A weighted integral approach to the mean square of Dirichlet L-functions

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1 Introduction

Let $s=\sigma+it$ be a complex variable, and $\zeta(s)$ the Riemann zeta-function. The mean square formula of the form

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 dt = T\log T-(\log 2\pi-2\gamma+1)T+E(T)$$

for $T\geq 2$, where γ is Euler's constant and $E(T)$ is the error term, has been known from 1920's. The estimate

$$E(T)=O(T^{\alpha+\varepsilon}) \tag{1.1}$$

with $\alpha<\frac{1}{3}$ was first proved by Balasubramanian [2]. Here, and throughout this paper, ε denotes an arbitrarily small positive number, not necessarily the same on each occurrence. Balasubramanian's argument, based on the Riemann-Siegel formula, is quite involved. Several years later, Jutila [10] found a simple elegant way of deriving (1.1) from the explicit formula of $E(T)$ proved by Atkinson [1]. At present $\alpha=\frac{72}{227}$ is known (Huxley [6]).

Atkinson's method is useful not only on the critical line $\sigma=\frac{1}{2}$, but also in the critical strip $\frac{1}{2}<\sigma<1$. As an analogue of (1.1), the second author [18] proved, among other things,

$$E_\sigma(T)=O\left(T^{1/(1+4\sigma)}(\log T)^2\right) \tag{1.2}$$

for $\frac{1}{2}<\sigma<\frac{3}{4}$, where

$$E_\sigma(T)=\int_0^T |\zeta(\sigma+it)|^2 dt-\zeta(2\sigma)T-(2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma},$$

and the implied constant depends only on σ. To show (1.2), the second author proved an analogue of Atkinson's explicit formula for $\frac{1}{2}<\sigma<\frac{3}{4}$; a similar analogue was also discussed by Laurinčikas [13] [14]. Motohashi then proved that (1.2) holds for any σ satisfying $\frac{1}{2}<\sigma<1$. His manuscript [25] is unpublished, but a modified version of his argument is presented in Section 2.7 of Ivić [8]. The fundamental idea of this method is to apply Atkinson's dissection device to the weighted integral

$$\frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\zeta(u+iy)\zeta(v-iy)e^{-(y/\Delta)^2}dy, \tag{1.3}$$

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where u and v are complex and $\Delta > 0$. The explicit formula of Atkinson's type itself is not necessary here. For further improvements and refinements of (1.2), see Ivić-Matsumoto [9], Kačėnas [11] [12] and Laurinčikas [16].

It is the purpose of the present paper to prove a generalization of the estimates (1.1) and (1.2) to the case of Dirichlet L-functions. Let q be a positive integer, χ a Dirichlet character mod q, and $L(s,\chi)$ the corresponding Dirichlet L-function. Define

$$\begin{aligned} E(T,\chi) &= E_{\frac{1}{2}}(T,\chi) = \int_0^T |L(\tfrac{1}{2}+it,\chi)|^2 dt - \frac{\varphi(q)}{q} T\log T \\ &\quad - \frac{\varphi(q)}{q}\left(\log\frac{q}{2\pi} + 2\gamma - 1 + 2\sum_{p|q}\frac{\log p}{p-1}\right) T, \end{aligned} \tag{1.4}$$

where $\varphi(q)$ is Euler's function and p runs over all prime divisors of q, and

$$\begin{aligned} E_\sigma(T,\chi) &= \int_0^T |L(\sigma+it,\chi)|^2 dt - L(2\sigma,\chi_0)T \\ &\quad - \left(\frac{2\pi}{q}\right)^{2\sigma-1} \frac{L(2-2\sigma,\chi_0)}{2-2\sigma} T^{2-2\sigma} \end{aligned} \tag{1.5}$$

for $\frac{1}{2} < \sigma < 1$, where χ_0 is the principal character mod q. We shall prove

Theorem 1. *Let $q = p$ be a prime, and χ a primitive character mod p. Then*

$$E(T,\chi) = O\left\{(pT)^{\frac{1}{3}}(\log pT)^2 + p^{\frac{1}{2}}(\log pT)^3 \log T\right\}. \tag{1.6}$$

Theorem 2. *Let $q = p$ be a prime, and χ a primitive character mod p. Then, for $\frac{1}{2} < \sigma < 1$,*

$$E_\sigma(T,\chi) = O\Big\{(pT)^{1/(1+4\sigma)}\log pT + p^{\frac{1}{2}}(\log pT)^{1+2\sigma}\Big\}. \tag{1.7}$$

Remark. In the above theorems we should restrict ourselves to the case that $q = p$ is a prime, because we use Weil's estimate in Section 9 (see Lemma 10). Our arguments are given for general modulus q up to Section 8. See also the end of this paper.

To prove these theorems, we introduce the weighted integral

$$\frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty} L(u+iy,\chi)L(v-iy,\overline{\chi})e^{-(y/\Delta)^2}dy$$

as a generalization of (1.3), where $\overline{\chi}$ is the complex conjugate of χ, and apply the argument similar to that given in Section 2.7 of Ivić's monograph [8]. The analogue of Ivić's Lemma 2.5 is our Lemma 4, stated at the end of Section 4, which reduces our problem to the evaluation of certain integrals. Then we evaluate them by Atkinson's saddle point lemma, the first and the second derivative tests, and Weil's estimate [28] of Kloosterman sums.

The usefulness of Weil's estimate in the present problem was already noticed in the second author's note [17], in which $E(T,\chi)$ is estimated by a generalization of Balasubramanian's method [2]. However, [17] includes a serious error. The corrected version [19] was published later, which includes the assertion $E(T,\chi) = O((qT)^{\frac{1}{3}+\varepsilon})$ for any odd integer q, when $T \gg q^{20}$. The details of the lengthy proof are given in [20] (unpublished).

Motohashi [23] announced that if $q = p$ is a prime, then

$$E(T,\chi) = O\left\{\left((pT)^{\frac{1}{3}} + p^{\frac{1}{2}}\right)(\log pT)^4\right\} \tag{1.8}$$

holds. He described a very brief sketch of his method in [23], which is a variant of Atkinson's method, combined with Weil's estimate. Actually he treated the error term

$$\begin{aligned}\widetilde{E}(T,\chi) &= \frac{1}{2}\int_{-T}^{T} |L(\tfrac{1}{2}+it,\chi)|^2 dt - \frac{\varphi(q)}{q} T\log T \\ &\quad -\frac{\varphi(q)}{q}\left(\log\frac{q}{2\pi} + 2\gamma - 1 + 2\sum_{p|q}\frac{\log p}{p-1}\right)T,\end{aligned}$$

hence (1.8) should be read as the estimate for $\widetilde{E}(T,\chi)$ instead of $E(T,\chi)$. If χ is real, then $\widetilde{E}(T,\chi) = E(T,\chi)$, but in general equality does not hold. The details of Motohashi's method are written in his unpublished manuscript [24], but it includes several gaps. (It is possible to fill these gaps.)

At about the same time, Meurman considered the same problem by Atkinson's method, but his manuscript [22] is also unpublished.

We mention that Atkinson's explicit formula has been satisfactorily generalized by Meurman [21] to the mean square of the form

$$\sum_{\chi(\bmod q)} \int_0^T |L(\tfrac{1}{2}+it,\chi)|^2 dt,$$

and its analogue for $\sigma > \frac{1}{2}$ was studied by Laurinčikas [15]. It is highly desirable to establish the explicit formula of Atkinson's type for our $E(T,\chi)$ and $E_\sigma(T,\chi)$.

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2 Atkinson's dissection

Let q be a positive integer, χ a primitive Dirichlet character mod q, Δ a positive parameter, and define

$$I(u,v;\Delta) = \frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty} L(u+iy,\chi)L(v-iy,\overline{\chi})e^{-(y/\Delta)^2}dy \tag{2.1}$$

for any complex u and v. At first we assume $\operatorname{Re} u > 1$ and $\operatorname{Re} v > 1$. Then

$$I(u,v;\Delta) = \frac{1}{\Delta\sqrt{\pi}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\chi(m)\overline{\chi}(n)\int_{-\infty}^{\infty} m^{-u-iy}n^{-v+iy}e^{-(y/\Delta)^2}dy.$$

We divide this double sum into three parts according to the conditions $m = n$, $m < n$ and $m > n$ to obtain

$$I(u,v;\Delta) = L(u+v,\chi_0) + I_1(u,v;\Delta) + \overline{I_1(\overline{v},\overline{u};\Delta)}, \tag{2.2}$$

where χ_0 is the principal character mod q and

$$I_1(u,v;\Delta) = \frac{1}{\Delta\sqrt{\pi}}\sum\sum_{m<n}\chi(m)\overline{\chi}(n)m^{-u}n^{-v}\int_{-\infty}^{\infty}\left(\frac{n}{m}\right)^{iy}e^{-(y/\Delta)^2}dy.$$

Applying the well-known formula

$$\int_{-\infty}^{\infty} \exp(At - Bt^2)dt = \left(\frac{\pi}{B}\right)^{\frac{1}{2}} \exp\left(\frac{A^2}{4B}\right) \qquad (\operatorname{Re} B > 0)$$

(see (A.38) of Ivić [7]), and then putting $m = kq + a$ and $n = m + lq + b$, we obtain

$$\begin{aligned} I_1(u,v;\Delta) &= \sum_{a,b=1}^{q} \chi(a)\overline{\chi}(a+b) \sum_{k,l=0}^{\infty} (qk+a)^{-u-v} \\ &\quad \times \left(1 + \frac{ql+b}{qk+a}\right)^{-v} \exp\left(-\frac{1}{4}\Delta^2 \log^2\left(1 + \frac{ql+b}{qk+a}\right)\right). \end{aligned} \tag{2.3}$$

Let, for $\operatorname{Re} v > \operatorname{Re} s > 0$,

$$M(s,v;\Delta) = \frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\Gamma(s)\Gamma(v+iy-s)}{\Gamma(v+iy)} e^{-(y/\Delta)^2} dy. \tag{2.4}$$

Then $M(s,v;\Delta)$ has the following properties, which have been shown in Section 5.2 of Ivić [8]:

(i) For $\operatorname{Re} s > 0$ and any v, we have

$$M(s,v;\Delta) = \int_0^{\infty} x^{s-1}(1+x)^{-v} \exp\left(-\frac{1}{4}\Delta^2 \log^2(1+x)\right) dx. \tag{2.5}$$

(ii) $M(s,v;\Delta)$ is entire in v, and can be continued to a meromorphic function of s, holomorphic except for the poles at $s = 0, -1, -2, \ldots$.

(iii) For any fixed $c > 0$, we have

$$M(s,v;\Delta) = O\left((1+|s|)^{-c}\right)$$

as $|\operatorname{Im} s| \to +\infty$, uniformly for bounded v and bounded $\operatorname{Re} s$.

(iv) If $\operatorname{Re} v > \alpha > 0$ and $x > 0$, then

$$\frac{1}{2\pi i} \int_{(\alpha)} M(s,v;\Delta) x^{-s} ds = (1+x)^{-v} \exp\left(-\frac{1}{4}\Delta^2 \log^2(1+x)\right), \tag{2.6}$$

where the path of integration is the vertical line from $\alpha - i\infty$ to $\alpha + i\infty$.

From (2.3) and (2.6) we have, for $\operatorname{Re} u > 1$ and $\operatorname{Re} v > \alpha > 1$,

$$\begin{aligned} I_1(u,v;\Delta) &= \sum_{a,b=1}^{q} \chi(a)\overline{\chi}(a+b) \frac{1}{2\pi i} \int_{(\alpha)} M(s,v;\Delta) \\ &\quad \times \sum_{k,l=0}^{\infty} (qk+a)^{-u-v+s}(ql+b)^{-s} ds \\ &= q^{-u-v} \sum_{a,b=1}^{q} \chi(a)\overline{\chi}(a+b) \frac{1}{2\pi i} \int_{(\alpha)} M(s,v;\Delta) \\ &\quad \times \zeta(u+v-s, a/q)\zeta(s, b/q) ds, \end{aligned} \tag{2.7}$$

where $\zeta(s,x)=\sum_{n=0}^{\infty}(n+x)^{-s}$ $(\mathrm{Re}\, s>1)$ is the Hurwitz zeta-function.

