac{f}{l} + \frac{n}{k} \right) \right)^{u-1} dy.$$

In case $n > -kf/l$, we can write

$$-y + 2\pi i \left( \frac{f}{l} + \frac{n}{k} \right) = e^{\pi i} y + 2\pi e^{\pi i/2} \left( \frac{f}{l} + \frac{n}{k} \right).$$

Hence, setting

$$y = 2\pi \left( \frac{f}{l} + \frac{n}{k} \right) e^{-\pi i/2} \eta, \qquad \arg \eta = \frac{\pi}{2},$$

we obtain

$$\begin{aligned} I(n; k, l, f) &= \left(e^{\pi i u} - e^{-\pi i u}\right) \int_0^{i\infty} \frac{\eta^{-u}(1+\eta)^{u-1}}{\exp\left(-2\pi i \left(\left(\frac{f}{l} + \frac{n}{k}\right)\eta + \frac{f}{l}\right)\right) - 1} \, d\eta \\ &= \left(e^{\pi i u} - e^{-\pi i u}\right) \sum_{m=1}^{\infty} \int_0^{i\infty} \eta^{-u}(1+\eta)^{u-1} \exp\left(2\pi i m \left(\left(\frac{f}{l} + \frac{n}{k}\right)\eta + \frac{f}{l}\right)\right) d\eta, \end{aligned} \tag{16}$$

where the second equality is valid if the summation and the integration can be interchanged. In case $n < -kf/l$, since

$$-y + 2\pi i \left( \frac{f}{l} + \frac{n}{k} \right) = e^{\pi i} y + 2\pi e^{3\pi i/2} \left| \frac{f}{l} + \frac{n}{k} \right|,$$

setting

$$y = 2\pi \left| \frac{f}{l} + \frac{n}{k} \right| e^{\pi i/2} \eta, \qquad \arg \eta = -\frac{\pi}{2},$$

we find that $I(n; k, l, f)$ has an expression similar to that in (16) with the integral over the interval $[0, -i\infty)$, instead of $[0, i\infty)$. Substituting these results into (15) and using the formula $\Gamma(u)\Gamma(1-u) = \pi / \sin(\pi u)$, we obtain

$$g(u, 1-u; A) = \sum_{k \le M} \sum_{l \le M} \frac{a(k)\overline{a(l)}}{kl} \left(J_+(k,l) + J_-(k,l)\right), \tag{17}$$

where

$$J_\pm(k,l) = \sum_{f=1}^{l} \sum_{n} \sum_{m=1}^{\infty} \int_0^{\pm i\infty} \eta^{-u}(1+\eta)^{u-1} \exp\left(2\pi i m \left(\left(\frac{f}{l} + \frac{n}{k}\right)\eta + \frac{f}{l}\right)\right) d\eta,$$

and the summation with respect to $n$ runs over all $n > -kf/l$ (resp. all $n < -kf/l$) for $J_+$ (resp. $J_-$).

Using the notations $\kappa$ and $\lambda$, we have

$$\exp\left(2\pi i m \left(\left(\frac{f}{l} + \frac{n}{k}\right)\eta + \frac{f}{l}\right)\right) = \exp\left(2\pi i \frac{mf}{l}\right) \exp\left(2\pi i \frac{m(f\kappa + n\lambda)}{(k,l)\kappa\lambda} \eta\right).$$

Put $h = f\kappa + n\lambda$. If $n > -kf/l$, then $h > 0$. On the other hand, for any positive integer $h$, we can find integers $f$ and $n$ such that $1 \le f \le l$ and $h = f\kappa + n\lambda$. In fact, since $(\kappa, \lambda) = 1$ we find integers $x$, $y$ with $x\kappa + y\lambda = 1$. Then $f = hx + \nu\lambda$ and $n = hy - \nu\kappa$ (for any integer $\nu$) satisfy $h = f\kappa + n\lambda$. Choosing $\nu$ suitably we have $1 \le f \le l$, as desired. Therefore

$$J_+(k,l) = \sum_{m=1}^{\infty} \sum_{h=1}^{\infty} \sum_{f} \exp\left(2\pi i \frac{mf}{l}\right) \int_0^{i\infty} \eta^{-u}(1+\eta)^{u-1} \exp\left(2\pi i \frac{mh}{(k,l)\kappa\lambda} \eta\right) d\eta,$$

where the innermost sum runs over all $f$ satisfying $1 \le f \le l$ and there exists an integer $n$ with $h = f\kappa + n\lambda$.

Recall that $\overline{\kappa}$ is an integer satisfying $\kappa\overline{\kappa} \equiv 1 \pmod{\lambda}$. Then $h = f\kappa + n\lambda$ implies $f \equiv h\overline{\kappa} \pmod{\lambda}$. The number of $f$ satisfying this congruence condition and $1 \le f \le l$ is $l/\lambda = (k,l)$. Hence

$$\sum_{f} \exp\left(2\pi i \frac{mf}{l}\right) = \sum_{j=1}^{(k,l)} \exp\left(2\pi i \frac{m}{l}(h\overline{\kappa} + j\lambda)\right) = \exp\left(2\pi i \frac{mh\overline{\kappa}}{l}\right) \sum_{j=1}^{(k,l)} \exp\left(2\pi i \frac{mj}{(k,l)}\right),$$

and the inner sum equals $(k,l)$ if $(k,l) \mid m$ and 0 otherwise. Therefore, setting $m = (k,l)\,\mu$, we obtain

$$J_+(k,l) = (k,l)\sum_{\mu=1}^{\infty}\sum_{h=1}^{\infty}\exp\left(2\pi i\,\frac{\mu h\overline{\kappa}}{\lambda}\right)\int_0^{i\infty}\eta^{-u}(1+\eta)^{u-1}\exp\left(2\pi i\,\frac{\mu h}{\kappa\lambda}\eta\right)d\eta.$$

Setting further $\mu h = n$, we obtain

$$J_+(k,l) = (k,l)\sum_{n=1}^{\infty} d(n)\,e^{2\pi i\overline{\kappa}n/\lambda}\int_0^{i\infty}\eta^{-u}(1+\eta)^{u-1}e^{2\pi in\eta/\kappa\lambda}\eta\,d\eta. \tag{18}$$

Similarly,

$$J_-(k,l) = (k,l)\sum_{n=1}^{\infty} d(n)\,e^{-2\pi i\overline{\kappa}n/\lambda}\int_0^{-i\infty}\eta^{-u}(1+\eta)^{u-1}e^{-2\pi in\eta/\kappa\lambda}\,d\eta. \tag{19}$$

The integrals on the right-hand sides of (18) and (19) are estimated as $O\left(n^{\Re u-1}\right)$, so the infinite series including those integrals are absolutely convergent for $\Re u < 0$. Hence we can justify the above interchanges of summation and integration. Lastly, we can rotate the paths of integration from $[0, \pm i\infty)$ to $[0,\infty)$. Substituting the resulting expressions into (17), we obtain

$$g(u,1-u;A) = \sum_{k\le M}\sum_{l\le M}\frac{a(k)\,\overline{a(l)}}{[k,l]}\sum_{n\ne 0} d(|n|)\,e^{2\pi i\overline{\kappa}n/\lambda}h\left(u,\frac{n}{\kappa\lambda}\right) \tag{20}$$

for $\Re u < 0$, where

$$h(u,x) = \int_0^{\infty} y^{-u}(1+y)^{u-1}\exp(2\pi ixy)\,dy. \tag{21}$$

Formulas (11), (13), (20) are stated by Motohashi in [16, p. 400]. The method used above to deduce these formulas was originally sketched in [16], and it is a variant of the idea presented by Motohashi in [15]. See also [11, Section 2 and 5].

Formula (20) is a generalization of the formula stated by Atkinson in [1, p. 357]. It is also possible to deduce (20) along the same line as in Atkinson [1], using Euler–Maclaurin's formula and (slightly modified) Poisson's summation formula, though we will not give further details here.



## 4. The analytic continuation

In this section we discuss the analytic continuation of $g(u,1-u;A)$. Our method is a direct generalization of Atkinson's original method [1]. In his argument, Atkinson used the asymptotic formula for $\sum_{n\le x} d(n)$. For our present purpose, we need an asymptotic formula for

$$D\left(x,\frac{\overline{\kappa}}{\lambda}\right) = \sum_{n\le x} d(n)\,e^{2\pi i\overline{\kappa}n/\lambda},$$

or $D^*(x,\overline{\kappa}/\lambda)$, whose expression is similar to that of $D(x,\overline{\kappa}/\lambda)$ but if $x$ is an integer then the last term of the sum is to be halved.

