

ON THE MEAN SQUARE OF THE PRODUCT OF $\zeta(s)$ AND A DIRICHLET POLYNOMIAL

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let $s = \sigma + it$ be a complex variable, $\zeta(s)$ the Riemann zeta-function, and

$$A(s) = \sum_{m \leq M} a(m)m^{-s}$$

be a Dirichlet polynomial, where $M \geq 1$ and $a(m)$'s are complex coefficients satisfying

$$a(m) = O(m^\varepsilon) \tag{1.1}$$

for any $\varepsilon > 0$. (In what follows, ε is always a small positive number, not necessarily the same at each occurrence.) The mean value

$$I(T, A) = \int_0^T |\zeta(\tfrac{1}{2} + it)A(\tfrac{1}{2} + it)|^2 dt \quad (T \geq 2)$$

has been studied by several mathematicians. Iwaniec [4] obtained an upper bound of $I(T, A)$, and then, Balasubramanian, Conrey and Heath-Brown [1] established the asymptotic formula

$$I(T, A) = \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{[k, \ell]} \left(\log \frac{(k, \ell)^2 T}{2\pi k\ell} + 2\gamma - 1 \right) T + E(T, A) \tag{1.2}$$

under the assumption $\log M \ll \log T$ (the symbol $f \ll g$ means $f = O(g)$), where (k, ℓ) is the greatest common divisor of k and ℓ , $[k, \ell] = k\ell/(k, \ell)$ is the least common multiple of k and ℓ , $\overline{a(\ell)}$ is the complex conjugate of $a(\ell)$, γ is Euler's constant, and $E(T, A)$ is the error term satisfying

$$E(T, A) \ll M^2 T^\varepsilon + T(\log T)^{-B} \quad (\text{for } \log M \ll \log T) \tag{1.3}$$

for any $B > 0$, where the implied constant depends on B and ε . They also gave several sharper estimates of $E(T, A)$ under further assumptions, and mentioned an application to the distribution of zeros of $\zeta(s)$. Their condition $\log M \ll \log T$ for (1.3) is actually not necessary (see the remark at the end of this section).

Motohashi [8] stated a different type of estimate, that is

$$E(T, A) \ll M^{4/3} T^{1/3+\varepsilon}, \tag{1.4}$$

with a brief sketch of the proof. His proof, different from that of [1], is a variant of Atkinson's method. Actually his argument is valid only for the integral from $-T$ to T , hence his claim should be understood as

$$\tilde{E}(T, A) \ll M^{4/3}T^{1/3+\varepsilon} \quad (\text{for } M \ll T^{1/2}(\log T)^{-3/4}), \quad (1.5)$$

where $\tilde{E}(T, A)$ is defined by

$$\begin{aligned} & \frac{1}{2} \int_{-T}^T |\zeta(\tfrac{1}{2} + it)A(\tfrac{1}{2} + it)|^2 dt \\ &= \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{[k, \ell]} \left(\log \frac{(k, \ell)^2 T}{2\pi k \ell} + 2\gamma - 1 \right) T + \tilde{E}(T, A). \end{aligned} \quad (1.6)$$

The left-hand side of (1.6) coincides with $I(T, A)$ if $a(m)$'s are real, but it is not true in general. Note that the condition $M \ll T^{1/2}(\log T)^{-3/4}$ is necessary, though Motohashi did not state it, because in the proof he used a parameter G which satisfies $G \leq T(\log T)^{-1}$, and at the last stage of the proof he chose $G = M^{4/3}T^{1/3}$.

The aim of the present paper is to develop another approach to this problem. Let A_0 be a sufficiently large positive number, $L = A_0(\log T)^{1/2}$, and μ, ρ be non-negative numbers satisfying $\rho < 1$, $\mu + \rho > 0$. We assume

$$L \leq M^\mu T^\rho \leq \frac{T}{A_0 L}. \quad (1.7)$$

We shall prove

Theorem 1. *For any $M \geq 1$ and $T \geq 2$ satisfying (1.7), we have*

$$E(T, A) \ll M^{2-\mu/2}T^{1/2-\rho/2+\varepsilon} + M^\mu T^{\rho+\varepsilon}. \quad (1.8)$$

In particular, replacing ε on the right-hand side of (1.8) by $\varepsilon/2$, and taking $\mu = 0$, $\rho = 1 - \varepsilon$, we obtain

Corollary 1. *For any $M \geq 1$ and $T \geq 2$, we have*

$$E(T, A) \ll M^2 T^\varepsilon + T^{1-\varepsilon/2}. \quad (1.9)$$

This gives a slight improvement of the estimate (1.3) of Balasubramanian, Conrey and Heath-Brown [1]. On the other hand, if $M \ll T^{1/2}(\log T)^{-3/8}$, we can choose $\mu = 4/3$, $\rho = 1/3$ to obtain the following corollary, which recovers Motohashi's claim.

Corollary 2. *Under the condition $M \ll T^{1/2}(\log T)^{-3/8}$, we have*

$$E(T, A) \ll M^{4/3}T^{1/3+\varepsilon}. \quad (1.10)$$

Our proof of the above theorem is an analogue of the argument developed in Katsurada and Matsumoto [5]. Therefore it is also a variant of Atkinson's method, and can be regarded as a generalization of the argument described in Section 2.7 of Ivić [3]. By the same method we can treat the case $1/2 < \sigma < 1$. In this case, the integral

$$I_\sigma(T, A) = \int_0^T |\zeta(\sigma + it)A(\sigma + it)|^2 dt$$

satisfies the asymptotic formula of the form

$$\begin{aligned} I_\sigma(T, A) &= \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{[k, \ell]^{2\sigma}} \zeta(2\sigma) T \\ &+ \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{k\ell} (k, \ell)^{2-2\sigma} (2\pi)^{2\sigma-1} \frac{\zeta(2-2\sigma)}{2-2\sigma} T^{2-2\sigma} \\ &+ E_\sigma(T, A), \end{aligned} \tag{1.11}$$

where $E_\sigma(T, A)$ is the error term. We can estimate this term as follows:

Theorem 2. *If $\rho < (4\sigma - 1)^{-1}$, then for any $M \geq 1$ and $T \geq 2$ satisfying (1.7), we have*

$$E_\sigma(T, A) \ll M^{2-(1-\rho)^{-1}\mu f(\sigma, \rho)} T^{f(\sigma, \rho)+\varepsilon} + M^\mu T^{\rho+\varepsilon} \tag{1.12}$$

for $1/2 < \sigma < 1$, where $f(\sigma, \rho) = \rho(\frac{1}{2} - 2\sigma) + \frac{1}{2}$.

Remark 1. The assumption $\rho < (4\sigma - 1)^{-1}$ implies $f(\sigma, \rho) > 0$.

The estimates corresponding to Corollaries 1 and 2 can be stated as

$$E_\sigma(T, A) \ll M^2 T^\varepsilon + T^{1/(4\sigma-1)-\varepsilon} \tag{1.13}$$

(under the choice $\mu = 0$, $\rho = 1/(4\sigma - 1) - \varepsilon$) and

$$E_\sigma(T, A) \ll M^{8\sigma/(1+4\sigma)} T^{1/(1+4\sigma)+\varepsilon} \quad (\text{for } M \ll T^{1/2} L^{-1/2-1/8\sigma}) \tag{1.14}$$

(under the choice $\mu = 8\sigma/(1+4\sigma)$, $\rho = 1/(1+4\sigma)$) for $1/2 < \sigma < 1$. The asymptotic formula (1.11) with the estimate (1.14) is clearly a generalization of Theorem 1 of the author [7].

Remark 2. Here we show that Theorems 1 and 2 are trivial if

$$M \gg T^b \tag{1.15}$$

with a sufficiently large positive constant b . In fact, since

$$\zeta(\sigma + it)A(\sigma + it) \ll M^{1-\sigma+\varepsilon} (|t| + 1)^{(1-\sigma)/3} \quad (1/2 \leq \sigma < 1), \tag{1.16}$$

we have $I(T, A) \ll M^{1+\varepsilon}T^{4/3}$ and $I_\sigma(T, A) \ll M^{2-2\sigma+\varepsilon}T^{2(1-\sigma)/3+1}$. In case $\sigma = 1/2$, by using (3.9) below, we see that the first term on the right-hand side of (1.2) is $O(M^\varepsilon T \log T)$. Hence trivially

$$E(T, A) \ll M^{1+\varepsilon}T^{4/3} + M^\varepsilon T \log T \ll M^{1+4/3b+\varepsilon},$$

which is clearly superseded by the right-hand side of (1.8) if b is sufficiently large. Similarly, in case $1/2 < \sigma < 1$, since the first and the second terms on the right-hand side of (1.11) are $O(T + M^\varepsilon T^{2-2\sigma})$, we have

$$E_\sigma(T, A) \ll M^{2-2\sigma+b^{-1}(2(1-\sigma)/3+1)+\varepsilon}. \quad (1.17)$$

Noting $(1-\rho)^{-1}f(\sigma, \rho) < 1$, we see that (1.17) implies (1.12) for sufficiently large b .