Let $\beta>\alpha$, and temporarily assume that $\mathrm{Re}\, u>1$, $\mathrm{Re}\, v>\alpha>1$, $\mathrm{Re}(u+v)<\beta+1$. Shifting the path of integration in (2.7) to $\mathrm{Re}\, s=\beta$, we obtain

$$
\begin{aligned}
(2.8)\qquad I_1(u,v;\Delta) &= q^{-u-v}M(u+v-1,v;\Delta)\\
&\quad\times\sum_{a,b=1}^{q}\chi(a)\overline{\chi}(a+b)\zeta(u+v-1,b/q)+P(u,v;\Delta)
\end{aligned}
$$

where

$$
\begin{aligned}
(2.9)\qquad P(u,v;\Delta) &= q^{-u-v}\sum_{a,b=1}^{q}\chi(a)\overline{\chi}(a+b)\frac{1}{2\pi i}\int_{(\beta)}M(s,v;\Delta)\\
&\quad\times\zeta(u+v-s,a/q)\zeta(s,b/q)ds.
\end{aligned}
$$

We note that

$$
(2.10)\qquad \sum_{a=1}^{q}\chi(a)\overline{\chi}(a+b)=\sum_{d|(q,b)}\mu(q/d)d,
$$

where $\mu(n)$ denotes the Möbius function. In fact, denoting by $\overline{a}$ the integer satisfying $a\overline{a}\equiv 1 \pmod q$, we have

$$
\begin{aligned}
\sum_{a=1}^{q}\chi(a)\overline{\chi}(a+b) &= \sum_{\substack{a=1\\(a,q)=1}}^{q}\chi(a)\overline{\chi}(a(1+\overline{a}b))=\sum_{\substack{a=1\\(a,q)=1}}^{q}\overline{\chi}(1+\overline{a}b)\\
&= \tau(\chi)^{-1}\sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{c=1}^{q}\chi(c)e((1+\overline{a}b)c/q),
\end{aligned}
$$

where $\tau(\chi)=\sum_{k=1}^{q}\chi(k)e(k/q)$ is the Gauss sum and $e(x)=\exp(2\pi ix)$. Here the last equality follows from the well-known property of the Gauss sum (cf. formula (2) in Chapter 9 of Davenport [3]). Changing the order of summation, we find that the resulting inner sum is equal to Ramanujan's sum

$$
\sum_{\substack{r=1\\(r,q)=1}}^{q}e(rb/q)=\sum_{d|(q,b)}\mu(q/d)d,
$$

hence (2.10) follows.

From (2.8) and (2.10) we obtain

$$
\begin{aligned}
(2.11)\qquad I_1(u,v;\Delta) &= q^{1-u-v}M(u+v-1,v;\Delta)\zeta(u+v-1)\\
&\quad\times\prod_{p|q}(1-p^{u+v-2})+P(u,v;\Delta),
\end{aligned}
$$

because

$$
\sum_{b=1}^{q}\sum_{d|(q,b)}\mu(q/d)d\cdot\zeta(u+v-1,b/q)
$$

$$
\begin{aligned}
&= \sum_{d|q} \mu(d) \frac{q}{d} \sum_{c=1}^{d} \zeta(u+v-1, c/d) \\
&= q \sum_{d|q} \mu(d) d^{u+v-2} \zeta(u+v-1) \\
&= q\zeta(u+v-1) \prod_{p|q} (1-p^{u+v-2}).
\end{aligned}
$$

The formula (2.11) gives the meromorphic continuation of $I_1(u,v;\Delta)$ to any (u,v) satisfying $\mathrm{Re}(u+v) < \beta+1$. From (2.2) and (2.11) we obtain

$$
\begin{aligned}
I(u,v;\Delta) = {} & L(u+v,\chi_0) + q^{1-u-v}\zeta(u+v-1)\prod_{p|q}(1-p^{u+v-2}) \\
& \times \{M(u+v-1,u;\Delta) + M(u+v-1,v;\Delta)\} \\
& + P(u,v;\Delta) + \overline{P(\overline{v},\overline{u};\Delta)}
\end{aligned} \tag{2.12}
$$

for $\mathrm{Re}(u+v) < \beta+1$.

3 The weighted mean square

Let $T \geq 2$, A_0 a sufficiently large positive number, $L = A_0(\log qT)^{\frac{1}{2}}$, and assume that Δ satisfies

$$
L \leq \Delta \leq \frac{T}{A_0 L}. \tag{3.1}
$$

To ensure the existence of such a Δ, we need

$$
A_0 L^2 \leq T, \tag{3.2}
$$

which we assume hereafter. As for the opposite case, see Remark 2 at the end of this section. Define

$$
J_\sigma(t;\Delta) = \frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} |L(\sigma+i(t+y),\chi)|^2 e^{-(y/\Delta)^2} dy.
$$

Lemma 1. *We have*

$$
\begin{aligned}
J_\sigma(t;\Delta) = {} & L(2\sigma,\chi_0) + \left(\frac{qt}{2\pi}\right)^{1-2\sigma} L(2-2\sigma,\chi_0) \\
& + P_\sigma(t;\Delta) + \overline{P_\sigma(t;\Delta)} + O(q^{1-2\sigma}T^{-2\sigma}\Delta)
\end{aligned} \tag{3.3}
$$

for $\frac{1}{2} < \sigma < 1$ *and* $t \asymp T$ *(i.e.* $T \ll t \ll T$*), where*

$$
P_\sigma(t;\Delta) = P(\sigma+it, \sigma-it;\Delta).
$$

Proof: Putting $u = \sigma+it$ and $v = \sigma-it$ in (2.12), we have

$$
\begin{aligned}
J_\sigma(t;\Delta) = {} & L(2\sigma,\chi_0) + q^{1-2\sigma}\zeta(2\sigma-1)\prod_{p|q}(1-p^{2\sigma-2}) \\
& \times \{M(2\sigma-1,\sigma+it;\Delta) + M(2\sigma-1,\sigma-it;\Delta)\} \\
& + P_\sigma(t;\Delta) + \overline{P_\sigma(t;\Delta)}.
\end{aligned} \tag{3.4}
$$

By the definition (2.4) we have

$$\begin{aligned}&M(2\sigma-1,\sigma+it;\Delta)+M(2\sigma-1,\sigma-it;\Delta)\\&\quad=2\,\mathrm{Re}\,M(2\sigma-1,\sigma+it;\Delta)\\&\quad=\frac{2\Gamma(2\sigma-1)}{\Delta\sqrt{\pi}}\,\mathrm{Re}\int_{-\infty}^{\infty}\frac{\Gamma(1-\sigma+i(t+y))}{\Gamma(\sigma+i(t+y)}e^{-(y/\Delta)^2}\,dy.\end{aligned}$$

Hence, in view of the functional equation

$$\zeta(2\sigma-1)=\frac{(2\pi)^{2\sigma-1}\zeta(2-2\sigma)}{2\sin(\pi\sigma)\Gamma(2\sigma-1)},$$

it suffices to show

$$\frac{1}{\Delta\sqrt{\pi}}\,\mathrm{Re}\int_{-\infty}^{\infty}\frac{\Gamma(1-\sigma+i(t+y)}{\Gamma(\sigma+i(t+y))}e^{-(y/\Delta)^2}\,dy=t^{1-2\sigma}\sin(\pi\sigma)+O\left(\frac{\Delta}{t^{2\sigma}}\right). \tag{3.5}$$

To show this, we divide the integral as

$$\int_{-\infty}^{-\Delta L}+\int_{-\Delta L}^{\Delta L}+\int_{\Delta L}^{\infty}.$$

The first and the third integrals can be easily seen to be small. By using Stirling's formula, we have

$$\begin{aligned}\int_{-\Delta L}^{\Delta L}&=e^{\frac{1}{2}\pi i(1-2\sigma)}\int_{-\Delta L}^{\Delta L}t^{1-2\sigma}\left\{1+O\left(t^{-1}(|y|+1)\right)\right\}e^{-(y/\Delta)^2}dy\\&=e^{\frac{1}{2}\pi i(1-2\sigma)}t^{1-2\sigma}\Delta\left\{\sqrt{\pi}+O(e^{-L^2})+O(t^{-1}\Delta)\right\},\end{aligned}$$

hence (3.5) follows. Lemma 1 is proved. □

Next we consider the case on the critical line. Put $\sigma=\frac{1}{2}+\delta$ in (3.4), where δ is small. Then

$$M(2\sigma-1,\sigma+it;\Delta)=\frac{1}{2\delta}-B(t;\Delta)-\gamma+O(\delta), \tag{3.6}$$

where

$$B(t;\Delta)=\frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+i(t+y)\right)e^{-(y/\Delta)^2}dy. \tag{3.7}$$

Hence, taking the limit $\delta\to0$ in (3.4), with noting that

$$\zeta(1+2\delta)=\frac{1}{2\delta}+\gamma+O(\delta),$$

$$\zeta(2\delta)=-\frac{1}{2}-(\log 2\pi)\delta+O(\delta^2),$$

and

$$\prod_{p|q}(1-p^{-1-2\delta})=\prod_{p|q}\left(1-\frac{1}{p}\right)\left\{1+2\sum_{p|q}\frac{\log p}{p-1}\delta+O(\delta^2)\right\},$$

we obtain

$$(3.8)\qquad J_{\frac{1}{2}}(t;\Delta) = \frac{\varphi(q)}{q}\Big\{\log\frac{q}{2\pi}+2\gamma+2\sum_{p|q}\frac{\log p}{p-1} +\frac{1}{2}B(t;\Delta)+\frac{1}{2}B(-t;\Delta)\Big\}+P_{\frac{1}{2}}(t;\Delta)+\overline{P_{\frac{1}{2}}(t;\Delta)}.$$

We can show

$$(3.9)\qquad B(\pm t;\Delta)=\pm\frac{1}{2}\pi i+\log t+O\left(\frac{\Delta}{t}\right)$$

for $t\asymp T$. This can be proved similarly to (3.5) by using Stirling's formula, so we omit the details. Combining (3.8) and (3.9) we obtain

Lemma 2. *For $t\asymp T$ we have*

$$J_{\frac{1}{2}}(t;\Delta) = \frac{\varphi(q)}{q}\left(\log t+\log\frac{q}{2\pi}+2\gamma+2\sum_{p|q}\frac{\log p}{p-1}\right) +P_{\frac{1}{2}}(t;\Delta)+\overline{P_{\frac{1}{2}}(t;\Delta)}+O\left(\frac{\varphi(q)}{q}\frac{\Delta}{T}\right).$$

Remark 1. The error terms in (3.5) and (3.9) can be sharpened to $O(t^{-2\sigma}+t^{-1-2\sigma}\Delta^2)$ by refining the argument.