These sums have been studied by Jutila [10]. Define the error term $\Delta(x,\overline{\kappa}/\lambda)$ by the formula

$$D\left(x,\frac{\overline{\kappa}}{\lambda}\right) = \frac{1}{\lambda}\left(x\log x + (2\gamma - 1 - 2\log\lambda)x\right) + E\left(0,\frac{\overline{\kappa}}{\lambda}\right) + \Delta\left(x,\frac{\overline{\kappa}}{\lambda}\right), \tag{22}$$

where $E(0,\overline{\kappa}/\lambda)$ is the constant term (which is the value at $s=0$ of the Estermann zeta-function). If $D(x,\overline{\kappa}/\lambda)$ is replaced by $D^*(x,\overline{\kappa}/\lambda)$ in (22), we denote the corresponding error term by $\Delta^*(x,\overline{\kappa}/\lambda)$. Clearly $\Delta^*(x,\overline{\kappa}/\lambda) = \Delta(x,\overline{\kappa}/\lambda)$ if $x$ is not an integer. Then we have (cf. [10, Theorem 1.6 and (1.5.20)])

### *Lemma 4.1.*

*For any $x > 0$,*

$$\Delta^*\left(x, \frac{\overline{\kappa}}{\lambda}\right) = -x^{1/2}\sum_{n=1}^{\infty} d(n)\, n^{-1/2}\left\{e^{-2\pi i\kappa n/\lambda}\, Y_1\left(\frac{4\pi\sqrt{nx}}{\lambda}\right) + \frac{2}{\pi}\, e^{2\pi i\kappa n/\lambda} K_1\left(\frac{4\pi\sqrt{nx}}{\lambda}\right)\right\}, \tag{23}$$

*where $Y_1$, $K_1$ are standard notation for Bessel functions and, moreover,*

$$\Delta\left(x, \frac{\overline{\kappa}}{\lambda}\right) = O\left(\lambda^{2/3}x^{1/3+\varepsilon}\right) \tag{24}$$

*if $x \geq 1$ and $\lambda \leq x$.*

Formula (23) is the Voronoï-type formula for $\Delta^*(x, \overline{\kappa}/\lambda)$.

Let $X = X(\kappa, \lambda) \geq 1$, whose value we will specify later, and write

$$\sum_{n>X} d(n)\, e^{2\pi i\overline{\kappa} n/\lambda} h\left(u, \frac{n}{\kappa\lambda}\right) = \int_X^{\infty} h\left(u, \frac{x}{\kappa\lambda}\right) dD\left(x, \frac{\overline{\kappa}}{\lambda}\right). \tag{25}$$

By rotating the path of integration of (21) to $[0, i\infty)$ we can easily see that

$$h(u, x) = O\left(x^{\Re u - 1}\right) \tag{26}$$

for $\Re u < 1$. Hence, if $\Re u < 0$,

$$h\left(u, \frac{x}{\kappa\lambda}\right) D\left(x, \frac{\overline{\kappa}}{\lambda}\right) \to 0$$

as $x \to \infty$. Therefore by integration by parts we have

$$\sum_{n>X} d(n)\, e^{2\pi i\overline{\kappa} n/\lambda}\, h\left(u, \frac{n}{\kappa\lambda}\right) = -h\left(u, \frac{X}{\kappa\lambda}\right) D\left(X, \frac{\overline{\kappa}}{\lambda}\right) - \int_X^{\infty} \frac{\partial h(u, x/\kappa\lambda)}{\partial x}\, D\left(x, \frac{\overline{\kappa}}{\lambda}\right) dx. \tag{27}$$

Substituting (22) into the integral on the right-hand side of (27), and applying integration by parts once more, we obtain

$$\sum_{n=1}^{\infty} d(n)\, e^{2\pi i\overline{\kappa} n/\lambda}\, h\left(u, \frac{n}{\kappa\lambda}\right) = g_1(u) - g_2(u) + g_3(u) - g_4(u)$$

for $\Re u < 0$, where

$$g_1(u) = \sum_{n\leq X} d(n)\, e^{2\pi i\overline{\kappa} n/\lambda} h\left(u, \frac{n}{\kappa\lambda}\right), \tag{28}$$

$$g_2(u) = h\left(u, \frac{X}{\kappa\lambda}\right) \Delta\left(X, \frac{\overline{\kappa}}{\lambda}\right), \tag{29}$$

$$g_3(u) = \frac{1}{\lambda}\int_X^{\infty} h\left(u, \frac{x}{\kappa\lambda}\right)\left(\log x + 2\gamma - 2\log\lambda\right) dx,$$

$$g_4(u) = \int_X^{\infty} \frac{\partial h(u, x/\kappa\lambda)}{\partial x}\, \Delta\left(x, \frac{\overline{\kappa}}{\lambda}\right) dx.$$

Similarly, we have

$$\sum_{n=1}^{\infty} d(n)\, e^{-2\pi i \bar{\kappa} n/\lambda}\, h\left(u, \frac{-n}{\kappa\lambda}\right) = \overline{g_1(\overline{u}) - g_2(\overline{u}) + g_3(\overline{u}) - g_4(\overline{u})},$$

hence from (20) we have

$$g(u, 1-u; A) = \sum_{k\le M}\sum_{l\le M} \frac{a(k)\,\overline{a(l)}}{[k,l]} \left(g_1(u) - g_2(u) + g_3(u) - g_4(u) + \overline{g_1(\overline{u}) - g_2(\overline{u}) + g_3(\overline{u}) - g_4(\overline{u})}\right) \tag{30}$$

for $\Re u < 0$.

In view of (26), $g_1(u)$ and $g_2(u)$ are holomorphic in the region $\Re u < 1$. Also, by [1, p. 359] we have

$$\frac{\partial h(u,x)}{\partial x} = O\left(x^{\Re u - 2}\right).$$

This estimate and (24) imply that $g_4(u)$ is holomorphic for $\Re u < 2/3$. As for $g_3(u)$, similarly to the argument in [1, pp. 359–360], we obtain

$$\begin{aligned} g_3(u) + \overline{g_3(\overline{u})} = &-\frac{\kappa}{\pi}(\log X + 2\gamma - 2\log\lambda)\int_0^\infty y^{-u-1}(1+y)^{u-1}\sin\left(2\pi\frac{Xy}{\kappa\lambda}\right)dy \\ &+ \frac{\kappa}{\pi u}\int_0^\infty y^{-u-1}(1+y)^u \sin\left(2\pi\frac{Xy}{\kappa\lambda}\right)dy. \end{aligned} \tag{31}$$

The two integrals on the right-hand side are convergent uniformly in $\Re u \le 1-\varepsilon$. Hence (30) with (31) gives an expression of $g(u, 1-u; A)$ valid for $\Re u < 2/3$.

By the same argument we have

$$\begin{aligned} \overline{g(1-\overline{u}, \overline{u}; A)} = \sum_{k\le M}\sum_{l\le M} \frac{\overline{a(k)}\,a(l)}{[k,l]} \Big(&g_1(1-u) - g_2(1-u) + g_3(1-u) - g_4(1-u) \\ &+ \overline{g_1(1-\overline{u}) - g_2(1-\overline{u}) + g_3(1-\overline{u}) - g_4(1-\overline{u})}\Big) \end{aligned} \tag{32}$$

which is valid for $\Re u > 1/3$.

## 5. The decomposition of the mean square

Now we start to consider the mean square integral of $\zeta(1/2+it)A(1/2+it)$. The rest of the proof of Theorem 1.1 is an analogue of the argument in [5]. Let $C_1, C_2$, and $C^*$ be the same as in the statement of Theorem 1.1. Assume $C_1 T < Y < C_2 T$, $T \ge C^*$, and let $X = X(\kappa,\lambda) = \kappa\lambda Y$.

Let $u = 1/2 + it$ in (13), and integrate both sides on the interval $[T, 2T]$. By Stirling's formula we obtain

$$\frac{1}{2}\int_T^{2T}\left(\frac{\Gamma'}{\Gamma}\left(\frac12+it\right) + \frac{\Gamma'}{\Gamma}\left(\frac12-it\right)\right)dt = \frac{1}{2i}\left(\log\Gamma\left(\frac12+it\right) - \log\Gamma\left(\frac12-it\right)\right)\Bigg|_{t=T}^{t=2T} = (t\log t - t)\Big|_{t=T}^{t=2T} + O(1).$$

Using this, (30), (32) and (49) below, we obtain

$$\int_T^{2T}\left|\zeta\left(\frac12+it\right)A\left(\frac12+it\right)\right|^2 dt = \mathcal{M}(2T,A) - \mathcal{M}(T,A) + I_1 - I_2 + I_3 - I_4 + O\left(M^\varepsilon\right), \tag{33}$$

where

$$I_\nu = \sum_{k\le M}\sum_{l\le M}\frac{2}{[k,l]}\Re\left\{a(k)\,\overline{a(l)}\int_T^{2T}\left(g_\nu\left(\frac{1}{2}+it\right)+\overline{g_\nu\left(\frac{1}{2}-it\right)}\right)dt\right\} \tag{34}$$

for $1\le \nu\le 4$.