2. THE WEIGHTED LOCAL INTEGRAL

Now we begin the proof of Theorems 1 and 2. Let Δ be a parameter satisfying

$$L \leq \Delta \leq \frac{T}{A_0 L}. \quad (2.1)$$

Moreover, in view of Remark 2 in Section 1, we may assume

$$M \leq T^b. \quad (2.2)$$

Let u, v be complex variables, and at first assume $\Re u > 1$, $\Re v > 1$. Consider

$$I(u, v; \Delta, A) = \frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} \zeta(u+iy)\zeta(v-iy)A(u+iy)\bar{A}(v-iy)e^{-(y/\Delta)^2} dy, \quad (2.3)$$

where $\bar{A}(s) = \sum_{m \leq M} \overline{a(m)}m^{-s}$. Substituting the Dirichlet series expressions, we have

$$\begin{aligned} I(u, v; \Delta, A) &= \frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} \sum_{r=1}^{\infty} r^{-u-iy} \sum_{s=1}^{\infty} s^{-v+iy} \sum_{k \leq M} a(k)k^{-u-iy} \\ &\quad \times \sum_{\ell \leq M} \overline{a(\ell)}\ell^{-v+iy} e^{-(y/\Delta)^2} dy \\ &= \frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \left(\sum_{k|m} a(k) \right) m^{-u-iy} \sum_{n=1}^{\infty} \left(\sum_{\ell|n} \overline{a(\ell)} \right) n^{-v+iy} e^{-(y/\Delta)^2} dy. \end{aligned}$$

The part corresponding to $m = n$ is equal to

$$\begin{aligned}
 & \frac{1}{\Delta\sqrt{\pi}} \sum_{m=1}^{\infty} \left(\sum_{k|m} a(k) \right) \left(\sum_{\ell|m} \overline{a(\ell)} \right) m^{-u-v} \int_{-\infty}^{\infty} e^{-(y/\Delta)^2} dy \\
 &= \sum_{m=1}^{\infty} \left(\sum_{k|m} a(k) \right) \left(\sum_{\ell|m} \overline{a(\ell)} \right) m^{-u-v} \\
 &= \sum_{k \leq M} \sum_{\ell \leq M} a(k) \overline{a(\ell)} \sum_{m \equiv 0 \pmod{[k, \ell]}} m^{-u-v} \\
 &= \zeta(u+v) \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k) \overline{a(\ell)}}{[k, \ell]^{u+v}}.
 \end{aligned}$$

Hence

$$I(u, v; \Delta, A) = \zeta(u+v) \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k) \overline{a(\ell)}}{[k, \ell]^{u+v}} + I_1(u, v; \Delta, A) + \overline{I_1(\bar{v}, \bar{u}; \Delta, \bar{A})}, \quad (2.4)$$

where

$$I_1(u, v; \Delta, A) = \frac{1}{\Delta\sqrt{\pi}} \sum_{m < n} \sum_{k|m} \left(\sum_{k|m} a(k) \right) \left(\sum_{\ell|n} \overline{a(\ell)} \right) \int_{-\infty}^{\infty} m^{-u-iy} n^{-v+iy} e^{-(y/\Delta)^2} dy.$$

Using the formula

$$\int_{-\infty}^{\infty} \exp(At - Bt^2) dt = \left(\frac{\pi}{B} \right)^{1/2} \exp\left(\frac{A^2}{4B} \right) \quad (\Re B > 0)$$

(see (A.38) of [2]), and then putting $m = km_1$ and $n = km_1 + n_1$, we have

$$\begin{aligned}
 I_1(u, v; \Delta, A) &= \sum_{m < n} \sum_{k|m} \left(\sum_{k|m} a(k) \right) \left(\sum_{\ell|n} \overline{a(\ell)} \right) m^{-u} n^{-v} \exp\left(-\frac{1}{4} \Delta^2 \log^2 \left(\frac{n}{m} \right) \right) \\
 &= \sum_{k \leq M} a(k) \sum_{m_1=1}^{\infty} \sum_{n_1=1}^{\infty} \left(\sum_{\ell|(km_1+n_1)} \overline{a(\ell)} \right) (km_1)^{-u} (km_1 + n_1)^{-v} \\
 &\quad \times \exp\left(-\frac{1}{4} \Delta^2 \log^2 \left(1 + \frac{n_1}{km_1} \right) \right) \\
 &= \sum_{k \leq M} a(k) k^{-u} \sum_{\ell \leq M} \overline{a(\ell)} \sum_{m_1=1}^{\infty} \sum_{n_1=1}^{\infty} m_1^{-u} (km_1 + n_1)^{-v} \\
 &\quad \times \exp\left(-\frac{1}{4} \Delta^2 \log^2 \left(1 + \frac{n_1}{km_1} \right) \right) \ell^{-1} \sum_{f=1}^{\ell} \exp\left(2\pi i \frac{(km_1 + n_1)f}{\ell} \right), \quad (2.5)
 \end{aligned}$$

because the innermost sum is

$$= \begin{cases} \ell & \text{if } \ell | (km_1 + n_1), \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$M(s, v; \Delta) = \frac{\Gamma(s)}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\Gamma(v + iy - s)}{\Gamma(v + iy)} e^{-(y/\Delta)^2} dy \quad (2.6)$$

for $\Re v > \Re s > 0$. In Section 5.2 of [3], Ivić proved the following properties of $M(s, v; \Delta)$.

Lemma 1. (i) $M(s, v; \Delta)$ can be continued to the whole space \mathbb{C}^2 , entire in v , and meromorphic in s . The poles with respect to s are only on $s = 0, -1, -2, \dots$

(ii) For $\Re s > 0$ and any v , we have

$$M(s, v; \Delta) = \int_0^{\infty} x^{s-1} (1+x)^{-v} \exp\left(-\frac{1}{4}\Delta^2 \log^2(1+x)\right) dx. \quad (2.7)$$

(iii) For any fixed $c > 0$, we have $M(s, v; \Delta) \ll (1 + |s|)^{-c}$ as $|\Im s| \rightarrow \infty$, uniformly for bounded v and bounded $\Re s$.

(iv) If $\Re v > \alpha > 0$ and $x > 0$, then

$$\frac{1}{2\pi i} \int_{(\alpha)} M(s, v; \Delta) x^{-s} ds = (1+x)^{-v} \exp\left(-\frac{1}{4}\Delta^2 \log^2(1+x)\right), \quad (2.8)$$

where the path of integration is the vertical line $\Re s = \alpha$.

From (2.5) and Lemma 1(iv) we have

$$\begin{aligned} I_1(u, v; \Delta, A) &= \sum_{k \leq M} a(k) k^{-u} \sum_{\ell \leq M} \overline{a(\ell)} \ell^{-1} \sum_{f=1}^{\ell} \sum_{m_1=1}^{\infty} \exp\left(2\pi i \frac{km_1 f}{\ell}\right) m_1^{-u} \\ &\times \sum_{n_1=1}^{\infty} \exp\left(2\pi i \frac{n_1 f}{\ell}\right) (km_1)^{-v} \frac{1}{2\pi i} \int_{(\alpha)} M(s, v; \Delta) \left(\frac{n_1}{km_1}\right)^{-s} ds, \end{aligned}$$

where $1 < \alpha < \Re v$. Summation and integration can be interchanged because of absolute convergence, hence

$$\begin{aligned} I_1(u, v; \Delta, A) &= \frac{1}{2\pi i} \int_{(\alpha)} \sum_{k \leq M} a(k) k^{-u-v+s} \sum_{\ell \leq M} \overline{a(\ell)} \ell^{-1} \\ &\times \sum_{f=1}^{\ell} \varphi\left(u+v-s, \frac{kf}{\ell}\right) \varphi\left(s, \frac{f}{\ell}\right) M(s, v; \Delta) ds \end{aligned} \quad (2.9)$$

for $\Re u > 1$, $\Re v > \alpha$, where

$$\varphi(s, x) = \sum_{n=1}^{\infty} \exp(2\pi i n x) n^{-s} \quad (2.10)$$

is the Lerch zeta-function with the real parameter x . If $x \in \mathbb{Z}$, then $\varphi(s, x) = \zeta(s)$, while if $x \notin \mathbb{Z}$, then $\varphi(s, x)$ is entire in s . Moreover, if $0 < x < 1$, $\varphi(s, x)$ satisfies

the functional equation

$$\begin{aligned} \varphi(s, x) = (2\pi)^{s-1} \Gamma(1-s) & \left\{ e\left(\frac{1-s}{4} - x\right) \zeta(1-s, x) \right. \\ & \left. + e\left(-\frac{1-s}{4} - x\right) \zeta(1-s, 1-x) \right\}, \end{aligned} \quad (2.11)$$

where $\zeta(s, x) = \sum_{n=0}^{\infty} (n+x)^{-s}$ is the Hurwitz zeta-function (see Chapter 2 of Laurinćikas and Garunkštis [6]), and $e(x) = \exp(2\pi i x)$.