Remark 2. Here we consider the case

$$(3.10)\qquad A_0L^2>T.$$

Let $q=p$ be a prime as in our theorems. Heath-Brown's estimate [5]

$$(3.11)\qquad L(\tfrac{1}{2}+it,\chi)=O\{(p(|t|+1))^{\frac{3}{16}+\varepsilon}\},$$

and the convexity argument show that

$$(3.12)\qquad L(\sigma+it,\chi)=O\{(p(|t|+1))^{\frac{3}{8}(1-\sigma)+\varepsilon}\}\qquad(\tfrac{1}{2}\le\sigma<1).$$

From (3.12) and trivial estimation we obtain $E_\sigma(T,\chi)=O(T(pT)^{\frac{3}{4}(1-\sigma)+\varepsilon})$ for $\frac{1}{2}\le\sigma<1$, which supersedes (1.6) and (1.7) when

$$(3.13)\qquad p\ge c_0T^{(7-3\sigma)/(-1+3\sigma)+\eta}$$

where $\eta=\eta(\varepsilon)>0$ is arbitrarily small and $c_0>0$. If c_0 is small enough, then (3.10) implies (3.13), hence Theorems 1 and 2 follow in the case of (3.10). Note that the classical convexity bound (4.6) (below) is insufficient for the above argument.

4 The integral of $J_\sigma(t;\Delta)$

We first prove the following

Lemma 3. *For $\frac{1}{2} \le \sigma < 1$, we have*

$$\int_{T-L\Delta}^{2T+L\Delta} J_\sigma(t;\Delta)dt \ge \int_T^{2T} |L(\sigma+it,\chi)|^2 dt + O(e^{-L^2}T(qT)^{1-\sigma+\varepsilon}), \tag{4.1}$$

and

$$\int_{T+L\Delta}^{2T-L\Delta} J_\sigma(t;\Delta)dt \le \int_T^{2T} |L(\sigma+it,\chi)|^2 dt + O(e^{-L^2}T(qT)^{1-\sigma+\varepsilon}). \tag{4.2}$$

Proof: By the definition of $J_\sigma(t;\Delta)$ we have

$$\begin{aligned} &\int_{T\mp L\Delta}^{2T\pm L\Delta} J_\sigma(t;\Delta)dt \\ &= \frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} |L(\sigma+i\eta,\chi)|^2 \int_{T\mp L\Delta}^{2T\pm L\Delta} \exp\left(-\left(\frac{\eta-t}{\Delta}\right)^2\right) dtd\eta. \end{aligned} \tag{4.3}$$

Hence

$$\begin{aligned} &\int_{T-L\Delta}^{2T+L\Delta} J_\sigma(t;\Delta)dt \\ &\ge \frac{1}{\Delta\sqrt{\pi}} \int_{T}^{2T} |L(\sigma+i\eta,\chi)|^2 \int_{T-L\Delta}^{2T+L\Delta} \exp\left(-\left(\frac{\eta-t}{\Delta}\right)^2\right) dtd\eta. \end{aligned}$$

The inner integral is equal to

$$\Delta\sqrt{\pi} + O(\Delta e^{-L^2}), \tag{4.4}$$

hence

$$\begin{aligned} \int_{T-L\Delta}^{2T+L\Delta} J_\sigma(t;\Delta)dt &\ge \int_T^{2T} |L(\sigma+i\eta,\chi)|^2 d\eta \\ &\quad + O\left(e^{-L^2}\int_T^{2T} |L(\sigma+i\eta,\chi)|^2 d\eta\right). \end{aligned} \tag{4.5}$$

The convexity bound

$$L(\sigma+it,\chi) = O\left((q(|t|+1))^{\frac{1}{2}(1-\sigma)+\varepsilon}\right) \tag{4.6}$$

is valid for $0 \le \sigma \le 1$. (see Prachar [26]). Estimating the error term in (4.5) by (4.6), we obtain (4.1).

Next, from (4.3) we have

$$\begin{aligned} \int_{T+L\Delta}^{2T-L\Delta} J_\sigma(t;\Delta)dt &= \frac{1}{\Delta\sqrt{\pi}}\left(\int_{-\infty}^{T} + \int_T^{2T} + \int_{2T}^{\infty}\right) |L(\sigma+i\eta,\chi)|^2 \\ &\quad \times \int_{T+L\Delta}^{2T-L\Delta} \exp\left(-\left(\frac{\eta-t}{\Delta}\right)^2\right) dtd\eta \\ &= J_1 + J_2 + J_3, \end{aligned}$$

say. Consider J_1. Changing the order of integration and then putting $(t-\eta)/\Delta = w$, we have

$$\begin{aligned} J_1 &= \frac{1}{\sqrt{\pi}} \int_{T+L\Delta}^{2T-L\Delta} \int_{(t-T)/\Delta}^{\infty} |L(\sigma + i(t-\Delta w), \chi)|^2 e^{-w^2} dw dt \\ &\le \frac{1}{\sqrt{\pi}} \int_{T+L\Delta}^{2T-L\Delta} \int_{L}^{\infty} |L(\sigma + i(t-\Delta w), \chi)|^2 e^{-w^2} dw dt \\ &= \frac{1}{\sqrt{\pi}} \int_{L}^{\infty} e^{-w^2} \int_{T+L\Delta-w\Delta}^{2T-L\Delta-w\Delta} |L(\sigma+it, \chi)|^2 dt dw. \end{aligned}$$

Hence, using (4.6), we obtain

$$\begin{aligned} J_1 &\ll \int_{L}^{\infty} e^{-w^2} T \left(q(T+|w|\Delta)\right)^{1-\sigma+\varepsilon} dw \\ &\ll e^{-L^2} T (qT)^{1-\sigma+\varepsilon}, \end{aligned}$$

and the same estimate also holds for J_3. Lastly, we see that

$$J_2 \le \frac{1}{\Delta\sqrt{\pi}} \int_{T}^{2T} |L(\sigma+i\eta, \chi)|^2 \int_{T-L\Delta}^{2T+L\Delta} \exp\left(-\left(\frac{\eta - t}{\Delta}\right)^2\right) d\eta dt,$$

and the inner integral is equal to (4.4). These results complete the proof of (4.2), hence of Lemma 3.

Next we establish the connection between the integral of $J_\sigma(t;\Delta)$ and $E_\sigma(T,\chi)$ (or $E(T,\chi)$). In the case of $\frac{1}{2} < \sigma < 1$, from Lemma 1 we have

$$\begin{aligned} (4.7) \qquad & \int_{T-L\Delta}^{2T+L\Delta} J_\sigma(t;\Delta) dt \\ &= L(2\sigma,\chi_0)(T+2L\Delta) + \frac{L(2-2\sigma,\chi_0)}{2-2\sigma} \left(\frac{q}{2\pi}\right)^{1-2\sigma} t^{2-2\sigma}\Big|_{t=T-L\Delta}^{2T+L\Delta} \\ &\quad + \int_{T-L\Delta}^{2T+L\Delta} \left\{P_\sigma(t;\Delta) + \overline{P_\sigma(t;\Delta)}\right\} dt + O\left((qT)^{1-2\sigma}\Delta\right) \\ &= L(2\sigma,\chi_0)T + \frac{L(2-2\sigma,\chi_0)}{2-2\sigma} \left(\frac{q}{2\pi}\right)^{1-2\sigma} t^{2-2\sigma}\Big|_{t=T}^{2T} \\ &\quad + \int_{T-L\Delta}^{2T+L\Delta} \left\{P_\sigma(t;\Delta) + \overline{P_\sigma(t;\Delta)}\right\} dt + O(L\Delta). \end{aligned}$$

Comparing this with the definition (1.5) of $E_\sigma(T,\chi)$, we find that

$$\begin{aligned} (4.8) \qquad E_\sigma(2T,\chi) - E_\sigma(T,\chi) &= \int_T^{2T} |L(\sigma+it,\chi)|^2 dt \\ &- \int_{T-L\Delta}^{2T+L\Delta} J_\sigma(t;\Delta) dt + \int_{T-L\Delta}^{2T+L\Delta} \left\{P_\sigma(t;\Delta) + \overline{P_\sigma(t;\Delta)}\right\} dt + O(L\Delta). \end{aligned}$$

In much the same way we can also derive

$$\begin{aligned} (4.9) \qquad E_\sigma(2T,\chi) - E_\sigma(T,\chi) &= \int_T^{2T} |L(\sigma+it,\chi)|^2 dt \\ &- \int_{T+L\Delta}^{2T-L\Delta} J_\sigma(t;\Delta) dt + \int_{T+L\Delta}^{2T-L\Delta} \left\{P_\sigma(t;\Delta) + \overline{P_\sigma(t;\Delta)}\right\} dt + O(L\Delta). \end{aligned}$$

Combining (4.8) with (4.1), we obtain

$$(4.10)\qquad E_\sigma(2T,\chi)-E_\sigma(T,\chi) \le \int_{T-L\Delta}^{2T+L\Delta}\left\{P_\sigma(t;\Delta)+\overline{P_\sigma(t;\Delta)}\right\}dt +O(L\Delta).$$

Similarly, from (4.9) and (4.2) we obtain

$$(4.11)\qquad E_\sigma(2T,\chi)-E_\sigma(T,\chi) \ge \int_{T+L\Delta}^{2T-L\Delta}\left\{P_\sigma(t;\Delta)+\overline{P_\sigma(t;\Delta)}\right\}dt +O(L\Delta).$$

By (4.10) and (4.11) we finish the proof of the case $\frac{1}{2}<\sigma<1$ of the following lemma. The proof of the case $\sigma=\frac{1}{2}$ can be done quite similarly, by using (1.4) and Lemma 2.

Lemma 4. *For $\frac{1}{2}\le\sigma<1$ we have*

$$|E_\sigma(2T,\chi)-E_\sigma(T,\chi)| \le \left|\int_{T-L\Delta}^{2T+L\Delta}\left\{P_\sigma(t;\Delta)+\overline{P_\sigma(t;\Delta)}\right\}dt\right| + \left|\int_{T+L\Delta}^{2T-L\Delta}\left\{P_\sigma(t;\Delta)+\overline{P_\sigma(t;\Delta)}\right\}dt\right| + O\left(L^{1+2\omega}\Delta\right),$$

where $\omega=1$ or 0 according as $\sigma=\frac{1}{2}$ or $\frac{1}{2}<\sigma<1$.

5 An infinite series expression of $P_\sigma(t;\Delta)$

Now our problem is reduced to the evaluation of the integral of $P_\sigma(t;\Delta)$. In this section we derive a useful infinite series expression of $P_\sigma(t;\Delta)$. From (2.9) we have

$$P_\sigma(t;\Delta) = q^{-2\sigma}\sum_{a,b=1}^{q}\chi(a)\overline{\chi}(a+b)\frac{1}{2\pi i}\int_{(\beta)}M(s,\sigma-it;\Delta) \times\zeta(2\sigma-s,a/q)\zeta(s,b/q)ds$$

for $\sigma<\frac{1}{2}(\beta+1)$. Substituting the functional equation (cf. formula (2.17.3) of Titchmarsh [27])

$$\zeta(2\sigma-s,a/q) = \frac{\Gamma(1-2\sigma+s)}{(2\pi)^{1-2\sigma+s}}\Big\{e^{-\frac{1}{2}\pi i(1-2\sigma+s)}\sum_{m=1}^{\infty}e(ma/q)m^{2\sigma-1-s} +e^{\frac{1}{2}\pi i(1-2\sigma+s)}\sum_{m=1}^{\infty}e(-ma/q)m^{2\sigma-1-s}\Big\}$$

into the above, and noting

$$\zeta(s,b/q)\sum_{m=1}^{\infty}e(m\alpha)m^{2\sigma-1-s} = q^s\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n;\alpha,q,b)n^{2\sigma-1-s}$$

with

$$\sigma_{1-2\sigma}(n;\alpha,q,b)=\sum_{\substack{l|n\\ l\equiv b(\bmod q)}} e(n\alpha/l)l^{1-2\sigma},$$

we obtain

$$\begin{aligned}P_\sigma(t;\Delta)=q^{-2\sigma}\sum_{a,b=1}^{q}\chi(a)\overline{\chi}(a+b)\frac{1}{2\pi i}\int_{(\beta)}M(s,\sigma-it;\Delta)\\ \times\frac{\Gamma(1-2\sigma+s)}{(2\pi)^{1-2\sigma+s}}\Big\{e^{-\frac{1}{2}\pi i(1-2\sigma+s)}q^s\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n;a/q,q,b)n^{2\sigma-1-s}\\ +e^{\frac{1}{2}\pi i(1-2\sigma+s)}q^s\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n;-a/q,q,b)n^{2\sigma-1-s}\Big\}ds.\end{aligned}$$