Let

$$C_l(y) = y^{1/2}(1+y)^l\log\frac{1+y}{y}.$$

Substituting (28) and (29) with (21) into (34) for $\nu = 1,2$, and changing the order of integration, we get

$$I_1 = \sum_{k\le M}\sum_{l\le M}\sum_{n\le X} S_1(t;n,k,l)\Big|_{t=T}^{2T}, \qquad I_2 = \sum_{k\le M}\sum_{l\le M} S_2(t;k,l)\Big|_{t=T}^{2T}, \tag{35}$$

where

$$\begin{aligned} S_1(t;n,k,l) &= 2\Im\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\,d(n)\,e^{2\pi i\bar{\kappa}n/\lambda}\int_0^\infty \frac{e^{2\pi iny/\kappa\lambda}}{C_{1/2}(y)}\,e^{it\log((1+y)/y)}\,dy\right\}\\ &\quad + 2\Im\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\,d(n)\,e^{-2\pi i\bar{\kappa}n/\lambda}\int_0^\infty \frac{e^{-2\pi iny/\kappa\lambda}}{C_{1/2}(y)}\,e^{it\log((1+y)/y)}\,dy\right\}\\ &= S_{11}(t;n,k,l) + S_{12}(t;n,k,l), \end{aligned} \tag{36}$$

and

$$\begin{aligned} S_2(t;k,l) &= 2\Im\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\,\Delta\left(X,\frac{\bar{\kappa}}{\lambda}\right)\int_0^\infty \frac{e^{2\pi iXy/\kappa\lambda}}{C_{1/2}(y)}\,e^{it\log((1+y)/y)}\,dy\right\}\\ &\quad + 2\Im\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\,\overline{\Delta\left(X,\frac{\bar{\kappa}}{\lambda}\right)}\int_0^\infty \frac{e^{-2\pi iXy/\kappa\lambda}}{C_{1/2}(y)}\,e^{it\log((1+y)/y)}\,dy\right\}. \end{aligned} \tag{37}$$

Using (31) we have

$$I_3 = \sum_{k\le M}\sum_{l\le M}\frac{2}{[k,l]}\Re\left\{a(k)\,\overline{a(l)}\left(-\frac{\kappa}{i\pi}\left(\log X + 2\gamma - 2\log\lambda\right)I_{31}(k,l) + \frac{\kappa}{i\pi}I_{32}(k,l)\right)\right\}, \tag{38}$$

where

$$I_{31}(k,l) = \int_0^\infty \frac{\sin(2\pi Xy/\kappa\lambda)}{y\,C_{1/2}(y)}\,e^{it\log((1+y)/y)}\Big|_{t=T}^{2T}\,dy$$

and

$$I_{32}(k,l) = \int_0^\infty \frac{\sin(2\pi Xy/\kappa\lambda)}{y}\int_{1/2+iT}^{1/2+2iT}\frac{1}{u}\left(\frac{1+y}{y}\right)^u\,du\,dy.$$

As for $I_4$, similarly to [1, p. 361], we first carry out the integration with respect to $t$, then put $xy=\eta$, change the order of differentiation and integration and differentiate with respect to $x$, and then again put $\eta/x = y$. Changing the variable temporarily to $\eta$ is necessary to ensure the interchange of differentiation and integration. The result is that

$$I_4 = \sum_{k\le M}\sum_{l\le M} S_4(t;k,l)\Big|_{t=T}^{2T}, \tag{39}$$

where

$$S_4(t;k,l) = S_{40} - S_{41} + S_{42} - S_{43}$$

with

$$S_{40} = 2\Re\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\int_X^\infty \frac{\Delta(x,\overline{\kappa}/\lambda)}{x}\int_0^\infty \frac{e^{2\pi ixy/\kappa\lambda}}{C_{3/2}(y)}\,t\,e^{it\log((1+y)/y)}\,dy\,dx\right\},$$
$$S_{41} = 2\Im\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\int_X^\infty \frac{\Delta(x,\overline{\kappa}/\lambda)}{x}\int_0^\infty \frac{e^{2\pi ixy/\kappa\lambda}}{C_{3/2}(y)}\,e^{it\log((1+y)/y)}\left(\frac{1}{2}+\frac{1}{\log((1+y)/y)}\right)dy\,dx\right\},$$

and $S_{42}$ (resp. $S_{43}$) is similar to $S_{40}$ (resp. $S_{41}$) with $\Delta(x,\overline{\kappa}/\lambda)$ and $e^{2\pi ixy/\kappa\lambda}$ replaced by their complex conjugates. We decompose

$$S_{40}\Big|_{t=T}^{2T} = J_1(k,l) + J_2(k,l) - J_3(k,l),$$

where

$$J_1(k,l) = 2\Re\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\int_{2X}^\infty \frac{\Delta(x,\overline{\kappa}/\lambda)}{x}\int_0^\infty \frac{e^{2\pi ixy/\kappa\lambda}}{C_{3/2}(y)}\,2T\,e^{2iT\log((1+y)/y)}\,dy\,dx\right\},$$
$$J_2(k,l) = 2\Re\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\int_X^{2X} \frac{\Delta(x,\overline{\kappa}/\lambda)}{x}\int_0^\infty \frac{e^{2\pi ixy/\kappa\lambda}}{C_{3/2}(y)}\,2T\,e^{2iT\log((1+y)/y)}\,dy\,dx\right\},$$

and

$$J_3(k,l) = 2\Re\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\int_X^\infty \frac{\Delta(x,\overline{\kappa}/\lambda)}{x}\int_0^\infty \frac{e^{2\pi ixy/\kappa\lambda}}{C_{3/2}(y)}\,T\,e^{iT\log((1+y)/y)}\,dy\,dx\right\}. \tag{40}$$

Therefore

$$I_4 = \sum_{k\le M}\sum_{l\le M}(J_1(k,l)+J_2(k,l)-J_3(k,l)) + \sum_{k\le M}\sum_{l\le M}(-S_{41}+S_{42}-S_{43})\Big|_{t=T}^{2T}. \tag{41}$$

In order to evaluate exponential integrals appearing in the above formulas, here we quote the following lemma, due to the first author [5].

**Lemma 5.1 ([5, Lemma 5 and Remark 5]).**
*(i) Let $\alpha,\beta,\gamma,a,b,k,T$ be real numbers such that $\alpha,\beta,\gamma$ are positive and bounded, $\alpha\ne 1$, $0<a<1/2$, $a<T/(8\pi k)$, $k>0$, $T\ge 1$, and*

$$b \ge \max\left\{T,\frac{1}{k},-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{T}{2\pi k}}\right\}.$$

*Then,*

$$\begin{aligned}\int_a^b \frac{\exp\left(\pm i\left(T\log((1+y)/y)+2\pi ky\right)\right)}{y^\alpha(1+y)^\beta\log^\gamma((1+y)/y)}\,dy &= \frac{T^{1/2}\exp\left(\pm i\left(TV+2\pi kU-\pi k+\pi/4\right)\right)}{2k\pi^{1/2}V^\gamma U^{1/2}(U-1/2)^\alpha(U+1/2)^\beta}\\ &+O\left(\frac{a^{1-\alpha}}{T}\right)+O\left(\frac{b^{\gamma-\alpha-\beta}}{k}\right)+R(T,k)+O\left(e^{-CT}\right)\\ &+O\left(e^{-C\sqrt{kT}}(T^{\gamma-\alpha-\beta+1}+T)\right)+O\left(k^{-1}e^{-CkT}(b^{\gamma-\alpha-\beta}+T^{\gamma-\alpha-\beta})\right)\end{aligned} \tag{42}$$

*uniformly for $|\alpha-1|>\varepsilon$, where $C$ is an absolute constant,*

$$U=\sqrt{\frac{1}{4}+\frac{T}{2\pi k}},\qquad V=2\,\mathrm{arcsinh}\sqrt{\frac{\pi k}{2T}},$$
$$R(T,k)\ll\begin{cases}T^{(\gamma-\alpha-\beta)/2-1/4}\,k^{-(\gamma-\alpha-\beta)/2-5/4} & \text{if } 0<k\le T,\\ T^{-1/2-\alpha}\,k^{\alpha-1} & \text{if } k\ge T.\end{cases}$$

*The term $O(e^{-CT})$ is to be omitted if $1 < \alpha$ or $0 < \alpha < 1$ with $k \ge T$. When we replace $k$ by $-k$ on the left-hand side of (42), then the explicit term and $R(T,k)$ for $k \le T$ on the right-hand side are to be omitted.*

(ii) *If we replace the assumption $k > 0$ by $k \ge B$, where $B$ is an absolute constant satisfying $0 < B \le 1$, then the last three error terms on the right-hand side of (42) are absorbed into the other error terms. When $k \ge B$ and $k$ on the left-hand side of (42) is replaced by $-k$, then the explicit term and the last three error terms on the right-hand side of (42) are to be omitted.*

This is a modified version of the saddle point lemma of Atkinson [1]. The important point of this modification is that it is available for small $k$.