Let $\beta > \max\{2, \alpha\}$, and assume $\Re(u+v) < \beta$. We shift the path of integration on the right-hand side of (2.9) to $\Re s = \beta$. Lemma 1(iii) implies that this shifting is possible. The function $\varphi(u+v-s, kf/\ell)$ has a pole at $s = u+v-1$ only when $kf/\ell \in \mathbb{Z}$. Hence we have

$$\begin{aligned} I_1(u, v; \Delta, A) &= P(u, v; \Delta, A) \\ &+ \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{k\ell} \sum_{f=1}^{\ell} \xi(f; k, \ell) \varphi\left(u+v-1, \frac{f}{\ell}\right) M(u+v-1, v; \Delta), \end{aligned} \quad (2.12)$$

where

$$\begin{aligned} P(u, v; \Delta, A) &= \frac{1}{2\pi i} \int_{(\beta)} \sum_{k \leq M} a(k) k^{-u-v+s} \sum_{\ell \leq M} \overline{a(\ell)} \ell^{-1} \\ &\times \sum_{f=1}^{\ell} \varphi\left(u+v-s, \frac{kf}{\ell}\right) \varphi\left(s, \frac{f}{\ell}\right) M(s, v; \Delta) ds \end{aligned} \quad (2.13)$$

and

$$\xi(f; k, \ell) = \begin{cases} 1 & \text{if } \ell | kf, \\ 0 & \text{otherwise.} \end{cases}$$

Since $M(s, v; \Delta)$ is entire in v (by Lemma 1(i)), the expression (2.13) is valid for any u, v satisfying $\Re(u+v) < \beta + 1$. Hence (2.12) gives the meromorphic continuation of $I_1(u, v; \Delta, A)$ to the region $\Re(u+v) < \beta + 1$. Therefore now we can put $u = \sigma + it$, $v = \sigma - it$ in (2.12), where $1/2 < \sigma < 1$, $t \geq 2$, and $T \ll t \ll T$. Substituting the resulting expression and its complex conjugate into (2.4), we obtain

$$\begin{aligned} I(\sigma + it, \sigma - it; \Delta, A) &= \zeta(2\sigma) \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{[k, \ell]^{2\sigma}} \\ &+ P_{\sigma}(t; \Delta, A) + \overline{P_{\sigma}(t; \Delta, A)} \\ &+ \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{k\ell} \sum_{f=1}^{\ell} \xi(f; k, \ell) \varphi\left(2\sigma-1, \frac{f}{\ell}\right) M(2\sigma-1, \sigma-it; \Delta) \\ &+ \sum_{k \leq M} \sum_{\ell \leq M} \frac{\overline{a(k)}a(\ell)}{k\ell} \sum_{f=1}^{\ell} \xi(f; k, \ell) \overline{\varphi\left(2\sigma-1, \frac{f}{\ell}\right) M(2\sigma-1, \sigma-it; \Delta)}, \end{aligned} \quad (2.14)$$

where $P_\sigma(t; \Delta, A) = P(\sigma + it, \sigma - it; \Delta, A)$. Changing the letters k and ℓ in the last member of the right-hand side, we obtain

$$\begin{aligned} I(\sigma + it, \sigma - it; \Delta, A) &= P_\sigma(t; \Delta, A) + \overline{P_\sigma(t; \Delta, A)} \\ &+ \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{k\ell} H_\sigma(t; k, \ell) \end{aligned} \quad (2.15)$$

for $1/2 < \sigma < 1$, where

$$\begin{aligned} H_\sigma(t; k, \ell) &= \frac{k\ell}{[k, \ell]^{2\sigma}} \zeta(2\sigma) \\ &+ \sum_{f=1}^{\ell} \xi(f; k, \ell) \varphi\left(2\sigma - 1, \frac{f}{\ell}\right) M(2\sigma - 1, \sigma - it; \Delta) \\ &+ \sum_{f=1}^k \xi(f; \ell, k) \varphi\left(2\sigma - 1, -\frac{f}{k}\right) M(2\sigma - 1, \sigma + it; \Delta). \end{aligned} \quad (2.16)$$

3. THE CASE ON THE CRITICAL LINE

In this section we show an expression, analogous to (2.15), for $\sigma = 1/2$. Let $\sigma = \frac{1}{2} + \delta$, where δ is a small positive number. Then from (2.6), noting $\Gamma(2\delta) = (2\delta)^{-1} - \gamma + O(\delta)$, we can easily see that

$$M(2\sigma - 1, \sigma + it; \Delta) = \frac{1}{2\delta} - B(t; \Delta) - \gamma + O(\delta),$$

where

$$B(t; \Delta) = \frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} + i(t+y)\right) e^{-(y/\Delta)^2} dy$$

((3.6) and (3.7) of [5]). Also,

$$\zeta(2\sigma)[k, \ell]^{-2\sigma} = \frac{1}{[k, \ell]} \left(\frac{1}{2\delta} - \log[k, \ell] + \gamma + O(\delta) \right),$$

$$\varphi(2\sigma - 1, x) = \varphi(0, x) + 2\delta\varphi'(0, x) + O(\delta^2).$$

Hence we have

$$\begin{aligned}
 H_\sigma(t; k, \ell) &= \frac{k\ell}{[k, \ell]} \left(\frac{1}{2\delta} - \log[k, \ell] + \gamma + O(\delta) \right) \\
 &+ \frac{1}{2\delta} \sum_{f=1}^{\ell} \xi(f; k, \ell) \varphi \left(0, \frac{f}{\ell} \right) + \frac{1}{2\delta} \sum_{f=1}^k \xi(f; \ell, k) \varphi \left(0, -\frac{f}{k} \right) \\
 &+ \sum_{f=1}^{\ell} \xi(f; k, \ell) \left\{ \varphi' \left(0, \frac{f}{\ell} \right) - (B(-t, \Delta) + \gamma) \varphi \left(0, \frac{f}{\ell} \right) \right\} \\
 &+ \sum_{f=1}^k \xi(f; \ell, k) \left\{ \varphi' \left(0, -\frac{f}{k} \right) - (B(t, \Delta) + \gamma) \varphi \left(0, -\frac{f}{k} \right) \right\} \\
 &+ O(\delta). \tag{3.1}
 \end{aligned}$$

We note that

$$\frac{k\ell}{[k, \ell]} + \sum_{f=1}^{\ell} \xi(f; k, \ell) \varphi \left(0, \frac{f}{\ell} \right) + \sum_{f=1}^k \xi(f; \ell, k) \varphi \left(0, -\frac{f}{k} \right) = 0. \tag{3.2}$$

In fact, if $(k, \ell) = d$, then we can write $k = d\kappa$, $\ell = d\lambda$, $(\kappa, \lambda) = 1$. Then $\xi(f; k, \ell) = 1$ if and only if $\ell|kf$, that is $\lambda|f$. Hence, putting $f = j\lambda$, we have

$$\sum_{f=1}^{\ell} \xi(f; k, \ell) \varphi \left(0, \frac{f}{\ell} \right) = \sum_{j=1}^d \varphi \left(0, \frac{j}{d} \right). \tag{3.3}$$

When $\Re s > 1$, we have

$$\sum_{j=1}^d \varphi \left(s, \frac{j}{d} \right) = d^{1-s} \zeta(s), \tag{3.4}$$

because the left-hand side is

$$= \sum_{n=1}^{\infty} \left(\sum_{j=1}^d \exp(2\pi i n j / d) \right) n^{-s} = d \sum_{n \equiv 0 \pmod{d}} n^{-s}.$$

The relation (3.4) is valid, by analytic continuation, at $s = 0$. Hence from (3.3) we have

$$\sum_{f=1}^{\ell} \xi(f; k, \ell) \varphi \left(0, \frac{f}{\ell} \right) = d\zeta(0) = -\frac{d}{2} = -\frac{k\ell}{2[k, \ell]}. \tag{3.5}$$

The value of $\sum_{f=1}^k \xi(f; \ell, k) \varphi(0, -f/k)$ is the same, hence (3.2) follows.