Changing the summation and integration, we have the following absolutely convergent infinite series expression:

$$\begin{aligned}P_\sigma(t;\Delta) &= q^{-1}\sum_{a,b=1}^{q}\chi(a)\overline{\chi}(a+b)\sum_{n=1}^{\infty}\Big\{\sigma_{1-2\sigma}(n;a/q,q,b)Q_\sigma^+(t;n,q)\\ &\quad+\sigma_{1-2\sigma}(n;-a/q,q,b)Q_\sigma^-(t;n,q)\Big\}\\ &= \sum_{n=1}^{\infty}a_{1-2\sigma}(n,\chi)Q_\sigma^+(t;n,q)+\sum_{n=1}^{\infty}\overline{a_{1-2\sigma}(n,\overline{\chi})}Q_\sigma^-(t;n,q)\end{aligned}\tag{5.1}$$

for $\sigma<\frac{1}{2}(\beta+1)$, where

$$\begin{aligned}Q_\sigma^{\pm}(t;n,q) &= \frac{1}{2\pi i}\int_{(\beta)}M(s,\sigma-it;\Delta)\Gamma(1-2\sigma+s)\\ &\quad\times(e^{\pm\frac{1}{2}\pi i}q^{-1}2\pi n)^{2\sigma-1-s}ds\end{aligned}\tag{5.2}$$

and

$$a_{1-2\sigma}(n,\chi)=q^{-1}\sum_{a=1}^{q}\sum_{l|n}\chi(a)\overline{\chi}(l+a)e(an/ql)l^{1-2\sigma}.\tag{5.3}$$

Note that

$$|a_{1-2\sigma}(n,\chi)|\le\sigma_{1-2\sigma}(n)=O(n^\varepsilon)\tag{5.4}$$

for any $\varepsilon>0$.

Let $l(\pm\theta)$ be the half-line which starts from the origin and proceeds to the direction $e^{\pm i\theta}$ $(0<\theta<\frac{1}{2}\pi)$. It is easily seen that the path of the integral (2.5) may be rotated to $l(\pm\theta)$. After this rotation, we substitute the resulting expression of $M(s,\sigma-it;\Delta)$ into (5.2), and then change the order of integration by Fubini's theorem to obtain

$$\begin{aligned}Q_\sigma^{\pm}(t;n,q) &= \frac{1}{2\pi i}\int_{l(\pm\theta)}y^{2\sigma-2}(1+y)^{-\sigma+it}\exp\left(-\frac{1}{4}\Delta^2\log^2(1+y)\right)\\ &\quad\times\int_{(\beta)}\Gamma(1-2\sigma+s)\{e^{\pm\frac{1}{2}\pi i}(qy)^{-1}2\pi n\}^{2\sigma-1-s}dsdy.\end{aligned}\tag{5.5}$$

Applying the well-known formula

$$\frac{1}{2\pi i}\int_{(\beta)}\Gamma(s)X^{-s}ds = e^{-X} \qquad (\mathrm{Re}\, X > 0)$$

to the inner integral of (5.5), we obtain

$$\begin{aligned} Q_\sigma^{\pm}(t;n,q) &= \int_{l(\pm\theta)} y^{2\sigma-2}(1+y)^{-\sigma+it}\exp\left(-\frac{1}{4}\Delta^2\log^2(1+y)\right) \\ &\quad \times \exp\{-e^{\pm\frac{1}{2}\pi i}(qy)^{-1}2\pi n\}dy. \end{aligned} \tag{5.6}$$

6 The integral of $Q_\sigma^{\pm}(t;n,q)$

We use the abbreviations $T_1 = T \mp L\Delta$ and $T_2 = 2T \pm L\Delta$ respectively. We note that $T_1 \asymp T$ and $T_2 \asymp T$. Let $\frac{1}{2} \le \sigma < 1$. From (5.1) we have

$$\begin{aligned} \int_{T_1}^{T_2} P_\sigma(t;\Delta)dt &= \sum_{n=1}^{\infty} a_{1-2\sigma}(n,\chi)\int_{T_1}^{T_2} Q_\sigma^{+}(t;n,q)dt \\ &\quad + \sum_{n=1}^{\infty}\overline{a_{1-2\sigma}(n,\overline{\chi})}\int_{T_1}^{T_2} Q_\sigma^{-}(t;n,q)dt. \end{aligned} \tag{6.1}$$

We have to evaluate the integrals on the right-hand side. Substituting (5.6) into those integrals, and then using Fubini's theorem, we obtain

$$\begin{aligned} \int_{T_1}^{T_2} Q_\sigma^{\pm}(t;n,q)dt &= \int_{l(\pm\theta)} y^{2\sigma-2}(1+y)^{-\sigma}\exp\left(-\frac{1}{4}\Delta^2\log^2(1+y)\right) \\ &\quad \times \exp\{-e^{\pm\frac{1}{2}\pi i}(qy)^{-1}2\pi n\}\left.\frac{\exp\left(it\log(1+y)\right)}{i\log(1+y)}\right|_{t=T_1}^{T_2} dy. \end{aligned} \tag{6.2}$$

Now we claim that the path $l(\pm\theta)$ in the above can be rotated back to the positive real axis. To prove this claim, denote the integrand on the right-hand side by $W(y)$. Then

$$W(y) = i(T_2 - T_1)y^{2\sigma-2}\exp\{-e^{\pm\frac{1}{2}\pi i}(qy)^{-1}2\pi n\}\left(1 + O(y)\right)$$

for small y. Hence, putting $y = \varepsilon e^{\pm i\varphi}$, we find that

$$\begin{aligned} &\int_0^\theta W(\varepsilon e^{\pm i\varphi})\varepsilon e^{\pm i\varphi}d\varphi \\ &= i(T_2 - T_1)\int_0^\theta (\varepsilon e^{\pm i\varphi})^{2\sigma-1}\exp\left(\frac{2\pi n}{q\varepsilon}(-\sin\varphi \mp i\cos\varphi)\right)\left(1 + O(\varepsilon)\right)d\varphi \\ &\ll (T_2 - T_1)\int_0^\theta \left\{\varepsilon^{2\sigma-1}\exp\left(-\frac{2\pi n}{q\varepsilon}\sin\varphi\right) + \varepsilon^{2\sigma}\right\}d\varphi \\ &\ll (T_2 - T_1)\varepsilon^{2\sigma}, \end{aligned}$$

which vanishes when ε tends to zero. This implies the above claim, hence we conclude that

$$\int_{T_1}^{T_2} Q_\sigma^{\pm}(t;n,q)dt = \frac{1}{i}\int_0^\infty \frac{\exp\left(-\frac{1}{4}\Delta^2\log^2(1+y)\right)}{y^{2-2\sigma}(1+y)^\sigma\log(1+y)} \tag{6.3}$$

$$\times \left\{ e\left(\frac{T_2}{2\pi}\log(1+y)\right) - e\left(\frac{T_1}{2\pi}\log(1+y)\right)\right\} e(\mp n/qy)dy$$

$$= \frac{1}{i}\int_0^\infty \frac{\exp\left(-\frac{1}{4}\Delta^2\log^2(1+\frac{1}{y})\right)}{y^\sigma(1+y)^\sigma\log\left(1+\frac{1}{y}\right)}$$

$$\times \left\{ e\left(\frac{T_2}{2\pi}\log\left(1+\frac{1}{y}\right)\right) - e\left(\frac{T_1}{2\pi}\log\left(1+\frac{1}{y}\right)\right)\right\} e(\mp ny/q)dy.$$

We may write

$$\int_{T_1}^{T_2} Q_\sigma^{\pm}(t;n,q)dt = \frac{1}{i}\lim_{Y\to\infty}\left\{\int_0^Y g_\sigma(y)e\left(f(y;T_2)\mp ny/q\right)dy - \int_0^Y g_\sigma(y)e\left(f(y;T_1)\mp ny/q\right)dy\right\}, \tag{6.4}$$

where

$$g_\sigma(y) = \frac{\exp\left(-\frac{1}{4}\Delta^2\log^2(1+\frac{1}{y})\right)}{y^\sigma(1+y)^\sigma\log(1+\frac{1}{y})}$$

and

$$f(y;T) = \frac{T}{2\pi}\log\left(1+\frac{1}{y}\right).$$

7 Application of Atkinson's saddle-point lemma

In this section we estimate the integrals on the right-hand side of (6.4) for large n, by applying the saddle-point lemma of Atkinson [1]. First we quote the lemma of Atkinson (Lemma 1 of [1]; see also Theorem 2.2 of Ivić [7]). In what follows, A denotes a positive constant, not necessarily the same on each occurrence.

Lemma 5 (Atkinson). *Let $f(z)$, $g(z)$ be two functions of the complex variable z, and $[a,b]$ a real interval, such that*

(i) $f(x)$ is real and $f''(x)>0$ for $a\le x\le b$;

(ii) there exists a positive differentiable function $\mu(x)$ defined on $[a,b]$, such that $f(z)$, $g(z)$ are analytic in the region $D=\{z \mid |z-x|\le\mu(x),\ a\le x\le b\}$;

(iii) there exist positive functions $F(x)$, $G(x)$ defined on $[a,b]$ such that $g(z)=O(G(x))$, $f'(z)=O(F(x)\mu^{-1}(x))$ and $f''(z)^{-1}=O(\mu^2(x)F^{-1}(x))$ on D, where the implied constants are absolute.

Let c be any real number, and if $f'(x)+c$ has a zero in $[a,b]$ denote it by x_0. Let the values of $f(x)$, $g(x)$, etc., at a, x_0, b be characterized by the suffixes a, 0 and b, respectively. Then

$$\int_a^b g(x)e\left(f(x)+cx\right)dx = g_0(f_0'')^{-\frac{1}{2}}e\left(f_0+cx_0+\frac{1}{8}\right) + O\left(\int_a^b G(x)\exp\left(-A|c|\mu(x)-AF(x)\right)(dx+|d\mu(x)|)\right) + O\left(G_0\mu_0F_0^{-\frac{3}{2}}\right) + O\left(\frac{G_a}{|f_a'+c|+(f_a'')^{\frac{1}{2}}}\right) + O\left(\frac{G_b}{|f_b'+c|+(f_b'')^{\frac{1}{2}}}\right).$$

If $f'(x)+c$ has no zero in $[a,b]$ then the terms involving x_0 are to be omitted.