## 6. Evaluation of $I_1$, $I_2$ and $I_3$

In this section we apply Lemma 5.1 to $I_1$, $I_2$ and $I_3$. From (35), we have

$$I_1 = \sum_{k\le M}\sum_{l\le M}\sum_{n\le 2X} S_1(2T;n,k,l) - \sum_{k\le M}\sum_{l\le M}\sum_{X<n\le 2X} S_1(2T;n,k,l) - \sum_{k\le M}\sum_{l\le M}\sum_{n\le X} S_1(T;n,k,l) = I_{11} - I_{12} - I_{13}. \tag{43}$$

Similarly to Section 8 of [5], we apply part (i) of Lemma 5.1 to $I_{13}$ to obtain

$$I_{13} = \Sigma_1(T,Y) + O\left(T^{-1/4}\sum_{k\le M}\sum_{l\le M}\frac{|a(k)\,\overline{a(l)}|}{[k,l]}(\kappa\lambda)^{5/4}\right). \tag{44}$$

The same type of formula, replacing $\Sigma_1(T,Y)$ with $\Sigma_1(2T,2Y)$, holds for $I_{11}$.

Next, again similarly to [5, Section 8], applying (24) (note that $T \ge C^*$) and part (ii) of Lemma 5.1 to (37), we obtain

$$I_2 \ll \sum_{k\le M}\sum_{l\le M}\frac{|a(k)\,\overline{a(l)}|}{[k,l]}\kappa^{1/3+\varepsilon}\lambda^{1+\varepsilon}T^{-1/6+\varepsilon}. \tag{45}$$

The quantity $I_3$ can be treated similarly to the argument in [1, pp. 368–371]. In fact, we have

$$I_{31}(k,l) \ll T^{-1/2}. \tag{46}$$

As for $I_{32}$, we decompose the integral with respect to $y$ at $y = 1$, and denote the contributions of the intervals $[0,1]$ and $[1,\infty)$ by $I_{32}'$ and $I_{32}''$, respectively. Then

$$I_{32}' \ll T^{-1/2}, \qquad I_{32}'' \ll T^{-1}\log T. \tag{47}$$

Note that in [1], the term corresponding to $I_{32}'$ is shown to be equal to $\pi^2 i + O(T^{-1/2})$. In [1] the integration with respect to $u$ is from $1/2 - iT$ to $1/2 + iT$, and the term $\pi^2 i$ appears from the residue at $u = 0$ when the path of integration is deformed. In the present situation, however, the integration with respect to $u$ is from $1/2 + iT$ to $1/2 + 2iT$, hence the residue does not appear. Therefore the first estimate of (47) follows.

From (38), (46) and (47), we obtain

$$I_3 \ll \sum_{k\le M}\sum_{l\le M}\frac{|a(k)\,\overline{a(l)}|}{[k,l]}\kappa\log(\kappa\lambda T)\,T^{-1/2}. \tag{48}$$

Using the equality $[k,l] = kl/(k,l)$ and putting $(k,l) = h$, $k = k'h$, $l = l'h$, we have

$$\sum_{k\le M}\sum_{l\le M}\frac{1}{[k,l]} \le \sum_{h\le M}\frac{1}{h}\sum_{k',l'\le M/h}\frac{1}{k'l'} \ll \log^3 M. \tag{49}$$

Applying this estimate and (1), we find that the error term on the right-hand side of (44) is $O\big(M^{5/2+\varepsilon}T^{-1/4}\big)$. Similarly, from (45) and (48) we have

$$I_2 \ll M^{4/3+\varepsilon}T^{-1/6+\varepsilon}, \qquad I_3 \ll M^{1+\varepsilon}T^{-1/2}\log T.$$

Collecting all the results obtained in this section, we have

$$I_1 - I_2 + I_3 = \Sigma_1(2T,2Y) - \Sigma_1(T,Y) - I_{12} + O\big(M^{5/2+\varepsilon}T^{-1/4} + M^{4/3+\varepsilon}T^{-1/6+\varepsilon}\big). \tag{50}$$

## 7. Evaluation of $I_4$

Now we proceed to the evaluation of $I_4$. Applying part Lemma 5.1(ii) to the inner integral of $S_{41}$, we have

$$S_{41} \ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]}\int_X^\infty \frac{|\Delta(x,\overline{\kappa}/\lambda)|}{x}\left(\frac{\kappa\lambda}{x}\right)^{1/2}dx = \frac{|a(k)\,\overline{a(l)}|}{[k,l]}(\kappa\lambda)^{1/2}\sum_{j=1}^{\infty}\int_{2^{j-1}X}^{2^jX} x^{-3/2}\left|\Delta\left(x,\frac{\overline{\kappa}}{\lambda}\right)\right|dx.$$

By the Cauchy–Schwarz inequality, each integral on the right-hand side is

$$\ll \left(\int_{2^{j-1}X}^{2^jX} x^{-3}dx\right)^{1/2}\left(\int_{2^{j-1}X}^{2^jX}\left|\Delta\left(x,\frac{\overline{\kappa}}{\lambda}\right)\right|^2dx\right)^{1/2}.$$

The second factor of the above is estimated using

$$\int_U^{2U}\left|\Delta\left(x,\frac{\overline{\kappa}}{\lambda}\right)\right|^2dx \ll \lambda U^{3/2} + \lambda^2U^{1+\varepsilon}, \qquad U\ge 1,$$

which is implied by [10, Theorem 1.2]. Hence

$$\begin{aligned} S_{41} &\ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]}(\kappa\lambda)^{1/2}\sum_{j=1}^{\infty}(2^jX)^{-1}\left\{\lambda^{1/2}(2^jX)^{3/4} + \lambda(2^jX)^{1/2+\varepsilon}\right\} \\ &\ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]}\left(\kappa^{1/4}\lambda^{3/4}T^{-1/4} + \kappa^{\varepsilon}\lambda^{1+\varepsilon}T^{-1/2+\varepsilon}\right), \end{aligned} \tag{51}$$

and the same estimate holds also for $S_{42}$, $S_{43}$.

Next consider $J_3(k,l)$. Here we use the formula

$$\begin{aligned} \Delta^*\left(x,\frac{\overline{\kappa}}{\lambda}\right) &= \frac{\lambda^{1/2}}{\sqrt{2}\pi}x^{1/4}\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/4}}e^{-2\pi i\overline{\kappa}n/\lambda}\cos\left(\frac{4\pi\sqrt{nx}}{\lambda}-\frac{\pi}{4}\right) \\ &\quad - \frac{3\lambda^{3/2}}{32\sqrt{2}\pi^2}x^{-1/4}\sum_{n=1}^{\infty}\frac{d(n)}{n^{5/4}}e^{-2\pi i\overline{\kappa}n/\lambda}\sin\left(\frac{4\pi\sqrt{nx}}{\lambda}-\frac{\pi}{4}\right) + O\left(\lambda^{5/2}x^{-3/4}\right), \end{aligned} \tag{52}$$

which can be easily obtained from (23) by using asymptotic expansion formulas for Bessel functions.