Therefore, letting $\delta \rightarrow 0$ in (3.1), we have

$$\begin{aligned}
H_\sigma(t; k, \ell)|_{\sigma \rightarrow 1/2+0} &= \frac{k\ell}{[k, \ell]} (-\log[k, \ell] + \gamma) \\
&+ \sum_{f=1}^{\ell} \xi(f; k, \ell) \left\{ \varphi' \left(0, \frac{f}{\ell} \right) - (B(-t, \Delta) + \gamma) \varphi \left(0, \frac{f}{\ell} \right) \right\} \\
&+ \sum_{f=1}^k \xi(f; \ell, k) \left\{ \varphi' \left(0, -\frac{f}{k} \right) - (B(t, \Delta) + \gamma) \varphi \left(0, -\frac{f}{k} \right) \right\}. \tag{3.6}
\end{aligned}$$

Differentiating the both sides of (3.4) and putting $s = 0$, we have

$$\sum_{j=1}^d \varphi' \left(0, \frac{j}{d} \right) = -\zeta(0)d \log d + \zeta'(0)d = \frac{1}{2}d \log d - \frac{1}{2} \log(2\pi)d. \tag{3.7}$$

Using (3.5) and (3.7), we see that the right-hand side of (3.6) is

$$\begin{aligned}
&\frac{k\ell}{[k, \ell]} (-\log[k, \ell] + \gamma) + d \log d - \log(2\pi)d + \frac{d}{2} (B(t, \Delta) + B(-t, \Delta) + 2\gamma) \\
&= \frac{k\ell}{[k, \ell]} \left(\log \frac{(k, \ell)^2}{2\pi k\ell} + 2\gamma + \frac{1}{2}B(t, \Delta) + \frac{1}{2}B(-t, \Delta) \right).
\end{aligned}$$

Moreover, using Stirling's formula we have

$$B(\pm t, \Delta) = \pm \frac{1}{2}\pi i + \log t + O\left(\frac{\Delta}{t}\right)$$

((3.9) of [5]). Hence

$$H_\sigma(t; k, \ell)|_{\sigma \rightarrow 1/2+0} = \frac{k\ell}{[k, \ell]} \left(\log \frac{(k, \ell)^2}{2\pi k\ell} + 2\gamma + \log t + O\left(\frac{\Delta}{t}\right) \right). \tag{3.8}$$

We let $\sigma \rightarrow 1/2+0$ in (2.15), with using (3.8). The contribution of the error term on the right-hand side of (3.8) to (2.14) is $O(\Delta t^{-1+\varepsilon})$. To prove this estimate, it suffices to show

$$\sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{[k, \ell]} \ll M^\varepsilon, \tag{3.9}$$

because $M^\varepsilon \ll t^\varepsilon$ in view of (2.2). The left-hand side of (3.9) is

$$\begin{aligned}
&\ll \sum_{k \leq M} \sum_{\ell \leq M} (k\ell)^{-1+\varepsilon} d \ll \sum_{d \leq M} d \sum_{\kappa \leq M/d} \sum_{\lambda \leq M/d} (\kappa\lambda d^2)^{-1+\varepsilon} \\
&\ll \sum_{d \leq M} d^{-1+\varepsilon} \left(\frac{M}{d}\right)^\varepsilon \ll M^\varepsilon,
\end{aligned}$$

hence (3.9) follows. Therefore now we obtain

$$\begin{aligned} I\left(\frac{1}{2} + it, \frac{1}{2} - it; \Delta, A\right) &= P_{1/2}(t; \Delta, A) + \overline{P_{1/2}(t; \Delta, A)} \\ &+ \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{[k, \ell]} \left(\log \frac{(k, \ell)^2 t}{2\pi k\ell} + 2\gamma \right) + O\left(\Delta t^{-1+\varepsilon}\right). \end{aligned} \quad (3.10)$$

4. THE WEIGHTED MEAN SQUARE

Now we go back to the integral (2.3), and continue it meromorphically by a different method. At first assume $\Re u > 1$ and $\Re v > 1$. Let $K > \max\{1, \Re v - 1\}$, and shift the path of integration on the right-hand side of (2.3) to $\Im y = -K$. The residue at $y = i(1 - v)$ appears. The resulting expression

$$\begin{aligned} I(u, v; \Delta, A) &= \frac{2\sqrt{\pi}}{\Delta} \zeta(u + v - 1) A(u + v - 1) \bar{A}(1) e^{(v-1)^2/\Delta^2} \\ &+ \frac{1}{\Delta\sqrt{\pi}} \int_{\Im y = -K} \zeta(u + iy) \zeta(v - iy) A(u + iy) \bar{A}(v - iy) e^{-(y/\Delta)^2} dy \end{aligned} \quad (4.1)$$

can be continued meromorphically to the region $\Re u > -K + 1$, $\Re v < K + 1$, which includes $\mathcal{D} = \{(u, v) \mid 0 < \Re u < 1, 0 < \Re v < 1\}$.

Now assume $(u, v) \in \mathcal{D}$, and shift back the path of integration on the right-hand side of (4.1) to $\Im y = 0$. This time the residue at $y = i(u - 1)$ appears, and

$$\begin{aligned} I(u, v; \Delta, A) &= \frac{2\sqrt{\pi}}{\Delta} \zeta(u + v - 1) \left\{ A(u + v - 1) \bar{A}(1) e^{(v-1)^2/\Delta^2} \right. \\ &\quad \left. + A(1) \bar{A}(u + v - 1) e^{(u-1)^2/\Delta^2} \right\} \\ &+ \frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} \zeta(u + iy) \zeta(v - iy) A(u + iy) \bar{A}(v - iy) e^{-(y/\Delta)^2} dy \end{aligned} \quad (4.2)$$

for $(u, v) \in \mathcal{D}$. In particular, we can now put $u = \sigma + it$, $v = \sigma - it$, where $1/2 \leq \sigma < 1$, $t \geq 2$ and $T \ll t \ll T$, in (4.2). Then

$$\begin{aligned} I(\sigma + it, \sigma - it; \Delta, A) &= J_{\sigma}(t; \Delta, A) \\ &+ \frac{2\sqrt{\pi}}{\Delta} \zeta(2\sigma - 1) \left\{ A(2\sigma - 1) \bar{A}(1) \exp((\sigma - 1 - it)^2/\Delta^2) \right. \\ &\quad \left. + A(1) \bar{A}(2\sigma - 1) \exp((\sigma - 1 + it)^2/\Delta^2) \right\}, \end{aligned} \quad (4.3)$$

where

$$J_{\sigma}(t; \Delta, A) = \frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} |\zeta(\sigma + i(t + y)) A(\sigma + i(t + y))|^2 e^{-(y/\Delta)^2} dy.$$

Since $\Delta \leq T/A_0L$ by (2.1), the second member on the right-hand side of (4.3) is

$$\ll M^{1+\varepsilon} \Delta^{-1} e^{-C_1 L^2} \ll T^{-C_2},$$

where C_1, C_2 are positive constants. Since we assume (2.2), if A_0 is sufficiently large, then C_1, C_2 are also large. Therefore

$$I(\sigma + it, \sigma - it; \Delta, A) = J_\sigma(t; \Delta, A) + O(T^{-C_2}) \quad (4.4)$$

with a large C_2 for $1/2 \leq \sigma < 1$.