Atkinson applied this lemma to the integral

$$\int_a^b y^{-\alpha}(1+y)^{-\beta}\left(\log\left(1+\frac{1}{y}\right)\right)^{-\gamma} e\left(\frac{T}{2\pi}\log\left(1+\frac{1}{y}\right)+cy\right)dy, \tag{7.1}$$

where $\alpha(\neq 1)$, β and γ are positive, and obtained an asymptotic formula for (7.1) (Lemma 2 of Atkinson [1]). Save for the factor $\exp(-\frac{1}{4}\Delta^2\log^2(1+\frac{1}{y}))$, our integral is the same as (7.1), therefore we can apply a modification of Atkinson's argument to our case. We use Lemma 5 with $f(z)=f(z;T)$, $g(z)=g_\sigma(z)$, which are defined at the end of Section 6, $[a,b]=[\eta,Y]$ where η is small positive and Y is large positive, $c=\mp n/q$, $\mu(x)=\frac{1}{4}x$, $F(x)=T/(1+x)$ and

$$G(x)=x^{-\sigma}(1+x)^{1-\sigma}\exp\left(-\frac{1}{20}\Delta^2\log^2\left(1+\frac{1}{x}\right)\right).$$

The conditions (i)–(iii) in Lemma 5 are clearly valid except for the inequality $g(z)=O(G(x))$. To verify the last inequality, we notice that

$$\left(\arg\left(1+\frac{1}{z}\right)\right)^2 \le \frac{1}{5}\left(\log\left|1+\frac{1}{z}\right|\right)^2 \tag{7.2}$$

and

$$\log^2\left(1+\frac{1}{x}\right) \le 4\left(\log\left|1+\frac{1}{z}\right|\right)^2 \tag{7.3}$$

for any $z\in D$ and $a\le x\le b$ with $|z-x|\le\mu(x)$. To show these inequalities, put $z=x_1+iy_1$. Then

$$\log\left|1+\frac{1}{z}\right| = \frac{1}{2}\log\left(1+\frac{1+2x_1}{|z|^2}\right). \tag{7.4}$$

Note that $|y_1|\le\frac{1}{4}x$ and $\frac{3}{4}x\le x_1\le\frac{5}{4}x$. From (7.4) and

$$\frac{1+2x_1}{|z|^2} \ge \frac{2x_1}{x_1^2+(x/4)^2} = 2\left(x_1+\frac{x^2}{16x_1}\right)^{-1} \ge \frac{3}{2x} > \frac{1}{x},$$

(7.3) follows. Let $c_1=(e^{\sqrt{5}\pi}-1)^{-\frac{1}{2}}$. If $|z|\le c_1$, then (7.2) follows from

$$\begin{aligned}\log\left|1+\frac{1}{z}\right| &\ge \frac{1}{2}\log\left(1+\frac{1}{|z|^2}\right) \ge \frac{1}{2}\log\left(1+\frac{1}{c_1^2}\right) \ge \sqrt{5}\cdot\frac{\pi}{2}\\ &> \sqrt{5}\left|\arg\left(1+\frac{1}{z}\right)\right|,\end{aligned}$$

where the first inequality comes from (7.4). If $|z|>c_1$, then

$$\frac{1+2x_1}{|z|^2} \le \frac{1+2|z|}{|z|^2} < \frac{1+2c_1}{c_1^2}(=0.06056\ldots),$$

so

$$\log\left(1+\frac{1+2x_1}{|z|^2}\right) \ge c_2\cdot\frac{1+2x_1}{|z|^2} > \frac{2c_2x_1}{|z|^2}$$

with $c_2 = c_1^{-1}\log(1+c_1) = 0.97088\ldots$. Hence we see

$$\begin{aligned}\left|\arg\left(1+\frac{1}{z}\right)\right| &= \left|\arctan\left(\frac{y_1}{|z|^2+x_1}\right)\right| \le \frac{|y_1|}{|z|^2+x_1} < \frac{|y_1|}{|z|^2} \le \frac{x_1}{3|z|^2}\\ &< \frac{1}{3c_2}\log\left|1+\frac{1}{z}\right|,\end{aligned}$$

which implies (7.2). By using (7.2) and (7.3), we have

$$\begin{aligned}\left|\exp\left(-\frac{1}{4}\Delta^2\log^2\left(1+\frac{1}{z}\right)\right)\right| &= \exp\left\{-\frac{1}{4}\Delta^2\left(\log\left|1+\frac{1}{z}\right|\right)^2\right.\\ &\qquad\left.+\frac{1}{4}\Delta^2\left(\arg\left(1+\frac{1}{z}\right)\right)^2\right\}\\ &\le \exp\left(-\frac{1}{5}\Delta^2\left(\log\left|1+\frac{1}{z}\right|\right)^2\right)\\ &\le \exp\left(-\frac{1}{20}\Delta^2\log^2\left(1+\frac{1}{x}\right)\right),\end{aligned}$$

hence the desired inequality $g(z) = O(G(x))$. Therefore we can use Lemma 5. Similarly to Lemma 2 of Atkinson [1] (see also Lemma 15.1 of Ivić [7]), we can show the following

Lemma 6. *For $\frac{1}{2} \le \sigma < 1$ we have*

(7.5)
$$\begin{aligned}&\int_\eta^Y g_\sigma(y)e\left(f(y;T)+cy\right)dy\\ &\quad= \delta(c)\Lambda(T;c) + O\left(\int_\eta^Y G(y)\exp\left(-A|c|y - \frac{AT}{1+y}\right)dy\right)\\ &\qquad+O(\eta^{1-\sigma}T^{-1}) + O(Y^{1-2\sigma}|c|^{-1}) + \delta(c)R(T;c),\end{aligned}$$

where $\delta(c) = 1$ or 0 according as $c > 0$ or $c < 0$,

$$\Lambda(T;c) = \frac{T^{\frac{1}{2}}\exp(-\frac{1}{4}\Delta^2V^2)}{2c\sqrt{\pi}VU^{\frac{1}{2}}(U^2-1/4)^\sigma}e\left(\frac{TV}{2\pi} + cU - \frac{1}{2}c + \frac{1}{8}\pi\right),$$

$$U = U(T) = \left(\frac{T}{2\pi c} + \frac{1}{4}\right)^{\frac{1}{2}},$$

$$V = V(T) = 2\,\mathrm{arsinh}\sqrt{\frac{\pi c}{2T}},$$

(7.6)
$$R(T;c) \ll \begin{cases} T^{\frac{1}{4}-\sigma}c^{\sigma-\frac{7}{4}}\exp(-A\Delta^2cT^{-1}) & \text{if } c \ll T,\\ T^{-\frac{1}{2}-\sigma}c^{\sigma-1}\exp\left(-\frac{1}{20}\Delta^2\log^2\left(1+\frac{Ac}{T}\right)\right) & \text{if } c \gg T,\end{cases}$$

and the implied constants in (7.5) and (7.6) are uniform for $|\sigma - 1| > \varepsilon$.

First assume $\frac{1}{2} < \sigma < 1$, and take the limit $\eta \to 0$, $Y \to \infty$. Then the second and the third error terms in (7.5) vanish. After this procedure, we take the limit $\sigma \to \frac{1}{2} + 0$ to get the result for $\sigma = \frac{1}{2}$. Consequently, from (6.4), we obtain

$$\begin{aligned} \int_{T_1}^{T_2} Q_\sigma^\pm(t; n, q)dt &= \frac{1}{i}\delta\left(\mp\frac{n}{q}\right)\left\{\Lambda\left(T_2; \mp\frac{n}{q}\right) - \Lambda\left(T_1; \mp\frac{n}{q}\right)\right\} \\ &+O\left(\int_0^\infty G(y)\exp\left(-A\frac{ny}{q} - \frac{AT_2}{1+y}\right)dy\right) \\ &+O\left(\int_0^\infty G(y)\exp\left(-A\frac{ny}{q} - \frac{AT_1}{1+y}\right)dy\right) \\ &+\frac{1}{i}\delta\left(\mp\frac{n}{q}\right)\left\{R\left(T_2; \mp\frac{n}{q}\right) - R\left(T_1; \mp\frac{n}{q}\right)\right\} \end{aligned} \tag{7.7}$$

for $\frac{1}{2} \le \sigma < 1$.

Let $N \gg qT\Delta^{-2}L^2$, and now we estimate the subsum $\sum_{n>N}$ on the right-hand side of (6.1) by using (7.7). In view of (5.4), it is enough to estimate the sum

$$\begin{aligned} &\sum_{n>N} \sigma_{1-2\sigma}(n)\delta\left(\mp\frac{n}{q}\right)\left|\Lambda\left(T_j; \mp\frac{n}{q}\right)\right| \\ &\quad + \sum_{n>N} \sigma_{1-2\sigma}(n)\int_0^\infty G(y)\exp\left(-\frac{An}{q}y - \frac{AT_j}{1+y}\right)dy \\ &\quad + \sum_{n>N} \sigma_{1-2\sigma}(n)\delta\left(\mp\frac{n}{q}\right)\left|R\left(T_j; \mp\frac{n}{q}\right)\right| \\ &= \Sigma_1(N) + \Sigma_2(N) + \Sigma_3(N), \end{aligned} \tag{7.8}$$

say, for $j = 1, 2$.

Lemma 7. *The above quantity (7.8) is estimated as $O(q^{1+\sigma+\varepsilon}e^{-AT} + (qT)^{-c})$, where c is a large positive constant.*

Proof: First consider $\Sigma_1(N)$. Since

$$U \asymp \left(\frac{qT}{n}\right)^{\frac{1}{2}}, \qquad V \ge A\left(\frac{\pi n}{2qT}\right)^{\frac{1}{2}}$$

for $n \le qT_j$, the contribution of the part $N < n \le qT_j$ to $\Sigma_1(N)$ is

$$\begin{aligned} &\ll \sum_{N<n\le qT_j} q^{\frac{1}{2}}(qT)^{\frac{3}{4}-\sigma}n^{-\frac{5}{4}+\sigma+\varepsilon}\exp\left(-\frac{A\Delta^2 n}{qT}\right) \\ &\le q^{\frac{1}{2}}(qT)^{\frac{3}{4}-\sigma}\exp\left(-\frac{A\Delta^2 N}{qT}\right)\sum_{N<n\le qT_j} n^{-\frac{5}{4}+\sigma+\varepsilon} \\ &\ll q^{\frac{1}{2}}(qT)^{\frac{1}{2}+\varepsilon}e^{-AL^2}. \end{aligned} \tag{7.9}$$

Also, noting

$$U \asymp 1, \qquad V \ge \log(\pi n/2qT_j) \gg 1$$

for $n > qT_j$, we find that the contribution of the remaining part is

$$\begin{aligned}
(7.10) \qquad & \ll \sum_{n>qT_j} q^{\frac{1}{2}}(qT)^{\frac{1}{2}-\sigma} n^{\sigma-1+\varepsilon} \exp\left(-\frac{1}{4}\Delta^2\left(\log\frac{\pi n}{2qT_j}\right)^2\right) \\
& \ll q^{\frac{1}{2}}(qT)^{\frac{1}{2}-\sigma}\int_{qT_j-1}^{\infty} x^{\sigma-1+\varepsilon}\exp\left(-\frac{1}{4}\Delta^2\left(\log\frac{\pi x}{2qT_j}\right)^2\right)dx \\
& \ll q^{\frac{1}{2}}(qT)^{\frac{1}{2}+\varepsilon}\exp(-A\Delta^2) \ll q^{\frac{1}{2}}(qT)^{\frac{1}{2}+\varepsilon}\exp(-AL^2).
\end{aligned}$$

Here the last inequality follows by using (3.1). From (7.9) and (7.10) we obtain

$$\Sigma_1(N) = O((qT)^{-c}) \tag{7.11}$$

if A_0 is large enough.