We proceed similarly to the argument in [5, pp. 58–59]. First applying Lemma 5.1(ii) to the inner integral on the right-hand side of (40), and estimating the error term by (the same argument as) (51). Then applying (52) to the explicit term we obtain

$$J_3(k,l) = J_{31}(k,l) - J_{32}(k,l) + J_{33}(k,l) + O\left(\frac{|a(k)\,\overline{a(l)}|}{[k,l]}\left(\kappa^{1/4}\lambda^{3/4}T^{-1/4} + \kappa^{\varepsilon}\lambda^{1+\varepsilon}T^{-1/2+\varepsilon}\right)\right), \tag{53}$$

where

$$J_{31}(k,l) = \frac{T}{\pi}\Re\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\kappa^{1/4}\lambda^{3/4}\int_{\sqrt{X/\kappa\lambda}}^{\infty}\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/4}}e^{-2\pi i\kappa n/\lambda}\cos\left(4\pi x\sqrt{\frac{\kappa n}{\lambda}} - \frac{\pi}{4}\right)\right.$$
$$\left.\times\,\psi_{3/2}(T,x)\exp\left(i\left(f(T,x^2) - \pi x^2 + \frac{\pi}{2}\right)\right)dx\right\}, \tag{54}$$

$$J_{32}(k,l) = \frac{3T}{32\pi^2}\Re\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\kappa^{-1/4}\lambda^{5/4}\int_{\sqrt{X/\kappa\lambda}}^{\infty}\sum_{n=1}^{\infty}\frac{d(n)}{n^{5/4}}e^{-2\pi i\kappa n/\lambda}\sin\left(4\pi x\sqrt{\frac{\kappa n}{\lambda}} - \frac{\pi}{4}\right)\right.$$
$$\left.\times\,\psi_{5/2}(T,x)\exp\left(i\left(f(T,x^2) - \pi x^2 + \frac{\pi}{2}\right)\right)dx\right\}, \tag{55}$$

and

$$J_{33}(k,l) = O\left(\frac{|a(k)\,\overline{a(l)}|}{[k,l]}T\int_X^{\infty}\lambda^{5/2}x^{-7/4}\left|\psi_1(T,\sqrt{x/\kappa\lambda})\right|dx\right)$$

with

$$\psi_\alpha(T,x) = x^{-\alpha}\left(\operatorname{arcsinh} x\sqrt{\frac{\pi}{2T}}\right)^{-1}\left(\sqrt{\frac{T}{2\pi x^2} + \frac{1}{4}} + \frac{1}{2}\right)^{-1}\left(\frac{T}{2\pi x^2} + \frac{1}{4}\right)^{-1/4}.$$

Since $\psi_1(T,\sqrt{x/\kappa\lambda}) \ll \sqrt{\kappa\lambda/x}$, we have

$$J_{33}(k,l) \ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]}\kappa^{-3/4}\lambda^{7/4}T^{-1/4}. \tag{56}$$

As for $J_{31}(k,l)$, we consider a finite truncation of the integral and apply [5, Lemma 6]. (Here we note that on the left-hand side of (42) in [5], the factor $g_{3/2}(T,x)\cos(\cdots)\exp i(\cdots)\,dx$ is to be on the numerator.) Similarly to (43) in [5], we obtain

$$J_{31}(k,l) = W_1(k,l) + W_2(k,l) + W_3(k,l) + W_4(k,l), \tag{57}$$

where

$$W_1(k,l) = \frac{T}{2\pi}\Re\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\kappa^{1/4}\lambda^{3/4}\sum_{n\le Z}\frac{d(n)}{n^{3/4}}e^{-2\pi i\kappa n/\lambda}\left(\frac{\kappa n}{\lambda}\right)^{1/4}\frac{4\pi\exp\left(i(g(T,\kappa n/\lambda))\right)}{T\log(\lambda T/2\pi\kappa n)}\right\}$$

(here we put $Z = (\lambda/\kappa)\,\xi(T,X/\kappa\lambda) = (\lambda/\kappa)\,\xi(T,Y)$),

$$W_2(k,l) \ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]}\kappa^{1/4}\lambda^{3/4}T^{-1/2}\sum_{n\le Z}\frac{d(n)}{n^{3/4}}\left(\frac{\kappa n}{\lambda}\right)^{1/4}\frac{1}{(T/2\pi - \kappa n/\lambda)^{1/2}},$$
$$W_3(k,l) \ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]}\kappa^{1/4}\lambda^{3/4}T^{1/4}\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/4}}\min\left\{1, \frac{1}{|2\sqrt{\kappa n/\lambda} - 2\sqrt{\xi(T,Y)}|}\right\}, \tag{58}$$

and

$$W_4(k,l) \ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]}\kappa^{1/4}\lambda^{3/4}\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/4}}e^{-CT}\left(\frac{\lambda}{\kappa n}\right)^{1/2}.$$

In the case of $J_{32}(k,l)$, it is not necessary to consider the finite truncation of the integral, because the infinite series on the right-hand side of (55) is absolutely convergent. Hence, we can apply [5, Lemma 6] to $J_{32}(k,l)$ to obtain

$$J_{32}(k,l) = W_5(k,l) + W_6(k,l) + W_7(k,l) + W_8(k,l), \tag{59}$$

where

$$W_5(k,l) = \frac{3T}{64\pi^2}\Re\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\,\kappa^{-1/4}\lambda^{5/4}\sum_{n\le Z}\frac{d(n)}{n^{5/4}}\,e^{-2\pi i\kappa n/\lambda}\left(\frac{\kappa n}{\lambda}\right)^{3/4}\frac{4\pi\exp\big(i\big(g(T,\kappa n/\lambda)\big)\big)}{T\log(\lambda T/2\pi\kappa n)\,(T/2\pi-\kappa n/\lambda)}\right\},$$

and $W_6(k,l)$, $W_7(k,l)$, $W_8(k,l)$ are similar to $W_2(k,l)$, $W_3(k,l)$, $W_4(k,l)$, respectively, just replacing the factor $\kappa^{1/4}\lambda^{3/4}$ by $\kappa^{-1/4}\lambda^{5/4}$, the factor $n^{3/4}$ by $n^{5/4}$, (in the case of $W_6(k,l)$) $(T/2\pi-\kappa n/\lambda)^{1/2}$ by $(T/2\pi-\kappa n/\lambda)^{3/2}$, and (in the case of $W_7(k,l)$) $T^{1/4}$ by $T^{-1/4}$.

It is immediate that

$$W_4(k,l) \ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]}\,\kappa^{-1/4}\lambda^{5/4}\,e^{-CT}, \tag{60}$$

$$W_8(k,l) \ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]}\,\kappa^{-3/4}\lambda^{7/4}\,e^{-CT}. \tag{61}$$

Also, replacing the factor $\min\{\cdots\}$ by 1, we get

$$W_7(k,l) \ll \frac{|a(k)\acute{o},\,\overline{a(l)}|}{[k,l]}\,\kappa^{-1/4}\lambda^{5/4}\,T^{-1/4}. \tag{62}$$

Next we note that, when $n\le Z$, the inequalities

$$\left(\frac{T}{2\pi}-\frac{\kappa n}{\lambda}\right)^{-1} \ll T^{-1}, \qquad \log^{-1}\frac{\lambda T}{2\pi\kappa n} \ll 1$$

hold. Since $T \ll \xi(T, X/\kappa\lambda) < T$, we have

$$\sum_{n\le Z}\frac{d(n)}{n^{1/2}} \ll \left(\frac{\lambda T}{\kappa}\right)^{1/2}\log\left(2+\frac{\lambda T}{\kappa}\right).$$

By using these facts, we can show

$$W_2(k,l),\ W_5(k,l) \ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]}\,\lambda T^{-1/2}\log\left(2+\frac{\lambda T}{\kappa}\right), \tag{63}$$

$$W_6(k,l) \ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]}\,\lambda T^{-3/2}\log\left(2+\frac{\lambda T}{\kappa}\right). \tag{64}$$

Consider $W_3(k,l)$. Split the sum on the right-hand side of (58) as

$$\sum_{n\le Z/2}+\sum_{Z/2<n\le Z-\sqrt{Z}}+\sum_{Z-\sqrt{Z}<n\le Z+\sqrt{Z}}+\sum_{Z+\sqrt{Z}<n\le 2Z}+\sum_{n>2Z} = S_1+S_2+S_3+S_4+S_5.$$

If $Z \geq 2$, we can estimate these sums similarly to the argument on the last page in [1]. We have

$$S_1,\ S_5 \ll \left(\frac{\lambda}{\kappa}\right)^{1/4} T^{-1/4} \log\left(2+\frac{\lambda T}{\kappa}\right), \qquad S_2,\ S_4 \ll \left(\frac{\lambda}{\kappa}\right)^{1/4} T^{-1/4} \log^2\left(2+\frac{\lambda T}{\kappa}\right),$$
$$\text{and} \qquad S_3 \ll \left(\frac{\kappa}{\lambda}\right)^{1/4} T^{-1/4} \log\left(2+\frac{\lambda T}{\kappa}\right).$$