Now, combining (4.4) with the results proved in Sections 2 and 3, we obtain the following expressions of $J_\sigma(t; \Delta, A)$, which are fundamental in our analysis. First, in the case $\sigma = 1/2$, from (3.10) and (4.4) we immediately obtain the following

Lemma 2. *For $T \ll t \ll T$, we have*

$$\begin{aligned} J_{1/2}(t; \Delta, A) &= \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{[k, \ell]} \left(\log \frac{(k, \ell)^2 t}{2\pi k \ell} + 2\gamma \right) \\ &+ P_{1/2}(t; \Delta, A) + \overline{P_{1/2}(t; \Delta, A)} + O(\Delta T^{-1+\varepsilon}). \end{aligned} \quad (4.5)$$

Next consider the case $1/2 < \sigma < 1$. Since

$$\sum_{f=1}^{\ell} \xi(f; k, \ell) \varphi \left(2\sigma - 1, \frac{f}{\ell} \right) = \sum_{j=1}^d \varphi \left(2\sigma - 1, \frac{j}{d} \right) = d^{2-2\sigma} \zeta(2\sigma - 1)$$

by (3.4), we can rewrite (2.16) as

$$\begin{aligned} H_\sigma(t; k, \ell) &= \frac{k\ell}{[k, \ell]^{2\sigma}} \zeta(2\sigma) \\ &+ d^{2-2\sigma} \zeta(2\sigma - 1) \{M(2\sigma - 1, \sigma - it; \Delta) + M(2\sigma - 1, \sigma + it; \Delta)\}. \end{aligned} \quad (4.6)$$

It has been shown in the proof of Lemma 1 of [5] that

$$M(2\sigma - 1, \sigma - it; \Delta) + M(2\sigma - 1, \sigma + it; \Delta) = 2\Gamma(2\sigma - 1) \left\{ t^{1-2\sigma} \sin(\pi\sigma) + O\left(\frac{\Delta}{t^{2\sigma}}\right) \right\},$$

and the functional equation of $\zeta(s)$ implies

$$2\Gamma(2\sigma - 1) \zeta(2\sigma - 1) \sin(\pi\sigma) = (2\pi)^{2\sigma-1} \zeta(2 - 2\sigma).$$

Hence from (4.6) we have

$$H_\sigma(t; k, \ell) = \frac{k\ell}{[k, \ell]^{2\sigma}} \zeta(2\sigma) + d^{2-2\sigma} (2\pi)^{2\sigma-1} \zeta(2 - 2\sigma) t^{1-2\sigma} + O\left(d^{2-2\sigma} \frac{\Delta}{t^{2\sigma}}\right). \quad (4.7)$$

Substitute this into the right-hand side of (2.15), and combine with (4.4). By using (3.9) we see that the contribution of the error term on the right-hand side of (4.7) is $O(\Delta t^{-2\sigma+\varepsilon})$. Hence we obtain

Lemma 3. For $1/2 < \sigma < 1$ and $T \ll t \ll T$, we have

$$\begin{aligned}
 J_\sigma(t; \Delta, A) &= \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{[k, \ell]^{2\sigma}} \zeta(2\sigma) \\
 &+ \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{k\ell} (k, \ell)^{2-2\sigma} (2\pi)^{2\sigma-1} \zeta(2-2\sigma) t^{1-2\sigma} \\
 &+ P_\sigma(t; \Delta, A) + \overline{P_\sigma(t; \Delta, A)} + O(\Delta T^{-2\sigma+\varepsilon}). \tag{4.8}
 \end{aligned}$$

We can connect $J_\sigma(t; \Delta, A)$ with the mean square of $\zeta(\sigma + it)A(\sigma + it)$ by the following inequalities:

$$\begin{aligned}
 \int_{T-L\Delta}^{2T+L\Delta} J_\sigma(t; \Delta, A) dt &\geq \int_T^{2T} |\zeta(\sigma + it)A(\sigma + it)|^2 dt \\
 &+ O(M^{2(1-\sigma)+\varepsilon} T^{2(1-\sigma)/3+1} e^{-L^2}) \tag{4.9}
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{T+L\Delta}^{2T-L\Delta} J_\sigma(t; \Delta, A) dt &\leq \int_T^{2T} |\zeta(\sigma + it)A(\sigma + it)|^2 dt \\
 &+ O(M^{2(1-\sigma)+\varepsilon} T^{2(1-\sigma)/3+1} e^{-L^2}) \tag{4.10}
 \end{aligned}$$

for $1/2 \leq \sigma < 1$. These can be proved analogously to Lemma 3 of [5], so we omit the details of the proof. We only note that, instead of (4.6) of [5], we use the bound (1.16) to obtain the above error estimates.

On the other hand, from Lemma 2 we have

$$\begin{aligned}
 &\int_{T \mp L\Delta}^{2T \pm L\Delta} J_{1/2}(t; \Delta, A) dt \\
 &= \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{[k, \ell]} \left(\log \frac{(k, \ell)^2 t}{2\pi k\ell} + 2\gamma - 1 \right) t \Big|_{t=T \mp L\Delta}^{2T \pm L\Delta} \\
 &+ \int_{T \mp L\Delta}^{2T \pm L\Delta} (P_{1/2}(t; \Delta, A) + \overline{P_{1/2}(t; \Delta, A)}) dt + O(\Delta T^\varepsilon) \\
 &= \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{[k, \ell]} \left\{ \left(\log \frac{(k, \ell)^2 t}{2\pi k\ell} + 2\gamma - 1 \right) t \Big|_{t=T}^{2T} + O(L\Delta \log T) \right\} \\
 &+ \int_{T \mp L\Delta}^{2T \pm L\Delta} (P_{1/2}(t; \Delta, A) + \overline{P_{1/2}(t; \Delta, A)}) dt + O(\Delta T^\varepsilon).
 \end{aligned}$$

Comparing this with (1.2), we obtain the case $\sigma = 1/2$ of

$$\begin{aligned} E_\sigma(2T, A) - E_\sigma(T, A) &= \int_T^{2T} |\zeta(\sigma + it)A(\sigma + it)|^2 dt \\ &\quad - \int_{T \mp L\Delta}^{2T \pm L\Delta} J_\sigma(t; \Delta, A) dt + \int_{T \mp L\Delta}^{2T \pm L\Delta} \left(P_\sigma(t; \Delta, A) + \overline{P_\sigma(t; \Delta, A)} \right) dt \\ &\quad + O(L\Delta(T^\varepsilon)^\omega), \end{aligned} \quad (4.11)$$

where $E_{1/2}(T, A) = E(T, A)$ and $\omega = 1$ or 0 according as $\sigma = 1/2$ or $1/2 < \sigma < 1$. The case $1/2 < \sigma < 1$ can be shown similarly from Lemma 3 and (1.11). It is an analogue of (4.8) and (4.9) of [5]. Combining (4.11) with (4.9) and (4.10), again similarly to [5], we obtain

Lemma 4. *For $1/2 \leq \sigma < 1$, we have*

$$\begin{aligned} |E_\sigma(2T, A) - E_\sigma(T, A)| &\leq \left| \int_{T-L\Delta}^{2T+L\Delta} \left(P_\sigma(t; \Delta, A) + \overline{P_\sigma(t; \Delta, A)} \right) dt \right| \\ &\quad + \left| \int_{T+L\Delta}^{2T-L\Delta} \left(P_\sigma(t; \Delta, A) + \overline{P_\sigma(t; \Delta, A)} \right) dt \right| + O(L\Delta(T^\varepsilon)^\omega). \end{aligned} \quad (4.12)$$

Therefore, now our problem is reduced to the evaluation of the integral

$$\int_{T'}^{T''} \left(P_\sigma(t; \Delta, A) + \overline{P_\sigma(t; \Delta, A)} \right) dt, \quad (4.13)$$

where $T' = T \mp L\Delta$ and $T'' = 2T \pm L\Delta$. Since A_0 is sufficiently large (see (2.1)), we see that $T \ll T' \ll T$, $T \ll T'' \ll T$.

5. AN INFINITE SERIES EXPRESSION OF $P_\sigma(t; \Delta, A)$

Let $1/2 \leq \sigma < 1$. From (2.13) we have

$$\begin{aligned} P_\sigma(t; \Delta, A) &= \frac{1}{2\pi i} \int_{(\beta)} \sum_{k \leq M} a(k) k^{-2\sigma+s} \sum_{\ell \leq M} \overline{a(\ell)} \ell^{-1} \\ &\quad \times \sum_{f=1}^{\ell} \varphi\left(2\sigma - s, \frac{kf}{\ell}\right) \varphi\left(s, \frac{f}{\ell}\right) M(s, \sigma - it; \Delta) ds. \end{aligned} \quad (5.1)$$

We rewrite the factor $\varphi(2\sigma - s, kf/\ell)$ by using the functional equation, that is (2.11) if $kf/\ell \notin \mathbb{Z}$, and

$$\zeta(2\sigma - s) = 2^{2\sigma-s} \pi^{-1+2\sigma-s} \cos\left(\frac{\pi}{2}(1 - 2\sigma + s)\right) \Gamma(1 - 2\sigma + s) \zeta(1 - 2\sigma + s)$$

if $kf/\ell \in \mathbb{Z}$. We have

$$\begin{aligned}
 P_\sigma(t; \Delta, A) &= \frac{1}{2\pi i} \int_{(\beta)} \sum_{k \leq M} a(k) k^{-2\sigma+s} \sum_{\ell \leq M} \overline{a(\ell)} \ell^{-1} (2\pi)^{-1+2\sigma-s} \Gamma(1-2\sigma+s) \\
 &\times \left\{ 2 \sum_{j=1}^d \cos\left(\frac{\pi}{2}(1-2\sigma+s)\right) \zeta(1-2\sigma+s) \varphi\left(s, \frac{j}{d}\right) \right. \\
 &+ \sum_f^* \left(e\left(\frac{1-2\sigma+s}{4} - \frac{kf}{\ell}\right) \zeta\left(1-2\sigma+s, \frac{\{kf\}_\ell}{\ell}\right) \right. \\
 &+ \left. \left. e\left(-\frac{1-2\sigma+s}{4} - \frac{kf}{\ell}\right) \zeta\left(1-2\sigma+s, \frac{\{-kf\}_\ell}{\ell}\right) \right) \varphi\left(s, \frac{f}{\ell}\right) \right\} \\
 &\times M(s, \sigma - it; \Delta) ds, \tag{5.2}
 \end{aligned}$$

where the summation \sum^* runs over all f satisfying $1 \leq f \leq \ell$ and $f \not\equiv 0 \pmod{\lambda}$, and $\{x\}_\ell$ means the integer determined uniquely by $\{x\}_\ell \equiv x \pmod{\ell}$, $0 < \{x\}_\ell < \ell$.