Next consider $\Sigma_2(N)$. We have

$$\begin{aligned}
&\Sigma_2(N) \\
&\ll \sum_{n>N}\sigma_{1-2\sigma}(n)\int_0^1 y^{-\sigma}\exp\left(-\frac{1}{20}\Delta^2\log^2\left(1+\frac{1}{y}\right)-\frac{Any}{q}-AT\right)dy \\
&\quad + \sum_{n>N}\sigma_{1-2\sigma}(n)\int_1^{\infty} y^{1-2\sigma}\exp\left(-\frac{1}{20}\Delta^2\log^2\left(1+\frac{1}{y}\right)-\frac{Any}{q}-\frac{AT}{1+y}\right)dy \\
&= \Sigma_{21}(N)+\Sigma_{22}(N),
\end{aligned}$$

say. We put $ny = \eta$ in the first integral and get

$$\begin{aligned}
\Sigma_{21}(N) \ll\ & e^{-AT}\sum_{n>N}\sigma_{1-2\sigma}(n)n^{\sigma-1} \\
& \times\left(\int_0^{\sqrt{n}}+\int_{\sqrt{n}}^{n}\right)\eta^{-\sigma}\exp\left(-\frac{1}{20}\Delta^2\log^2\left(1+\frac{n}{\eta}\right)-\frac{A\eta}{q}\right)d\eta.
\end{aligned}$$

We see that

$$\begin{aligned}
\int_0^{\sqrt{n}} &\le \exp\left(-\frac{1}{20}\Delta^2\log^2(1+\sqrt{n})\right)\int_0^{\sqrt{n}}\eta^{-\sigma}e^{-A\eta/q}d\eta \\
&\ll q^{1-\sigma}\exp\left(-\frac{1}{20}\Delta^2\log^2(1+\sqrt{n})\right)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\sqrt{n}}^{n} &\le \exp\left(-\frac{A\sqrt{n}}{2q}\right)\int_{\sqrt{n}}^{n}\eta^{-\sigma}\exp\left(-\frac{A\eta}{2q}\right)d\eta \\
&\le q^{1-\sigma}\exp\left(-\frac{A\sqrt{n}}{2q}\right),
\end{aligned}$$

and the contribution of the latter integral to $\Sigma_{21}(N)$ can be estimated by the same manner as in (7.10). Hence we have

$$\Sigma_{21}(N) = O(q^{1+\sigma+\varepsilon}e^{-AT}). \tag{7.12}$$

As for $\sum_{22}(N)$, we split the integral at $y = TL^{-2}$. The contribution of the integral on the interval $[1, TL^{-2}]$ to $\sum_{22}(N)$ is

$$\begin{aligned} &\ll \sum_{n>N} \sigma_{1-2\sigma}(n) TL^{-2} \exp\left(-\frac{An}{q} - \frac{AT}{1+TL^{-2}}\right) \\ &\ll TL^{-2} e^{-AL^2} \sum_{n>N} \sigma_{1-2\sigma}(n) e^{-An/q} \\ &\ll q^{1+\varepsilon} TL^{-2} e^{-AL^2}. \end{aligned}$$

By using the estimate

$$\int_X^\infty u^{-\alpha} e^{-u} du \le X^{-\alpha} e^{-X} \qquad (X > 0,\ \alpha \ge 0),$$

we find that the contribution of the remaining integral is

$$\begin{aligned} &\ll \sum_{n>N} \sigma_{1-2\sigma}(n) \int_{TL^{-2}}^\infty y^{1-2\sigma} e^{-Any/q} dy \\ &\ll q(TL^{-2})^{1-2\sigma} \sum_{n>N} n^{-1+\varepsilon} \exp\left(-\frac{AnT}{qL^2}\right) \\ &\ll q^2 (TL^{-2})^{-2\sigma} N^{-1+\varepsilon} \exp\left(-\frac{ANT}{qL^2}\right) \\ &\ll q^2 (TL^{-2})^{-2\sigma} (qT\Delta^{-2}L^2)^{-1+\varepsilon} \exp(-AL^2). \end{aligned}$$

Here the last inequality follows by using (3.1). Hence we have

$$\Sigma_{22}(N) = O((qT)^{-c}) \tag{7.13}$$

if A_0 is large enough.

Lastly consider $\Sigma_3(N)$. Using (7.6), we find that the contribution of the part $N < n \le qT_j$ to $\Sigma_3(N)$ is

$$\begin{aligned} &\ll q^{\frac{3}{2}} (qT)^{\frac{1}{4}-\sigma} \exp\left(-A\Delta^2 \cdot \frac{N}{qT_j}\right) \sum_{N<n\le qT_j} n^{\sigma-\frac{7}{4}+\varepsilon} \\ &\ll q^{\frac{3}{2}} e^{-AL^2}. \end{aligned}$$

The contribution of the remaining part is

$$\begin{aligned} &\ll q^{\frac{3}{2}} (qT)^{-\frac{1}{2}-\sigma} \sum_{n>qT_j} n^{\sigma-1+\varepsilon} \exp\left(-\frac{1}{20}\Delta^2 \log^2\left(1+\frac{An}{qT}\right)\right) \\ &\ll q^{\frac{3}{2}} (qT)^{-\frac{1}{2}+\varepsilon} \exp(-A\Delta^2) \\ &\ll q^{\frac{3}{2}} \exp(-AL^2), \end{aligned}$$

again by using (3.1). Hence we obtain $\Sigma_3(N) = O((qT)^{-c})$ if A_0 is large enough. Combining this with (7.11)–(7.13), we complete the proof of Lemma 7. □

8 Estimates for the integrals of $Q^{\pm}_\sigma(t;n,q)$

We set

$$\Sigma_{\sigma,N}(T_1,T_2;\chi) = \sum_{n\le N} a_{1-2\sigma}(n,\chi)\int_{T_1}^{T_2} Q^{+}_\sigma(t;n,q)dt + \sum_{n\le N}\overline{a_{1-2\sigma}(n,\overline{\chi})}\int_{T_1}^{T_2} Q^{-}_\sigma(t;n,q)dt, \tag{8.1}$$

which is a truncation of (6.1), and define

$$h^{\pm}_\sigma(y;n) = g_\sigma(y)\{e(f(y;\pm T_2)) - e(f(y;\pm T_1))\}e(-ny/q).$$

Then we see by (6.3) that

$$\Sigma_{\sigma,N}(T_1,T_2;\chi) = \frac{1}{i}\sum_{n\le N} a_{1-2\sigma}(n,\chi)\int_0^\infty h^{+}_\sigma(y;n)dy + \frac{1}{i}\sum_{n\le N}\overline{a_{1-2\sigma}(n,\overline{\chi})}\int_0^\infty \overline{h^{-}_\sigma(y;n)}dy. \tag{8.2}$$

The purpose of this section is to give an upper-bound estimate for the right-hand side of (8.2).

We first show

Lemma 8. *The estimates*

$$\sum_{n\le N} a_{1-2\sigma}(n,\chi)\int_0^{L^{-1}\Delta} h^{+}_\sigma(y;n)dy = O(e^{-AL^2}NT(\log N\log T)^\omega) \tag{8.3}$$

and

$$\sum_{n\le N}\overline{a_{1-2\sigma}(n,\overline{\chi})}\int_0^{L^{-1}\Delta}\overline{h^{-}_\sigma(y,n)}dy = O(e^{-AL^2}NT(\log N\log T)^\omega) \tag{8.4}$$

hold, where A is a positive absolute constant and ω is defined in the statement of Lemma 4.

Proof: Note that $L^{-1}\Delta \ge 1$ by (3.1). Then using the inequalities

$$\exp\left(-\frac{1}{4}\Delta^2\log^2\left(1+\frac{1}{y}\right)\right) \ll e^{-AL^2} \qquad (0<y\le L^{-1}\Delta),$$

$$e(f(y;T_2)) - e(f(y;T_1)) \ll y^{-1}T \qquad (y\ge 1),$$

and (5.4), we find that the left-hand side of (8.3) can be estimated as

$$\begin{aligned} &\ll \sum_{n\le N}\sigma_{1-2\sigma}(n)\left\{\int_0^1 \frac{e^{-AL^2}}{y^\sigma(|\log y|+1)}dy + \int_0^{L^{-1}\Delta}\frac{e^{-AL^2}T}{y^{2\sigma}}dy\right\} \\ &\ll e^{-AL^2}TN(\log T\log N)^\omega. \end{aligned}$$

Here the last inequality follows from

$$\sum_{n\le x}\sigma_{1-2\sigma}(n) \ll x(\log x)^\omega \qquad (\sigma\ge \tfrac{1}{2}). \tag{8.5}$$

Hence we obtain (8.3). The estimate (8.4) can be derived by the same manner. □

We next estimate the integrals over the interval $[L^{-1}\Delta, \infty)$. Defining

$$\varphi_\sigma^\pm(y;n) = \int_{L^{-1}\Delta}^{y} \frac{e(-n\eta/q)}{(\eta(\eta+1))^{\sigma-\frac{1}{2}}}\{e(f(\eta;\pm T_2)) - e(f(\eta;\pm T_1))\}d\eta,$$

we have, by integration by parts,

$$\int_{L^{-1}\Delta}^{\infty} h_\sigma^\pm(y;n)dy = \varphi_\sigma^\pm(\infty;n) - \int_{L^{-1}\Delta}^{\infty} \varphi_\sigma^\pm(y;n)g'_{\frac{1}{2}}(y)dy \tag{8.6}$$

with

$$\varphi_\sigma^\pm(\infty;n) = \lim_{y\to\infty} \varphi_\sigma^\pm(y;n).$$

Here the existence of the limit is ensured by the fact

$$e(f(\eta;\pm T_2)) - e(f(\eta;\pm T_1)) = \pm i(T_2 - T_1)\frac{1}{\eta} + O\left(\frac{T^2}{\eta^2}\right) \qquad (\eta \to +\infty)$$

and the second mean value theorem.

To estimate the second term on the right-hand side of (8.6) we first prove, for $y \geq L^{-1}\Delta$,

$$\varphi_\sigma^\pm(y;n) \ll \begin{cases} (L^{-1}\Delta)^{1-2\sigma} y^{\frac{3}{2}} T^{-\frac{1}{2}} & \text{if } n \leq qT_1L^2\Delta^{-2}, \\ (L^{-1}\Delta)^{1-2\sigma}\left(\dfrac{q}{n} + y^{\frac{3}{2}}T^{-\frac{1}{2}}\right) & \text{if } qT_1L^2\Delta^{-2} < n \leq qT_2L^2\Delta^{-2}, \\ (L^{-1}\Delta)^{1-2\sigma}\dfrac{q}{n} & \text{if } n > qT_2L^2\Delta^{-2}. \end{cases} \tag{8.7}$$

Setting

$$\Phi_\pm(\eta;T_j) = -\frac{2\pi n}{q}\eta \pm T_j \log\left(1 + \frac{1}{\eta}\right)$$

and

$$\Psi(\eta) = (\eta(\eta+1))^{\frac{1}{2}-\sigma},$$

we have

$$\begin{aligned} \varphi_\sigma^\pm(y;n) &= \int_{L^{-1}\Delta}^{y} \Psi(\eta)\exp(i\Phi_\pm(\eta;T_2))d\eta \\ &\quad - \int_{L^{-1}\Delta}^{y} \Psi(\eta)\exp(i\Phi_\pm(\eta;T_1))d\eta. \end{aligned} \tag{8.8}$$

The inequalities

$$|\Phi'_+(\eta;T_j)| \gg \frac{n}{q} \qquad (j = 1, 2)$$

hold trivially for $\eta > 0$ and $n \geq 1$. Hence by the first derivative test (cf. (2.3) of Ivić [7]) we have

$$\int_{L^{-1}\Delta}^{y} \Psi(\eta)\exp(i\Phi_+(\eta;T_j))d\eta \ll (L^{-1}\Delta)^{1-2\sigma}\frac{q}{n}. \tag{8.9}$$

Let η_j $(j = 1, 2)$ be the unique positive zero of $\Phi_-(\eta;T_j)$. If $n > qT_2L^2\Delta^{-2}$, then

$$\Phi'_-(L^{-1}\Delta;T_j) = -\frac{2\pi n}{q} + \frac{T_j}{L^{-1}\Delta(L^{-1}\Delta+1)} < 0,$$

and so $\eta_j < L^{-1}\Delta$. Hence we have, for $\eta \in [L^{-1}\Delta, y]$,

$$|\Phi'_-(\eta; T_j)| \geq |\Phi'_-(L^{-1}\Delta; T_j)| \gg \frac{n}{q}.$$

Therefore the same estimate as (8.9) holds for the integral involving $\Phi_-(\eta; T_j)$ instead of $\Phi_+(\eta; T_j)$. The case $n > qT_2L^2\Delta^{-2}$ of (8.7) now follows. Next, to treat the case $n \leq qT_1L^2\Delta^{-2}$, we apply the second derivative test (cf. (2.5) of Ivić [7]). Noting that

$$\Phi''_\pm(\eta; T_j) = \mp 2T_j\left(\frac{1}{\eta^3} - \frac{1}{(\eta+1)^3}\right)$$

are monotonic for $\eta > 0$, we have

$$|\Phi''_\pm(\eta; T_j)| \geq |\Phi''_\pm(y; T_j)| \gg Ty^{-3}$$

for $\eta \in [L^{-1}\Delta, y]$. Hence the case $n \leq qT_1L^2\Delta^{-2}$ of (8.7) follows. The result for the remaining case can be derived by combining the estimates for the other cases.