Hence

$$W_3(k,l) \ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]} \left\{\lambda \log^2\left(2+\frac{\lambda T}{\kappa}\right) + (\kappa\lambda)^{1/2}\, T^{-1/4} \log\left(2+\frac{\lambda T}{\kappa}\right)\right\} \tag{65}$$

if $Z \geq 2$. For $Z < 2$, we decompose the sum on the right-hand side of (58) as

$$\sum_{n \leq 2Z} + \sum_{n > 2Z}.$$

The first sum is obviously $O(1)$, and the second sum is easily estimated as $O\big((\lambda/\kappa)^{1/2}\big)$. Hence, in this case, we have

$$W_3(k,l) \ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]}\, \kappa^{1/4}\lambda^{3/4}\, T^{1/4} \left(1+\left(\frac{\lambda}{\kappa}\right)^{1/2}\right) \ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]}\, (\kappa\lambda)^{1/2}, \tag{66}$$

where the second inequality follows from

$$\frac{\lambda}{\kappa} \ll \frac{\lambda}{\kappa} T \ll \frac{\lambda}{\kappa}\xi(T,Y) = Z < 2.$$

Combining (65) and (66), we obtain the conclusion that

$$W_3(k,l) \ll \frac{|a(k)\,\overline{a(l)}|}{[k,l]} \left\{\lambda \log^2\left(2+\frac{\lambda T}{\kappa}\right) + (\kappa\lambda)^{1/2} T^{-1/4} \log\left(2+\frac{\lambda T}{\kappa}\right) + (\kappa\lambda)^{1/2}\right\}. \tag{67}$$

Now, from (53), (56), (57), (59), (60), (61), (62), (63), (64) and (67), we obtain

$$\begin{aligned} J_3(k,l) &= W_1(k,l) \\ &+ O\Bigg(\frac{|a(k)\,\overline{a(l)}|}{[k,l]} \Bigg\{(\kappa\lambda)^{1/2}\, T^{-1/4} \log\left(2+\frac{\lambda T}{\kappa}\right) + \lambda \log^2\left(2+\frac{\lambda T}{\kappa}\right) + (\kappa\lambda)^{1/2} \\ &\qquad + \kappa^{\varepsilon}\lambda^{1+\varepsilon}\, T^{-1/2+\varepsilon} + \left(\kappa^{1/4}\lambda^{3/4} + \kappa^{-1/4}\lambda^{5/4} + \kappa^{-3/4}\lambda^{7/4}\right) T^{-1/4}\Bigg\}\Bigg). \end{aligned} \tag{68}$$

The quantity $J_1(k,l)$ can be treated similarly. Note that the term $\kappa^{-1/4}\lambda^{5/4}\, T^{-1/4}$ on the right-hand side can be omitted, because

$$\kappa^{-1/4}\lambda^{5/4} \ll \kappa^{1/4}\lambda^{3/4} + \kappa^{-3/4}\lambda^{7/4}.$$

Since

$$\sum_{k \leq M} \sum_{l \leq M} W_1(k,l) = -\Sigma_2\big(T, \xi(T,Y)\big),$$

substituting the obtained results into (41) we get

$$I_4 = -\Sigma_2\big(2T, \xi(2T,2Y)\big) + \Sigma_2\big(T, \xi(T,Y)\big) + \sum_{k \leq M} \sum_{l \leq M} J_2(k,l) + O\left(\sum_{k \leq M} \sum_{l \leq M} \frac{|a(k)\,\overline{a(l)}|}{[k,l]} \left\{\cdots\right\}\right), \tag{69}$$

where $\cdots$ means the same as in the curly parentheses on the right-hand side of (68). Using (1) and (49), we find that the above error term is $O\big(M^{1+\varepsilon} \log^2 T + M^{7/4+\varepsilon} T^{-1/4}\big)$, and hence

$$I_4 = -\Sigma_2\big(2T, \xi(2T,2Y)\big) + \Sigma_2\big(T, \xi(T,Y)\big) + \sum_{k \leq M} \sum_{l \leq M} J_2(k,l) + O\big(M^{1+\varepsilon} \log^2 T + M^{7/4+\varepsilon} T^{-1/4}\big). \tag{70}$$

## 8. The cancellation

Substituting (50) and (70) into (33), we obtain

$$\begin{aligned}\int_T^{2T}\left|\zeta\left(\frac{1}{2}+it\right)A\left(\frac{1}{2}+it\right)\right|^2 dt &= \mathcal{M}(2T,A)-\mathcal{M}(T,A)\\ &+\Sigma_1(2T,2Y)-\Sigma_1(T,Y)+\Sigma_2\big(2T,\xi(2T,2Y)\big)-\Sigma_2\big(T,\xi(T,Y)\big)-I_{12}\\ &-\sum_{k\le M}\sum_{l\le M}J_2(k,l)+O\big(M^{5/2+\varepsilon}T^{-1/4}+M^{4/3+\varepsilon}T^{-1/6+\varepsilon}+M^{1+\varepsilon}\log^2 T\big).\end{aligned} \tag{71}$$

Hence the remaining task is to consider $I_{12}$ and $\sum\sum J_2(k,l)$. In this section we will show that there is a big cancellation between these two quantities and consequently

$$I_{12}=-\sum_{k\le M}\sum_{l\le M}J_2(k,l)+O\big(M^{5/2+\varepsilon}T^{-1/4}+M^{4/3+\varepsilon}T^{-1/6+\varepsilon}\big). \tag{72}$$

This type of result was proved, in the case of Dirichlet $L$-functions, by the first author (in [5, Section 10]). Our proof of (72) follows an argument similar to that in [5].

***Remark 8.1.***
In the case of Dirichlet $L$-functions, this cancellation argument is the most novel part of [5]. When Hafner and Ivić [3] studied the asymptotic behavior of the integral of $E(T)$, the same type of cancellation also happened, and they proved this fact by applying Jutila's transformation method [9]. However, their argument does not seem to be applicable in the general $L$-function case (and also to our present case), so the first author developed an alternative method of proving the cancellation process in [5], which we also use here.

Let

$$S_{13}(t;n,k,l)=2\Im\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\,d(n)\,e^{2\pi i\overline{\kappa}n/\lambda}\int_0^\infty\frac{e^{2\pi iny/\kappa\lambda}}{C_{1/2}(y)}\,e^{-it\log((1+y)/y)}\,dy\right\}$$

and rewrite (36) (with $t=2T$) as

$$S_1(2T;n,k,l)=S_{11}(2T;n,k,l)-S_{13}(2T;n,k,l)+S_{13}(2T;n,k,l)+S_{12}(2T;n,k,l). \tag{73}$$

Substitute this expression into the definition of $I_{12}$. Denote by $R_1$ the contribution of the last two terms on the right-hand side of (73). As for the first two terms, we have

$$\int_0^\infty\frac{e^{2\pi iny/\kappa\lambda}}{C_{1/2}(y)}\left(e^{2iT\log((1+y)/y)}-e^{-2iT\log((1+y)/y)}\right)dy=\int_0^\infty\frac{e^{2\pi iny/\kappa\lambda}}{1+y}\int_{1/2-2iT}^{1/2+2iT}\left(\frac{1+y}{y}\right)^u du\,dy=\int_{1/2-2iT}^{1/2+2iT}h\left(u,\frac{n}{\kappa\lambda}\right)du$$

(here the second equality is justified because $h(u,n/\kappa\lambda)$ is uniformly convergent in $\Re u<1$), so

$$I_{12}=2\sum_{k\le M}\sum_{l\le M}\Im\left\{\int_{1/2-2iT}^{1/2+2iT}\frac{a(k)\,\overline{a(l)}}{[k,l]}\sum_{X<n\le 2X}d(n)\,e^{2\pi i\overline{\kappa}n/\lambda}\,h\left(u,\frac{n}{\kappa\lambda}\right)du\right\}+R_1. \tag{74}$$

We express the inner sum as a Stieltjes integral, as in (25), and apply (22) to obtain

$$\begin{aligned}\sum_{X<n\le 2X}d(n)\,e^{2\pi i\overline{\kappa}n/\lambda}h\left(u,\frac{n}{\kappa\lambda}\right)&=\frac{1}{\lambda}\int_X^{2X}h\left(u,\frac{x}{\kappa\lambda}\right)\big(\log x+2\gamma-2\log\lambda\big)\,dx\\ &+h\left(u,\frac{x}{\kappa\lambda}\right)\Delta\left(x,\frac{\overline{\kappa}}{\lambda}\right)\Big|_X^{2X}-\int_X^{2X}\frac{\partial h(u,x/\kappa\lambda)}{\partial x}\Delta\left(x,\frac{\overline{\kappa}}{\lambda}\right)dx.\end{aligned} \tag{75}$$