From the assumption $\beta > 2$ it follows that $\Re(1-2\sigma+s) > 1$, hence the zeta factors on the right-hand side can be written down as Dirichlet series. Let

$$\sigma_a(n; x) = \sum_{m|n} e^{2\pi i m x} m^a$$

and

$$\sigma_a(n; x, \ell, b) = \sum_{\substack{m|n \\ n/m \equiv b \pmod{\ell}}} e^{2\pi i m x} m^a,$$

where a, x are real numbers and n, ℓ, b are positive integers. We see that

$$\begin{aligned}
 &\zeta(1-2\sigma+s, b/\ell) \varphi(s, x) \\
 &= \ell^{1-2\sigma+s} \sum_{\substack{1 \leq k < \infty \\ k \equiv b \pmod{\ell}}} k^{-1+2\sigma-s} \sum_{m=1}^{\infty} e^{2\pi i m x} m^{1-2\sigma} m^{-1+2\sigma-s} \\
 &= \ell^{1-2\sigma+s} \sum_{n=1}^{\infty} \sigma_{1-2\sigma}(n; x, \ell, b) n^{-1+2\sigma-s}, \tag{5.3}
 \end{aligned}$$

and especially

$$\zeta(1-2\sigma+s) \varphi(s, x) = \sum_{n=1}^{\infty} \sigma_{1-2\sigma}(n; x) n^{-1+2\sigma-s}. \tag{5.4}$$

Applying (5.3), (5.4), and the formula

$$\cos\left(\frac{\pi}{2}(1-2\sigma+s)\right) = \frac{1}{2} \left(e\left(\frac{1-2\sigma+s}{4}\right) + e\left(-\frac{1-2\sigma+s}{4}\right) \right)$$

to (5.2), and changing the order of integration and summation (which can be verified by absolute convergence), we obtain

$$\begin{aligned}
P_\sigma(t; \Delta, A) &= \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{k\ell} \\
&\times \left\{ \sum_{j=1}^d \sum_{n=1}^{\infty} \sigma_{1-2\sigma} \left(n; \frac{j}{d} \right) (Q_\sigma^+(t; n, k) + Q_\sigma^-(t; n, k)) \right. \\
&+ \sum_f^* e \left(-\frac{kf}{\ell} \right) \sum_{n=1}^{\infty} \sigma_{1-2\sigma} \left(n; \frac{f}{\ell}, \ell, \{kf\}_\ell \right) Q_\sigma^-(t; n, k\ell) \\
&\left. + \sum_f^* e \left(-\frac{kf}{\ell} \right) \sum_{n=1}^{\infty} \sigma_{1-2\sigma} \left(n; \frac{f}{\ell}, \ell, \{-kf\}_\ell \right) Q_\sigma^+(t; n, k\ell) \right\}, \quad (5.5)
\end{aligned}$$

where

$$\begin{aligned}
Q_\sigma^\pm(t; n, q) &= \frac{1}{2\pi i} \int_{(\beta)} e \left(\mp \frac{1-2\sigma+s}{4} \right) \Gamma(1-2\sigma+s) \\
&\times \left(\frac{q}{2\pi n} \right)^{1-2\sigma+s} M(s, \sigma - it; \Delta) ds. \quad (5.6)
\end{aligned}$$

The above (5.5) is the analogue of (5.1) of [5]. (The condition $\sigma < (\beta + 1)/2$ stated there is to be read as $\sigma < \beta/2$.)

Define

$$S_\sigma(n; k, \ell) = \sum_{\mu|n}^\# \mu^{2\sigma-1} e(\mu/\lambda),$$

where the summation $\sum^\#$ runs over all positive integer μ satisfying $\mu|n$, $\mu^{-1}n \equiv k \pmod{d}$ and $\mu \not\equiv 0 \pmod{\lambda}$. Then from (5.5) we can show the following

Lemma 5. *For $1/2 \leq \sigma < 1$, we have*

$$\begin{aligned}
P_\sigma(t; \Delta, A) &= \sum_{k \leq M} \sum_{\ell \leq M} \frac{a(k)\overline{a(\ell)}}{[k, \ell]} \\
&\times \left\{ d^{1-2\sigma} \sum_{n=1}^{\infty} \sigma_{1-2\sigma}(n) (Q_\sigma^+(t; dn, k) + Q_\sigma^-(t; dn, k)) \right. \\
&+ \sum_{n=1}^{\infty} n^{1-2\sigma} e \left(\frac{n\bar{\kappa}}{d\lambda} \right) \overline{S_\sigma(n; k, \ell)} Q_\sigma^-(t; dn, k\ell) \\
&\left. + \sum_{n=1}^{\infty} n^{1-2\sigma} e \left(-\frac{n\bar{\kappa}}{d\lambda} \right) S_\sigma(n; k, \ell) Q_\sigma^+(t; dn, k\ell) \right\}, \quad (5.7)
\end{aligned}$$

where $\sigma_{1-2\sigma}(n) = \sum_{m|n} m^{1-2\sigma}$ and $\bar{\kappa}$ is determined by $\kappa\bar{\kappa} \equiv 1 \pmod{\lambda}$.

Proof. We first note that

$$\sum_{j=1}^d \sigma_{1-2\sigma} \left(n; \frac{j}{d} \right) = \sum_{m|n} m^{1-2\sigma} \sum_{j=1}^d e^{2\pi i m j / d} = d \sum_{\substack{m|n \\ m \equiv 0 \pmod{d}}} m^{1-2\sigma}$$

vanishes unless $d|n$. If $d|n$, putting $n = d\nu$ and $m = d\mu$, we find that the above is

$$= d \sum_{\mu|\nu} (d\mu)^{1-2\sigma} = d^{2-2\sigma} \sigma_{1-2\sigma}(\nu).$$

Hence

$$\begin{aligned} & \sum_{j=1}^d \sum_{n=1}^{\infty} \sigma_{1-2\sigma} \left(n; \frac{j}{d} \right) (Q_{\sigma}^{+}(t; n, k) + Q_{\sigma}^{-}(t; n, k)) \\ &= \sum_{\nu=1}^{\infty} d^{2-2\sigma} \sigma_{1-2\sigma}(\nu) (Q_{\sigma}^{+}(t; d\nu, k) + Q_{\sigma}^{-}(t; d\nu, k)). \end{aligned} \quad (5.8)$$

Next consider

$$\begin{aligned} & \sum_f^* e \left(-\frac{kf}{\ell} \right) \sigma_{1-2\sigma} \left(n; \frac{f}{\ell}, \ell, \{kf\}_{\ell} \right) \\ &= \sum_f^* e \left(-\frac{kf}{\ell} \right) \sum_{\substack{m|n \\ m \equiv kf \pmod{\ell}}} e \left(\frac{nf}{m\ell} \right) \left(\frac{n}{m} \right)^{1-2\sigma}. \end{aligned} \quad (5.9)$$