It is easily seen that

$$g'_{\frac{1}{2}}(y) \ll \Delta^2 y^{-3} \qquad (y \geq 1). \tag{8.10}$$

Using the bounds (8.7) and (8.10), we find that

$$\begin{aligned}
&\int_{L^{-1}\Delta}^{\infty} \varphi_\sigma^\pm(y; n) g'_{\frac{1}{2}}(y)dy \\
&\ll \begin{cases} \Delta^2(L^{-1}\Delta)^{\frac{1}{2}-2\sigma}T^{-\frac{1}{2}} & \text{if } n \leq qT_1L^2\Delta^{-2}, \\ \Delta^2(L^{-1}\Delta)^{-1-2\sigma}\dfrac{q}{n} & \\ \quad +\Delta^2(L^{-1}\Delta)^{\frac{1}{2}-2\sigma}T^{-\frac{1}{2}} & \text{if } qT_1L^2\Delta^{-2} < n \leq qT_2L^2\Delta^{-2}, \\ \Delta^2(L^{-1}\Delta)^{-1-2\sigma}\dfrac{q}{n} & \text{if } n \geq qT_2L^2\Delta^{-2}. \end{cases}
\end{aligned} \tag{8.11}$$

The treatment of $\varphi_\sigma^\pm(\infty; n)$ on the right-hand side of (8.6) is more involved. We show the estimate

$$\begin{aligned}
&\varphi_\sigma^\pm(\infty; n) \\
&\ll \begin{cases} (L^{-1}\Delta)^{1-2\sigma}\dfrac{q}{n}\left(1 + \left(\dfrac{nT}{q}\right)^{\frac{1}{4}}\right) & \text{if } n \leq qT_2L^2\Delta^{-2}, \\ (L^{-1}\Delta)^{1-2\sigma}\dfrac{q}{n} & \text{if } n > qT_2L^2\Delta^{-2}. \end{cases}
\end{aligned} \tag{8.12}$$

If $n > qT_2L^2\Delta^{-2}$, then the result immediately follows from (8.7). Suppose next that $n \leq qT_1L^2\Delta^{-2}$. Let δ be a positive parameter which will be specified later, and we temporarily assume that

$$L^{-1}\Delta < \eta_j - \delta \qquad (j = 1, 2). \tag{8.13}$$

We divide

$$\begin{aligned}
\int_{L^{-1}\Delta}^{y} \Psi(\eta)\exp(i\Phi_-(\eta; T_j))d\eta &= \int_{L^{-1}\Delta}^{\eta_j-\delta} + \int_{\eta_j-\delta}^{\eta_j+\delta} + \int_{\eta_j+\delta}^{y} \\
&= K_1 + K_2 + K_3,
\end{aligned} \tag{8.14}$$

say. We see that

$$\Phi_-'(\eta_j \pm \delta; T_j) = -\frac{T_j\delta(\delta \pm (2\eta_j+1))}{\eta_j(\eta_j+1)(\eta_j \pm \delta)(\eta_j \pm \delta + 1)}. \tag{8.15}$$

Hence if $\delta \le \frac{1}{2}\eta_j$ then

$$|\Phi_-'(\eta_j \pm \delta; T_j)| \gg \eta_j^{-3} T_j \delta,$$

so by the first derivative test we have

$$K_1,\ K_3 \ll (L^{-1}\Delta)^{1-2\sigma}\eta_j^3(\delta T_j)^{-1}.$$

Clearly $K_2 \ll (L^{-1}\Delta)^{1-2\sigma}\delta$ holds. Therefore if we can choose $\delta = \eta_j^{\frac{3}{2}} T_j^{-\frac{1}{2}}$, noting that $\eta_j \asymp (qT_j/n)^{\frac{1}{2}}$, we obtain

$$\int_{L^{-1}\Delta}^{y} \Psi(\eta)\exp(i\Phi_-(\eta;T_j))d\eta \ll (L^{-1}\Delta)^{1-2\sigma}\frac{q}{n}\left(\frac{nT}{q}\right)^{\frac{1}{4}}.$$

From this estimate and (8.9), the case $n \le qT_1L^2\Delta^{-2}$ of (8.12) follows under the assumption $\delta \le \frac{1}{2}\eta_j$. The choice $\delta = \eta_j^{\frac{3}{2}} T_j^{-\frac{1}{2}}$ agrees with this assumption only if $\eta_j \le \frac{1}{4}T_j$. If $\eta_j > \frac{1}{4}T_j$, we choose $\delta = \frac{1}{2}\eta_j$. Then from (8.15) we have

$$|\Phi_-'(\eta_j + \delta; T_j)| \gg \eta_j^{-2} T_j,$$

so that

$$\begin{aligned}\int_{L^{-1}\Delta}^{y} \Psi(\eta)\exp(i\Phi_-(\eta;T_j))d\eta &= \int_{L^{-1}\Delta}^{\eta_j+\delta} + \int_{\eta_j+\delta}^{y} \\ &\ll (L^{-1}\Delta)^{1-2\sigma}\eta_j + (L^{-1}\Delta)^{1-2\sigma}\eta_j^2 T_j^{-1}.\end{aligned} \tag{8.16}$$

Using $\eta_j \asymp (qT_j/n)^{\frac{1}{2}}$ and the assumption $\eta_j > \frac{1}{4}T_j$, we see that

$$\eta_j \ll \frac{q}{n}\left(\frac{nT_j}{q}\right)^{\frac{1}{4}} \qquad \text{and} \qquad \eta_j^2 T_j^{-1} \ll \frac{q}{n}.$$

Therefore, from these estimates and (8.9), the case $n \le qT_1L^2\Delta^{-2}$ of (8.12) follows under the assumption $\eta_j > \frac{1}{4}T_j$. The remaining case $qT_1L^2\Delta^{-2} < n \le qT_2L^2\Delta^{-2}$ of (8.12) can be treated by combining the estimates for the other cases.

Finally, when the assumption (8.13) does not hold, we replace the lower (and upper) limits $\eta_j \pm \delta$ of the integrals appearing in (8.14) and (8.16) by $\max(\eta_j \pm \delta, L^{-1}\Delta)$, and proceed similarly. This completes the proof of (8.12).

We substitute the bounds (8.11) and (8.12) into the right-hand side of (8.6) to obtain

$$\int_{L^{-1}\Delta}^{\infty} h_\sigma^{\pm}(y;n)dy \ll \begin{cases} (L^{-1}\Delta)^{1-2\sigma}\Big\{\frac{q}{n}\Big(1+\big(\frac{nT}{q}\big)^{\frac{1}{4}}\Big) + \Delta^{\frac{3}{2}}L^{\frac{1}{2}}T^{-\frac{1}{2}}\Big\} & \text{if } n \le qT_1L^2\Delta^{-2}, \\ (L^{-1}\Delta)^{1-2\sigma}\Big\{\frac{q}{n}\Big(L^2+\big(\frac{nT}{q}\big)^{\frac{1}{4}}\Big) + \Delta^{\frac{3}{2}}L^{\frac{1}{2}}T^{-\frac{1}{2}}\Big\} & \text{if } qT_1L^2\Delta^{-2} < n \le qT_2L^2\Delta^{-2}, \\ (L^{-1}\Delta)^{1-2\sigma}L^2\frac{q}{n} & \text{if } n > qT_2L^2\Delta^{-2}. \end{cases} \tag{8.17}$$

This together with Lemma 8 gives

Lemma 9. *For $\frac{1}{2} \le \sigma < 1$ and $N \gg qTL^2\Delta^{-2}$, it holds that*

$$\begin{aligned}
&\sum_{n\le N} a_{1-2\sigma}(n,\chi)\int_0^\infty h_\sigma^+(y;n)dy \\
\ll &\sum_{n\le qT_1L^2\Delta^{-2}} |a_{1-2\sigma}(n,\chi)|(L^{-1}\Delta)^{1-2\sigma} \\
&\quad\times\left\{\frac{q}{n}\Big(1+\Big(\frac{nT}{q}\Big)^{\frac{1}{4}}\Big)+\Delta^{\frac{3}{2}}L^{\frac{1}{2}}T^{-\frac{1}{2}}\right\} \\
&+\sum_{qT_1L^2\Delta^{-2}<n\le qT_2L^2\Delta^{-2}} |a_{1-2\sigma}(n,\chi)|(L^{-1}\Delta)^{1-2\sigma} \\
&\quad\times\left\{\frac{q}{n}\Big(L^2+\Big(\frac{nT}{q}\Big)^{\frac{1}{4}}\Big)+\Delta^{\frac{3}{2}}L^{\frac{1}{2}}T^{-\frac{1}{2}}\right\} \\
&+\sum_{qT_2L^2\Delta^{-2}<n\le N} |a_{1-2\sigma}(n,\chi)|(L^{-1}\Delta)^{1-2\sigma}L^2\frac{q}{n} \\
&+e^{-AL^2}NT(\log N\log T)^\omega,
\end{aligned}$$

and the same estimate holds for

$$\sum_{n\le N}\overline{a_{1-2\sigma}(n,\overline{\chi})}\int_0^\infty \overline{h_\sigma^-(y;n)}dy.$$

9 Application of Weil's estimate

In the last two sections we assume that $q = p$ is a prime number. Then we have the following lemma, which is a consequence of Weil's estimate [28] of Kloosterman sums.

Lemma 10. *Let $q = p$ be a prime, and $\frac{1}{2} \le \sigma < 1$. Then we have*

$$|a_{1-2\sigma}(n,\chi)| \le 2\sigma_{1-2\sigma}(n)(p,n)^{\frac{1}{2}}p^{-\frac{1}{2}}.$$

Proof: We write

$$a_{1-2\sigma}(n,\chi) = p^{-1}\sum_{d|n} d^{1-2\sigma}S(p;\overline{\chi},d,n/d) \tag{9.1}$$

where

$$S(p;\chi,d,k) = \sum_{r=0}^{d-1}\chi(r+d)\overline{\chi}(r)e(rk/p).$$

Heath-Brown [4] studied this sum in detail, and Lemma 8 of [4] asserts that, if $(p,d) = 1$, then

$$|S(p;\chi,d,k)| \le 2\sqrt{p}. \tag{9.2}$$

Heath-Brown used Weil's estimate essentially to prove (9.2). To obtain the assertion of Lemma 10, we apply (9.2) to (9.1) when $(p,n) = 1$, while the trivial estimate

$$|S(p;\chi,d,k)| \le p$$

is enough when $p|n$. □

Applying Lemma 10, we get

$$(9.3) \qquad \sum_{n\le pT_jL^2\Delta^{-2}} |a_{1-2\sigma}(n,\chi)|$$
$$\le 2p^{-\frac12}\left\{ \sum_{\substack{n\le pT_jL^2\Delta^{-2}\\ p|n}} \sigma_{1-2\sigma}(n)p^{\frac12} + \sum_{\substack{n\le pT_jL^2\Delta^{-2}\\ (p,n)=1}} \sigma_{1-2\sigma}(n)\right\}$$
$$\le 2\sum_{m\le T_jL^2\Delta^{-2}} \sigma_{1-2\sigma}(pm) + 2p^{-\frac12}\sum_{n\le pT_jL^2\Delta^{-2}} \sigma_{1-2\sigma}(n)$$

for $j=1,2$. We can show

$$(9.4) \qquad \sigma_{1-2\sigma}(pm) \ll \sigma_{1-2\sigma}(m),$$

where the implied constant depends only on σ. In fact, writing $m=p^am'$ with $a\ge 0$ and $(p,m')=1$, we have

$$\begin{aligned}\sigma_{1-2\sigma}(pm) &= \sum_{j=0}^{a+1}\sum_{d'|m'}(p^jd')^{1-2\sigma}\\ &= \begin{cases} (a+2)\sigma_{1-2\sigma}(m') & \text{if } \sigma=\dfrac12,\\ \dfrac{1-(p^{1-2\sigma})^{a+2}}{1-p^{1-2\sigma}}\sigma_{1-2\sigma}(m') & \text{if } \sigma>\dfrac12.\end{cases}\end{aligned}$$

Comparing this with the corresponding expression of $\sigma_{1-2\sigma}(m)$, we obtain (9.4).