Substitute this expression into (74) and denote the contribution of the first (resp. second) term on the right-hand side of (75) to (74) by $R_2$ (resp. $R_3$). We have

$$I_{12} = -2\sum_{k\le M}\sum_{l\le M}\Im\left\{\int_{1/2-2iT}^{1/2+2iT}\frac{a(k)\,\overline{a(l)}}{[k,l]}\int_X^{2X}\frac{\partial h(u,x/\kappa\lambda)}{\partial x}\,\Delta\left(x,\frac{\overline{\kappa}}{\lambda}\right)\,dx\,du\right\} + R_1 + R_2 + R_3. \tag{76}$$

Change the order of integration on the right-hand side and use

$$\int_{1/2-2iT}^{1/2+2iT}\frac{\partial h(u,x/\kappa\lambda)}{\partial x}\,du = \frac{1}{x}\int_0^{\infty}\frac{e^{2\pi ixy/\kappa\lambda}}{y^{1/2}(1+y)^{3/2}\log((1+y)/y)}\,e^{it\log((1+y)/y)}\left(it-\frac{1}{2}-\frac{1}{\log((1+y)/y)}\right)\Bigg|_{t=-2T}^{2T}dy$$

(which is similar to the formula given in [1, p. 361]; cf. (39)). Let us decompose

$$\begin{aligned} e^{it\log((1+y)/y)}\left(it-\frac{1}{2}-\frac{1}{\log((1+y)/y)}\right)\Bigg|_{t=-2T}^{2T} &= 2iT\,e^{2iT\log((1+y)/y)} \\ &\quad -(-2iT)\,e^{-2iT\log((1+y)/y)} + e^{it\log((1+y)/y)}\left(-\frac{1}{2}-\frac{1}{\log((1+y)/y)}\right)\Bigg|_{t=-2T}^{2T}, \end{aligned} \tag{77}$$

and denote the contribution of the second (resp. third) term on the right-hand side of (77) to (76) by $R_4$ (resp. $R_5$). Since the contribution of the first term on the right-hand side of (77) to (76) is $-\sum_{k\le M}\sum_{l\le M} J_2(k,l)$, we obtain

$$I_{12} = -\sum_{k\le M}\sum_{l\le M} J_2(k,l) + \sum_{j=1}^{5} R_j. \tag{78}$$

We estimate $R_j$, $1\le j\le 5$. By using Lemma 5.1(ii), (1) and (49), we have

$$R_1 \ll \sum_{k\le M}\sum_{l\le M}\sum_{X<n\le 2X}\frac{|a(k)\,\overline{a(l)}|}{[k,l]}\,d(n)\,T^{-1/4}\left(\frac{\kappa\lambda}{n}\right)^{5/4} \ll M^{5/2+\varepsilon}\,T^{-1/4}. \tag{79}$$

Carrying out the integration with respect to $u$ in the definition of $R_3$, we find that the resulting expression can be treated similarly to the case of $I_2$. Hence, as in Section 6, we have

$$R_3 \ll M^{4/3+\varepsilon}\,T^{-1/6+\varepsilon}. \tag{80}$$

We can treat $R_4$ similarly to $S_{42}$, and treat $R_5$ similarly to $S_{41}$, $S_{43}$. Hence the same estimate in (51) holds for $R_4$ and $R_5$, which implies

$$R_4,\ R_5 \ll M^{1+\varepsilon}\,T^{-1/4}. \tag{81}$$

Finally consider $R_2$. Changing the order of integration and applying Lemma 5.1(ii) to the inner integral, we have $R_2 = R_{21} + R_{22}$, where

$$R_{21} = 2\sum_{k\le M}\sum_{l\le M}\Im\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\,\frac{1}{\lambda}\int_X^{2X} G_1(x;\lambda)\,G_2(x;\kappa,\lambda)\,e^{iF(x;\kappa,\lambda)}\,dx\right\} \tag{82}$$

with

$$\begin{aligned} F(x;\kappa,\lambda) &= 2f(T,x/2\kappa\lambda) - \frac{\pi x}{\kappa\lambda} + \frac{3\pi}{4}, \\ G_1(x;\lambda) &= \log x + 2\gamma - 2\log\lambda, \\ G_2(x;\kappa,\lambda) &= \frac{1}{2\sqrt{2}}\left(\operatorname{arcsinh}\sqrt{\frac{\pi x}{4T\kappa\lambda}}\right)^{-1}\left(\frac{2Tx}{\pi\kappa\lambda} + \frac{x^2}{4\kappa^2\lambda^2}\right)^{-1/4}, \end{aligned}$$

and

$$R_{22} = 2\sum_{k\le M}\sum_{l\le M}\Im\left\{\frac{a(k)\,\overline{a(l)}}{[k,l]}\,\frac{1}{\lambda}\int_X^{2X} G_1(x;\lambda)\,R(2T,x/\kappa\lambda)\,dx\right\}.$$

It is easy to see that

$$R_{22} \ll \sum_{k\le M}\sum_{l\le M}\frac{1}{\lambda}\frac{|a(k)\,\overline{a(l)}|}{[k,l]}\frac{(X\kappa\lambda)^{1/2}}{T}(\log X + \log\lambda) \ll M^{1+\varepsilon}\,T^{-1/2}\log T. \tag{83}$$

To estimate $R_{21}$, we quote the following

### *Lemma 8.2 (Heath-Brown [4]).*

*Let $F(x)$, $G_j(x)$, $1 \le j \le J$, be continuous functions defined on an interval $[a,b]$, which are monotone. Assume $F'(x)$ is also monotone, $|F'(x)| \ge M_0^{-1}$ and $|G_j(x)| \le M_j$, $1 \le j \le J$, on $[a,b]$. Then*

$$\left|\int_a^b G_1(x)\cdots G_J(x)\,e^{iF(x)}\,dx\right| \le 2^{J+3}\prod_{j=0}^{J} M_j.$$

In the present case, we see that

$$F'(x;\kappa,\lambda) = \sqrt{\frac{4\pi T}{x\kappa\lambda} + \left(\frac{\pi}{\kappa\lambda}\right)^2} - \frac{\pi}{\kappa\lambda},$$

which is positive and decreasing. Therefore $F(x;\kappa,\lambda)$ is also monotone and we can take $M_0 = F'(2X;\kappa,\lambda)^{-1}$. Hence by Lemma 8.2, we find that the integral on the right-hand side of (82) is

$$\ll \frac{(\log X + \log\lambda)\,G_2(X)}{F'(2X;\kappa,\lambda)} \ll \kappa\lambda T^{-1/2}\log(\kappa\lambda T).$$

This implies that $R_{21} \ll M^{1+\varepsilon}\,T^{-1/2}\log T$, and this with (83) gives

$$R_2 \ll M^{1+\varepsilon}\,T^{-1/2}\log T. \tag{84}$$

From (78), (79), (80), (81) and (84) we obtain the assertion of (72).

Substituting (72) into (71), and noting that

$$\begin{aligned} M^{4/3+\varepsilon}T^{-1/6+\varepsilon} &= M^{1/2+\varepsilon/2}\left(M^{5/6+\varepsilon/2}T^{-1/6+\varepsilon}\right) \\ &\ll M^{1+\varepsilon} + M^{5/3+\varepsilon}\,T^{-1/3+\varepsilon} \ll M^{1+\varepsilon}\log^2 T + M^{5/2+\varepsilon}\,T^{-1/4}, \end{aligned}$$

we complete the proof of Theorem 1.1.