The condition $m \equiv kf \pmod{\ell}$ implies $d|m$, hence the above double sum vanishes unless $d|n$. If $d|n$, we write $n = d\nu$, $m = d\mu$ to obtain that the right-hand side of (5.9) is

$$\begin{aligned} &= \sum_f^* e \left(-\frac{kf}{\ell} \right) \sum_{\substack{\mu|\nu \\ \mu \equiv kf \pmod{\ell}}} e \left(\frac{\nu f}{\mu\ell} \right) \left(\frac{\nu}{\mu} \right)^{1-2\sigma} \\ &= \nu^{1-2\sigma} \sum_{\mu|\nu} \mu^{2\sigma-1} \sum_f^{**} e \left(\left(\frac{\nu}{\mu} - k \right) \frac{f}{\ell} \right), \end{aligned} \quad (5.10)$$

where \sum^{**} runs over all f such that $1 \leq f \leq \ell$, $f \not\equiv 0 \pmod{\ell}$, and $f \equiv \mu\bar{k} \pmod{\ell}$. From these conditions it follows that $\mu \not\equiv 0 \pmod{\ell}$. We see that

$$\begin{aligned} \sum_f^{**} e \left(\left(\frac{\nu}{\mu} - k \right) \frac{f}{\ell} \right) &= \sum_{j=0}^{d-1} e \left(\left(\frac{\nu}{\mu} - k \right) \frac{\{\mu\bar{k}\}_{\ell} + j\ell}{\ell} \right) \\ &= e \left(\left(\frac{\nu}{\mu} - k \right) \frac{\{\mu\bar{k}\}_{\ell}}{\ell} \right) \times \begin{cases} d & \text{if } d | (\mu^{-1}\nu - k), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the right-hand side of (5.10) is

$$\begin{aligned}
&= d\nu^{1-2\sigma} \sum_{\mu|\nu}^{\#} \mu^{2\sigma-1} e\left(\left(\frac{\nu}{\mu} - k\right) \frac{\{\mu\bar{\kappa}\}_\lambda}{d\lambda}\right) \\
&= d\nu^{1-2\sigma} \sum_{\mu|\nu}^{\#} \mu^{2\sigma-1} e\left(\left(\frac{\nu}{\mu} - k\right) \frac{\mu\bar{\kappa}}{d\lambda}\right) \\
&= d\nu^{1-2\sigma} e\left(\frac{\nu\bar{\kappa}}{d\lambda}\right) \overline{S_\sigma(\nu; k, \ell)}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\sum_f^* e\left(-\frac{kf}{\ell}\right) \sum_{n=1}^{\infty} \sigma_{1-2\sigma}\left(n; \frac{f}{\ell}, \ell, \{kf\}_\ell\right) Q_\sigma^-(t; n, k\ell) \\
&= d \sum_{\nu=1}^{\infty} \nu^{1-2\sigma} e\left(\frac{\nu\bar{\kappa}}{d\lambda}\right) \overline{S_\sigma(\nu; k, \ell)} Q_\sigma^-(t; d\nu, k\ell),
\end{aligned}$$

and similarly we can show that the last double sum in the curly parenthesis on the right-hand side of (5.5) is equal to

$$d \sum_{\nu=1}^{\infty} \nu^{1-2\sigma} e\left(-\frac{\nu\bar{\kappa}}{d\lambda}\right) S_\sigma(\nu; k, \ell) Q_\sigma^+(t; d\nu, k\ell).$$

Substituting these formulas and (5.8) into (5.5), and using $d/k\ell = [k, \ell]^{-1}$, we arrive at the assertion of Lemma 5.

6. COMPLETION OF THE PROOF

Now we evaluate the integral (4.13) by using Lemma 5. Since

$$\left|n^{1-2\sigma} e\left(-\frac{n\bar{\kappa}}{d\lambda}\right) S_\sigma(n; k, \ell)\right| \leq n^{1-2\sigma} \sum_{\mu|n} \mu^{2\sigma-1} = \sigma_{1-2\sigma}(n), \quad (6.1)$$

from Lemma 5 we have

$$\begin{aligned}
&\int_{T'}^{T''} P_\sigma(t; \Delta, A) dt \ll \sum_{k \leq M} \sum_{\ell \leq M} (k\ell)^{-1+\varepsilon} d \\
&\times \left\{ d^{1-2\sigma} \sum_{n=1}^{\infty} \sigma_{1-2\sigma}(n) \left(\left| \int_{T'}^{T''} Q_\sigma^+(t; dn, k) dt \right| + \left| \int_{T'}^{T''} Q_\sigma^-(t; dn, k) dt \right| \right) \right. \\
&\quad \left. + \sum_{n=1}^{\infty} \sigma_{1-2\sigma}(n) \left(\left| \int_{T'}^{T''} Q_\sigma^+(t; dn, k\ell) dt \right| + \left| \int_{T'}^{T''} Q_\sigma^-(t; dn, k\ell) dt \right| \right) \right\}. \quad (6.2)
\end{aligned}$$

The quantity $Q_\sigma^\pm(t; n, q)$, defined by (5.6), is exactly the same as $Q_\sigma^\pm(t; n, q)$ introduced in Section 5 of [5] and studied in Sections 6, 7 and 8 of [5]. In Section

6 of [5], by using Lemma 1(ii), it is shown that

$$\int_{T'}^{T''} Q_{\sigma}^{+}(t; n, q) dt = \frac{1}{i} \int_0^{\infty} h_{\sigma}^{+}(y; n, q) dy \quad (6.3)$$

and

$$\int_{T'}^{T''} Q_{\sigma}^{-}(t; n, q) dt = \frac{1}{i} \int_0^{\infty} \overline{h_{\sigma}^{-}(y; n, q)} dy, \quad (6.4)$$

where

$$\begin{aligned} h_{\sigma}^{\pm}(y; n, q) &= \frac{\exp\left(-\frac{1}{4}\Delta^2 \log^2(1+y^{-1})\right)}{y^{\sigma}(1+y)^{\sigma} \log(1+y^{-1})} \\ &\times \left\{ e\left(\frac{\pm T''}{2\pi} \log(1+y^{-1})\right) - e\left(\frac{\pm T'}{2\pi} \log(1+y^{-1})\right) \right\} e(-ny/q) \end{aligned} \quad (6.5)$$

(see (6.3) and (8.2) of [5]). Let $N \gg qTL^2\Delta^{-2}$. Then Lemma 7 of [5] implies

$$\sum_{n>N} \sigma_{1-2\sigma}(n) \left| \int_{T'}^{T''} Q_{\sigma}^{\pm}(t; n, q) dt \right| \ll q^{1+\sigma+\varepsilon} e^{-AT} + (qT)^{-C_3}, \quad (6.6)$$

where A is a positive constant and C_3 is a large positive constant. The remaining part $n \leq N$ has been discussed in Section 8 of [5]. From Lemma 8 of [5] (and its proof) we have

$$\sum_{n \leq N} \sigma_{1-2\sigma}(n) \left| \int_0^{L^{-1}\Delta} h_{\sigma}^{\pm}(y; n, q) dy \right| \ll e^{-AL^2} NT (\log N \log T)^{\omega}, \quad (6.7)$$

where A is a positive constant and ω is as in Section 4. Also we have

$$\begin{aligned} &\int_{L^{-1}\Delta}^{\infty} h_{\sigma}^{\pm}(y; n, q) dy \\ &\ll (L^{-1}\Delta)^{1-2\sigma} \left\{ \frac{q}{n} \left(1 + \left(\frac{nT}{q} \right)^{1/4} \right) + \Delta^{3/2} L^{1/2} T^{-1/2} \right\} \end{aligned} \quad (6.8)$$

if $n \leq qT_1 L^2 \Delta^{-2}$ ((8.17) of [5]).

By k^* we denote k or $k\ell$. From (5.6) it follows that $Q_{\sigma}^{\pm}(t; dn, k^*) = Q_{\sigma}^{\pm}(t; n, d^{-1}k^*)$. Hence

$$\begin{aligned} &\sum_{n=1}^{\infty} \sigma_{1-2\sigma}(n) \left| \int_{T'}^{T''} Q_{\sigma}^{\pm}(t; dn, k^*) dt \right| \leq \sum_{n>N^*} \sigma_{1-2\sigma}(n) \left| \int_{T'}^{T''} Q_{\sigma}^{\pm}(t; n, d^{-1}k^*) dt \right| \\ &+ \sum_{n \leq N^*} \sigma_{1-2\sigma}(n) \left| \int_0^{L^{-1}\Delta} h_{\sigma}^{\pm}(y; n, d^{-1}k^*) dy \right| + \sum_{n \leq N^*} \sigma_{1-2\sigma}(n) \left| \int_{L^{-1}\Delta}^{\infty} h_{\sigma}^{\pm}(y; n, d^{-1}k^*) dy \right| \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3, \end{aligned} \quad (6.9)$$

say, where $N^* = d^{-1}k^*T_1 L^2 \Delta^{-2}$.