Applying (8.5) and (9.4) to (9.3), we get

$$\sum_{n\le pT_jL^2\Delta^{-2}} |a_{1-2\sigma}(n,\chi)| \ll p^{\frac12}TL^{2+2\omega}\Delta^{-2}.$$

Similarly, with partial summation, we get

$$\sum_{n\le pT_jL^2\Delta^{-2}} |a_{1-2\sigma}(n,\chi)|n^{-\frac34} \ll p^{-\frac14}T^{\frac14}L^{\frac12+2\omega}\Delta^{-\frac12},$$

$$\sum_{n\le pT_jL^2\Delta^{-2}} |a_{1-2\sigma}(n,\chi)|n^{-1} \ll p^{-\frac12}L^{2+2\omega},$$

and

$$\sum_{pT_2L^2\Delta^{-2}<n\le N} |a_{1-2\sigma}(n,\chi)|n^{-1} \ll p^{-\frac12}(\log N)^{1+\omega}.$$

We combine these estimates with the assertion of Lemma 9. Then from (8.2) we obtain

Lemma 11. *For $\frac12\le\sigma<1$ and $N\gg pTL^2\Delta^{-2}$ we have*

$$\begin{aligned}&\Sigma_{\sigma,N}(T_1,T_2;\chi)\\ &\ll \Delta^{\frac12-2\sigma}p^{\frac12}T^{\frac12}L^{2\sigma+\frac32+2\omega} + \Delta^{1-2\sigma}p^{\frac12}L^{2\sigma+3+2\omega}\\ &\quad +\Delta^{1-2\sigma}p^{\frac12}L^{2\sigma+1}(\log N)^{1+\omega} + e^{-AL^2}NT(\log N\log T)^{\omega}.\end{aligned}$$

10 Completion of the proofs

We are now ready for the final stage of the proofs of Theorems 1 and 2. The collection of Lemmas 4, 7 and 11, with noting (6.1), (7.7) and (8.1), implies that

(10.1)
$$\begin{aligned} &E_\sigma(2T,\chi)-E_\sigma(T,\chi)\\ &\ll \Delta^{\frac12-2\sigma}p^{\frac12}T^{\frac12}L^{2\sigma+\frac32+2\omega}+\Delta^{1-2\sigma}p^{\frac12}L^{2\sigma+3+2\omega}\\ &\quad+\Delta^{1-2\sigma}p^{\frac12}L^{2\sigma+1}(\log N)^{1+\omega}+e^{-AL^2}NT(\log N\log T)^{\omega}\\ &\quad+p^{1+\sigma+\varepsilon}e^{-AT}+(pT)^{-c}+\Delta L^{1+2\omega}\end{aligned}$$

for $\frac12\le\sigma<1$ and $N\gg pTL^2\Delta^{-2}$, with a large positive c.

We first prove Theorem 1. We take $N=pT$, and assume that $T\ge c_1p^{\frac{1}{12}}$ with some $c_1>0$, since otherwise the theorem follows trivially from Heath-Brown's estimate (3.11) (see Remark 2 at the end of Section 3). Then we find from (10.1) that

(10.2)
$$E(2T,\chi)-E(T,\chi)\ll\Delta^{-\frac12}p^{\frac12}T^{\frac12}L^{\frac92}+p^{\frac12}L^6+\Delta L^3.$$

If we can choose

(10.3)
$$\Delta=p^{\frac13}T^{\frac13}L,$$

we obtain

(10.4)
$$E(2T,\chi)-E(T,\chi)\ll(pT)^{\frac13}(\log pT)^2+p^{\frac12}(\log pT)^3.$$

The choice (10.3) agrees with (3.1) only when $T^2\ge A_0^3pL^6$. Otherwise we take $\Delta=T/A_0L$. Then (10.2) implies the estimate $E(2T,\chi)-E(T,\chi)\ll p^{\frac12}(\log pT)^3$ in this case. Replacing T by $2^{-j}T$ in (10.4) with $j=1,2,\ldots,J(=[\log(T/c_1p^{1/12})/\log 2])$, and summing them up, we have

$$E(T,\chi)-E(2^{-J}T,\chi)\ll(pT)^{\frac13}(\log pT)^2+p^{\frac12}(\log pT)^3\log T.$$

Noting the estimate $E(2^{-J}T,\chi)\ll p^{47/96+\varepsilon}$ (see Remark 2 at the end of Section 3), we obtain the assertion of Theorem 1.

The proof of Theorem 2 is similar. We take $N=pT$ and assume that $T\ge c_2p^{\alpha}$ with some $c_2>0$ where $\alpha=(-1+3\sigma)/(7.5-3\sigma)$. If $T^{4\sigma}\ge A_0^{1+4\sigma}pL^{2+8\sigma}$ then we can choose $\Delta=(pT)^{1/(1+4\sigma)}L$ and obtain

$$E_\sigma(2T,\chi)-E_\sigma(T,\chi)\ll(pT)^{1/(1+4\sigma)}\log pT+(pT)^{(1-2\sigma)/(1+4\sigma)}p^{\frac12}(\log pT)^2.$$

Otherwise we take $\Delta=T/A_0L$ to get

$$E_\sigma(2T,\chi)-E_\sigma(T,\chi)\ll T^{1-2\sigma}p^{\frac12}(\log pT)^{1+2\sigma}.$$

Finally we carry out the summing up process similarly to the proof of Theorem 1, with noting the estimate $E_\sigma(2^{-J}T,\chi)\ll p^{\frac12-(3\sigma-1)/12(5-2\sigma)+\varepsilon}$ $(J=[\log(T/c_2p^{\alpha})/\log 2])$; in this case the extra log-factor does not appear, because of the existence of the negative exponents of T. This completes the proof of Theorem 2.

Finally we mention the case of arbitrary modulus q. The second author's papers [17] [19] [20] contain the treatment of the case of odd composite modulus q. Motohashi [24] discussed the case of general q, including the even modulus case. Both of them are based on the idea in Heath-Brown's paper [4]. It is probably possible to combine Heath-Brown's method with our weighted integral approach to obtain the result for the general case.

References

[1] F. V. Atkinson, The mean-value of the Riemann zeta function, Acta Math. **81** (1949), 353–376.

[2] R. Balasubramanian, An improvement on a theorem of Titchmarsh on the mean square of $|\zeta(\frac{1}{2}+it)|$, Proc. London Math. Soc. (3) **36** (1978), 540–576.

[3] H. Davenport, Multiplicative Number Theory, 2nd ed., Springer, 1980.

[4] D. R. Heath-Brown, Hybrid bounds for Dirichlet L-functions, Invent. Math. **47** (1978), 149–170.

[5] D. R. Heath-Brown, Hybrid bounds for Dirichlet L-functions II, Quart. J. Math. Oxford (2) **31** (1980), 157–167.

[6] M. N. Huxley, A note on exponential sums with a difference, Bull. London Math. Soc. **26** (1994), 325–327.

[7] A. Ivić, The Riemann Zeta-Function, Wiley, 1985.

[8] A. Ivić, Mean Values of the Riemann Zeta Function, Lectures on Math. and Phys. **82**, Tata Inst. Fund. Res., Springer, 1991.

[9] A. Ivić and K. Matsumoto, On the error term in the mean square formula for the Riemann zeta-function in the critical strip, Monatsh. Math. **121** (1996), 213–229.

[10] M. Jutila, Riemann's zeta-function and the divisor problem, Ark. Mat. **21** (1983), 75–96.

[11] A. Kačėnas, The asymptotic behaviour of the second power moment of the Riemann zeta-function in the critical strip, Liet. Mat. Rinkinys **35** (1995), 315–331 (in Russian); Lithuanian Math. J. **35** (1995), 249–261.

[12] A. Kačėnas, A note on the mean square of $\zeta(s)$ in the critical strip, in "New Trends in Probability and Statistics Vol 4, Analytic and Probabilistic Methods in Number Theory" (A. Laurinčikas, E. Manstavičius and V. Stakėnas, Eds.), VSP/TEV, 1997, 107–117.

[13] A. Laurinčikas, The Atkinson formula near the critical line, in "New Trends in Probability and Statistics Vol 2, Analytic and Probabilistic Methods in Number Theory" (F. Schweiger and E. Manstavičius, Eds.), VSP/TEV, 1992, 335–354.

[14] A. Laurinčikas, The Atkinson formula near the critical line II, Liet. Mat. Rinkinys **33** (1993), 302–313 (in Russian); Lithuanian Math. J. **33** (1993), 234–242.

[15] A. Laurinčikas, The Atkinson formula for L-functions near the critical line, Liet. Mat. Rinkinys **33** (1993), 435–454 (in Russian); Lithuanian Math. J. **33** (1993), 337–351.

[16] A. Laurinčikas, A uniform estimate of the error term in the mean square of the Riemann zeta-function, Liet. Mat. Rinkinys **35** (1995), 508–517 (in Russian); Lithuanian Math. J. **35** (1995), 403-410.

[17] K. Matsumoto, The mean square of Dirichlet L-functions, Proc. Japan Acad. **58A** (1982), 443–446; Corrections, ibid. **65A** (1989), 344.

[18] K. Matsumoto, The mean square of the Riemann zeta-function in the critical strip, Japanese J. Math. **15** (1989), 1–13.

[19] K. Matsumoto, The mean square of Dirichlet L-functions (A generalization of Balasubramanian's method), Proc. Japan Acad. **66A** (1990), 204–208.

[20] K. Matsumoto, The mean square of Dirichlet L-functions (A generalization of Balasubramanian's method), unpublished manuscript, 1990, 288pp.

[21] T. Meurman, A generalization of Atkinson's formula to L-functions, Acta Arith. **47** (1986), 351–370.

[22] T. Meurman, Identities of the Atkinson type for L-functions, unpublished manuscript.

[23] Y. Motohashi, A note on the mean value of the zeta and L-functions II, Proc. Japan Acad. **61A** (1985), 313–316.

[24] Y. Motohashi, On the mean square of L-functions, unpublished manuscript, 1986, 42pp.+11pp.

[25] Y. Motohashi, The mean square of $\zeta(s)$ off the critical line, unpublished manuscript, 1990, 11pp.

[26] K. Prachar, Primzahlverteilung, Springer, 1957.

[27] E. C. Titchmarsh (revised by D. R. Heath-Brown), The Theory of the Riemann Zeta-Function, 2nd ed., Oxford, 1986.

[28] A. Weil, On some exponential sums, Proc. Nat. Acad. Sci. USA **34** (1948), 204–207.

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