## 9. Proof of Theorem 1.2

Let $L$ be a positive integer satisfying

$$2^{-L}T \geq C^*. \tag{85}$$

Then replacing $T$ (resp. $Y$) by $2^{-j}T$ (resp. $2^{-j}Y$) in (4) makes the formula valid for $1 \leq j \leq L$. Adding up, we obtain

$$\begin{aligned}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)A\left(\frac{1}{2}+it\right)\right|^2 dt &= \mathcal{M}(T,A) + \Sigma_1(T,Y) + \Sigma_2\big(T,\xi(T,Y)\big)\\ &\quad - \mathcal{M}\big(2^{-L}T,A\big) - \Sigma_1\big(2^{-L}T,2^{-L}Y\big) - \Sigma_2\big(2^{-L}T,\xi(2^{-L}T,2^{-L}Y)\big)\\ &\quad + O\left(M^{1+\varepsilon}\sum_{j=1}^{L}\log^2(2^{-j}T) + M^{5/2+\varepsilon}\sum_{j=1}^{L}(2^{-j}T)^{-1/4}\right) + \int_0^{2^{-L}T}\left|\zeta\left(\frac{1}{2}+it\right)A\left(\frac{1}{2}+it\right)\right|^2 dt.\end{aligned} \tag{86}$$

The condition (85) is equivalent to

$$L \leq \left[\frac{\log T - \log C^*}{\log 2}\right], \tag{87}$$

where $[x]$ is the integer part of $x$. Since $T \geq C^* \geq e$, we have $\log\log T \geq 0$. Therefore the choice

$$L = \left[\frac{\log T - \log C^* - \alpha\log\log T}{\log 2}\right]$$

satisfies (87) for any positive $\alpha$. This choice implies

$$T\log^{-\alpha}T \ll 2^L \ll T\log^{-\alpha}T. \tag{88}$$

Using (49) and (88), we have

$$\mathcal{M}\big(2^{-L}T,A\big) \ll M^{\varepsilon}\frac{T}{2^L}\log\frac{T}{2^L} \ll M^{\varepsilon}\log^{\alpha}T\cdot\log\log T. \tag{89}$$

Next, since $\operatorname{arcsinh} x \asymp x$ for small $x$, we have

$$\begin{aligned}\Sigma_1\big(2^{-L}T,2^{-L}Y\big) &\ll M^{\varepsilon}\left(\frac{T}{2^L}\right)^{1/2}\sum_{k,l\leq M}\frac{\kappa\lambda}{[k,l]}\sum_{n\leq\kappa\lambda Y/2^L}\frac{d(n)}{n}\\ &\ll M^{\varepsilon}\log^{\alpha/2}T\sum_{k,l\leq M}\frac{\kappa\lambda}{[k,l]}\log^2\big(\kappa\lambda Y/2^L\big) \ll M^{2+\varepsilon}\log^{\alpha/2}T\cdot(\log\log T)^2\end{aligned} \tag{90}$$

and

$$\Sigma_2\big(2^{-L}T,2^{-L}Y\big) \ll M^{\varepsilon}\sum_{k,l\leq M}\frac{(\kappa\lambda)^{1/2}}{[k,l]}\sum_{n\leq(\lambda/\kappa)Y/2^L}\frac{d(n)}{n^{1/2}} \ll M^{1+\varepsilon}\log^{\alpha/2}T\cdot\log\log T. \tag{91}$$

Also,

$$M^{1+\varepsilon}\sum_{j=1}^{L}\log^2\big(2^{-j}T\big) \ll M^{1+\varepsilon}L\log^2 T \ll M^{1+\varepsilon}\log^3 T, \tag{92}$$

$$M^{5/2+\varepsilon}\sum_{j=1}^{L}\big(2^{-j}T\big)^{-1/4} \ll M^{5/2+\varepsilon}\,T^{-1/4}\,2^{L/4} \ll M^{5/2+\varepsilon}\log^{-\alpha/4}T. \tag{93}$$

Finally, using the well-known mean square estimate for $\zeta(s)$ [18, Theorem 7.3], we obtain

$$\int_0^{2^{-L}T} \left|\zeta\left(\frac{1}{2}+it\right) A\left(\frac{1}{2}+it\right)\right|^2 dt \ll M^{1+\varepsilon} \frac{T}{2^L} \log \frac{T}{2^L} \ll M^{1+\varepsilon} \log^\alpha T \cdot \log\log T. \tag{94}$$

Substituting (89)–(94) into (86), we obtain

$$R(T,A) \ll M^{1+\varepsilon} \log^3 T + M^{1+\varepsilon} \log^\alpha T \cdot \log\log T + M^{2+\varepsilon} \log^{\alpha/2} T \cdot (\log\log T)^2 + M^{5/2+\varepsilon} \log^{-\alpha/4} T.$$

This is the general form of the estimation of $R(T,A)$ we have proved. Theorem 1.2 is the special case $\alpha = 3-4\delta$.

## 10. Proof of Theorem 1.3

In this section, we discuss the case when $a(m) \in \mathbb{R}$ (for all $m \geq 1$) briefly. In this case the analogue of (3) holds for $\zeta(s)A(s)$. Therefore, as an analogue of (33), we can show

$$I(T,A) = \frac{1}{2}\int_{-T}^{T} \left|\zeta\left(\frac{1}{2}+it\right) A\left(\frac{1}{2}+it\right)\right|^2 dt = \mathcal{M}(T,A) + \frac{1}{2}(I_1^* - I_2^* + I_3^* - I_4^*) + O(1),$$

where $I_\nu^*$, $1 \leq \nu \leq 4$, are almost the same as $I_\nu$, just replacing the integral from $T$ to $2T$ by that from $-T$ to $T$. The analogues of (35), (38), (39) hold for $I_\nu^*$, replacing $T$ (resp. $2T$) by $-T$ (resp. $T$).

First consider $I_1^*$. We see that

$$I_1^* = I_{13} - I_{13}',$$

where $I_{13}$ is the same as in (43), and $I_{13}'$ is almost the same as $I_{13}$, just replacing $T$ by $-T$. From (44) and the corresponding result for $I_{13}'$, we obtain

$$I_1^* = 2\Sigma_1(T,Y) + O\left(T^{-1/4} \sum_{k\leq M}\sum_{l\leq M} \frac{|a(k)\,a(l)|}{[k,l]} (\kappa\lambda)^{5/4}\right).$$

Next, it is easy to see that exactly the same estimate (45) for $I_2$ holds for $I_2^*$. As for $I_3^*$, define $I_{31}^*(k,l)$ as an analogue of $I_{31}(k,l)$ replacing $T$ (resp. $2T$) by $-T$ (resp. $T$). Similarly, we define $I_{32}^*(k,l)$, and further define $I_{32}'^*(k,l)$ and $I_{32}''^*(k,l)$ splitting the integral at $y=1$. Then, similarly to (46) and (47), the estimates

$$I_{31}^*(k,l) \ll T^{-1/2}, \qquad I_{32}''^*(k,l) \ll T^{-1}\log T$$

hold. Since the inner integral of $I_{32}'^*(k,l)$ is from $1/2 - iT$ to $1/2 + iT$, this time a residue appears in the course of the argument, so

$$I_{32}'^*(k,l) = \pi^2 i + O\big(T^{-1/2}\big).$$

Hence, corresponding to (48), we obtain

$$I_3^* = 2\pi \sum_{k\leq M}\sum_{l\leq M} \frac{a(k)\,a(l)\,\kappa}{[k,l]} + O\left(\sum_{k\leq M}\sum_{l\leq M} \frac{|a(k)\,a(l)|}{[k,l]} \kappa \log(\kappa\lambda T)\, T^{-1/2}\right).$$

At last, we consider

$$I_4^* = \sum_{k\leq M}\sum_{l\leq M} \big(S_4(T;k,l) - S_4(-T;k,l)\big).$$

As in Section 5, we decompose

$$S_4(\pm T; k, l) = S_{40}(\pm T) - S_{41}(\pm T) + S_{42}(\pm T) - S_{43}(\pm T).$$

The term $S_{40}(T)$ is exactly the same as $J_3(k, l)$ defined by (40), hence (68) can be used for $S_{40}(T)$. Therefore, the term $-\Sigma_2\big(T, \xi(T, Y)\big)$ appears from this part. As for $S_{41}(T)$, $S_{42}(T)$, $S_{43}(T)$, we use the estimate (51). We obtain

$$\sum_{k\le M}\sum_{l\le M} S_4(T; k, l) = -\Sigma_2\big(T, \xi(T, Y)\big) + \mathcal{E}_1(T),$$

where $\mathcal{E}_1(T)$ is the error term satisfying the same estimate as the error term on the right-hand side of (69). Similarly, the term $\Sigma_2\big(T, \xi(T, Y)\big)$ again appears from the sum of $S_{42}(-T)$, and so

$$\sum_{k\le M}\sum_{l\le M} S_4(-T; k, l) = \Sigma_2\big(T, \xi(T, Y)\big) + \mathcal{E}_2(T),$$

where $\mathcal{E}_2(T)$ satisfies the same estimate as that of $\mathcal{E}_1(T)$. We thus obtain

$$I_4^* = -2\Sigma_2\big(T, \xi(T, Y)\big) + \mathcal{E}_1(T) + \mathcal{E}_2(T).$$

Collecting the above results, we obtain the conclusion of Theorem 1.3.



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