Applying (6.8), we have

$$\begin{aligned}
\Sigma_3 &\ll \sum_{n \leq N^*} \sigma_{1-2\sigma}(n) (L^{-1}\Delta)^{1-2\sigma} \left\{ \frac{k^*}{nd} \left(1 + \frac{ndT}{k^*}\right)^{1/4} + \Delta^{3/2} L^{1/2} T^{-1/2} \right\} \\
&\ll (L^{-1}\Delta)^{1-2\sigma} \left\{ \sum_{n \leq N^*} \sigma_{1-2\sigma}(n) \left(\frac{k^*}{nd} + \left(\frac{k^*}{nd}\right)^{3/4} T^{1/4} \right) \right. \\
&\quad \left. + \Delta^{3/2} L^{1/2} T^{-1/2} \sum_{n \leq N^*} \sigma_{1-2\sigma}(n) \right\} \\
&\ll \Delta^{1-2\sigma} \left\{ \frac{k^*}{d} + \left(\frac{k^*}{d}\right)^{3/4} (TN^*)^{1/4} + \Delta^{3/2} T^{-1/2} N^* \right\} T^\varepsilon \\
&\ll \frac{k^*}{d} \Delta^{1/2-2\sigma} T^{1/2+\varepsilon},
\end{aligned}$$

hence the contribution of Σ_3 to the right-hand side of (6.2) is

$$\ll \sum_{k \leq M} \sum_{\ell \leq M} (k\ell)^{-1+\varepsilon} k^* \Delta^{1/2-2\sigma} T^{1/2+\varepsilon} \ll M^2 \Delta^{1/2-2\sigma} T^{1/2+\varepsilon}.$$

Using (6.6) and (6.7) we can see that the contributions of Σ_1 and Σ_2 are negligible. Hence

$$\int_{T'}^{T''} P_\sigma(t; \Delta, A) dt \ll M^2 \Delta^{1/2-2\sigma} T^{1/2+\varepsilon}. \quad (6.10)$$

Combining (6.10) with Lemma 4, we obtain

$$|E_\sigma(2T, A) - E_\sigma(T, A)| \ll M^2 \Delta^{1/2-2\sigma} T^{1/2+\varepsilon} + L\Delta(T^\varepsilon)^\omega \quad (6.11)$$

for $1/2 \leq \sigma < 1$.

Recall that we assume (2.2) to obtain (6.11). The inequality $M \leq (2^{-j}T)^b$ is valid for $0 \leq j \leq j_1$, where $j_1 = \lceil \log(M^{-1/b}T) / \log 2 \rceil$. Let

$$\Delta_j = \left(\max\{2^{-cj}M, T_j^\varepsilon\} \right)^\mu T_j^\rho \quad (0 \leq j \leq j_1),$$

where $T_j = 2^{-j}T$ and $c > 0$. Note that $2^{-cj}M \leq T_j^\varepsilon$ for $j > j_2$, where $j_2 = \lceil \log(MT^{-\varepsilon}) / (c - \varepsilon) \log 2 \rceil$. We prove that, if we choose c suitably, then there exists a constant $C_0 > 0$ such that

$$L_j \leq \Delta_j \leq \frac{T_j}{A_0 L_j} \quad (6.12)$$

for $T_j \geq C_0$, where $L_j = A_0(\log T_j)^{1/2}$. In fact, we see that $\Delta_j \geq T_j^{\mu\varepsilon+\rho} \gg L_j$, which yields the first inequality of (6.12). Since $\rho < 1$, it is clear that

$$T_j^{\mu\varepsilon+\rho} \ll \frac{T_j}{L_j}.$$

Next, if $\mu > 0$, we choose c for which $c\mu + \rho = 1$ holds, then

$$(2^{-cj}M)^\mu T_j^\rho = 2^{-(c\mu+\rho)j} M^\mu T^\rho = 2^{-j} M^\mu T^\rho \leq \frac{2^{-j}T}{A_0L} \leq \frac{T_j}{A_0L_j}$$

by (1.7). If $\mu = 0$, then clearly

$$(2^{-cj}M)^\mu T_j^\rho = T_j^\rho \ll \frac{T_j}{L_j}$$

because $\rho < 1$. Hence the second inequality of (6.12) follows.

Let j_3 be the largest integer for which $2^{-j_3}T \geq C_0$ holds, and put $j_0 = \min\{j_1, j_3\}$. Then (6.12) implies that (2.1) and (2.2) are valid for T_j and Δ_j ($0 \leq j \leq j_0$) instead of T and Δ , respectively. Hence we may replace T and Δ in (6.11) by T_j and Δ_j respectively to obtain

$$\begin{aligned} & |E_\sigma(T_{j-1}, A) - E_\sigma(T_j, A)| \\ & \ll M^{2+\mu(1/2-2\sigma)} (2^{-j})^{(c\mu+\rho)(1/2-2\sigma)+1/2+\varepsilon} T^{f(\sigma,\rho)+\varepsilon} \\ & \quad + L_j M^\mu (2^{-j})^{c\mu+\rho+\varepsilon\omega} T^{\rho+\varepsilon\omega} \end{aligned} \quad (6.13)$$

if $\mu > 0$ and $1 \leq j \leq \min\{j_0, j_2\}$, and

$$\ll M^2 (2^{-j})^{f(\sigma,\rho)+\varepsilon} T^{f(\sigma,\rho)+\varepsilon} + L_j (2^{-j})^{\rho+\varepsilon} T^{\rho+\varepsilon} \quad (6.14)$$

if $\mu > 0$ and $j_2 < j \leq j_0$, or if $\mu = 0$.

We sum up these inequalities for $j = 1, 2, \dots, j_0$. In case $\mu > 0$, noting $c\mu + \rho = 1$, $f(\sigma, \rho) > 0$ and $M^{-1/c}T^\varepsilon \ll 2^{-j_2} \ll M^{-1/c}T^\varepsilon$, we find that

$$\begin{aligned} & |E_\sigma(T, A) - E_\sigma(T_{j_0}, A)| \\ & \ll M^{2+\mu(1/2-2\sigma)} T^{f(\sigma,\rho)+\varepsilon} \sum_{j=1}^{j_2} (2^{-j})^{1-2\sigma+\varepsilon} + M^\mu T^{\rho+\varepsilon\omega} L \sum_{j=1}^{j_2} (2^{-j})^{1+\varepsilon\omega} \\ & \quad + M^2 T^{f(\sigma,\rho)+\varepsilon} \sum_{j=j_2+1}^{j_0} (2^{-j})^{f(\sigma,\rho)+\varepsilon} + T^{\rho+\varepsilon} \sum_{j=j_2+1}^{j_0} (2^{-j})^{\rho+\varepsilon} \\ & \ll M^{2+\mu(1/2-2\sigma)+c^{-1}(2\sigma-1)} T^{f(\sigma,\rho)+\varepsilon} + M^\mu T^{\rho+\varepsilon} \\ & \quad + M^{2-c^{-1}f(\sigma,\rho)} T^{f(\sigma,\rho)+\varepsilon} + M^{-c^{-1}\rho} T^{\rho+\varepsilon}, \end{aligned}$$

hence

$$|E_\sigma(T, A) - E_\sigma(T_{j_0}, A)| \ll M^{2-(1-\rho)^{-1}\mu f(\sigma,\rho)} T^{f(\sigma,\rho)+\varepsilon} + M^\mu T^{\rho+\varepsilon}, \quad (6.15)$$

because

$$\mu \left(\frac{1}{2} - 2\sigma \right) + \frac{2\sigma - 1}{c} = -\frac{f(\sigma, \rho)}{c} = -\frac{\mu f(\sigma, \rho)}{1 - \rho}.$$

From (6.14) we can easily see that (6.15) is also valid in case $\mu = 0$.

On the other hand, if $j_0 = j_1$, from Remark 2 in Section 1 we have

$$\begin{aligned} |E_\sigma(T_{j_0}, A)| &\ll M^{2-(1-\rho)^{-1}\mu f(\sigma,\rho)} T_{j_0}^{f(\sigma,\rho)+\varepsilon} + M^\mu T_{j_0}^{\rho+\varepsilon} \\ &\ll M^{2-(1-\rho)^{-1}\mu f(\sigma,\rho)} T^{f(\sigma,\rho)+\varepsilon} + M^\mu T^{\rho+\varepsilon}. \end{aligned} \quad (6.16)$$

If $j_0 = j_3$, then $T_{j_0} \ll 1$. Hence (1.15) is valid for $M \gg 1$, and in this case (6.16) again follows from Remark 2. If $T_{j_0} \ll 1$ and $M \ll 1$ then $E_\sigma(T_{j_0}, A)$ is clearly bounded. Hence (6.16) is true in all the cases. Combining (6.15) and (6.16), we obtain the assertions of Theorems 1 and 2.

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