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Functional equations for double L-functions and values at non-positive integers

Yasushi Komori

Y. Komori, Department of Mathematics, Rikkyo University, Nishi-Ikebukuro, Toshima-ku, Tokyo 171-8501, Japan
komori@rikkyo.ac.jp

Kohji Matsumoto

K. Matsumoto, Graduate School of Mathematics Nagoya University, Chikusa-ku, Nagoya, 464-8602, Japan
kohjimat@math.nagoya-u.ac.jp

Hirofumi Tsumura

H. Tsumura, Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1, Minami-Ohsawa, Hachioji, Tokyo 192-0397 Japan
tsumura@tmu.ac.jp

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We consider double L-functions with periodic coefficients and complex parameters. We prove functional equations for them, which is of traditional symmetric form on certain hyperplanes. These are character analogues of our previous result on double zeta-functions. We further evaluate double L-functions at non-positive integers and construct certain p-adic double L-functions.

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1. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers and $i = \sqrt{-1}$.

For $f \in \mathbb{N}$, let $a : \mathbb{Z} \to \mathbb{C}$ be a periodic function with period f. We assume that a is an even or odd function, namely $a(-m) = \lambda(a)a(m)$ for any $m \in \mathbb{Z}$, where $\lambda(a) \in \{\pm 1\}$. Let

$$\mathfrak{R}_f^{\pm} = \left\{ a : \mathbb{Z} \to \mathbb{C} \,|\, a(m+f) = a(m),\, a(-m) = \pm a(m) \ (m \in \mathbb{Z}) \right\}, \tag{1.1}$$

and let $\mathcal{R}_f = \mathcal{R}_f^+ \cup \mathcal{R}_f^-$.

For $a_j \in \mathcal{R}_{f_j}$ $(j = 1, 2)$, we choose any $f \in \mathbb{N}$ with $\mathrm{lcm}(f_1, f_2) \mid f$. Then we see that $a_j \in \mathcal{R}_f$ $(j = 1, 2)$. In view of this observation, from now on, we assume $f > 1$.

For $a_1, a_2 \in \mathcal{R}_f$, we define the double Dirichlet series associated with a_1 and a_2 by

$$L_2^{ш}(s_1, s_2; a_1, a_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_1(m)a_2(n)}{m^{s_1}(m+n)^{s_2}} \tag{1.2}$$

for $s_1, s_2 \in \mathbb{C}$. This series is absolutely convergent when

$$\sigma_2 > 1, \quad \sigma_1 + \sigma_2 > 2, \tag{1.3}$$

where $\sigma_j = \Re s_j$ $(j = 1, 2)$. In particular when $(s_1, s_2) \in \mathbb{N}^2$, we find that $L_2^{ш}(s_1, s_2; a_1, a_2)$, if convergent, is the double L-value denoted by $L_{ш}(s_2, s_1; a_2, a_1)$ which was first studied by Arakawa and Kaneko [4]. They considered not only double L-values, but also more general multiple L-values (abbreviated as MLVs). After their work, several properties for MLVs have been studied (see, for example, [25,26,28]). In addition, another type of multiple L-functions was studied by Akiyama and Ishikawa [2,13].

More generally we define the double series of Eisenstein type

$$L_2^{ш}(s_1, s_2; a_1, a_2; \omega_1, \omega_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_1(m)a_2(n)}{(m\omega_1)^{s_1}(m\omega_1 + n\omega_2)^{s_2}}, \tag{1.4}$$

where $\omega_1, \omega_2 \in \mathbb{C}$ with $\Re\omega_1 > 0$, $\Re\omega_2 > 0$. This is also absolutely convergent when (1.3) holds. The special case $a_1 \equiv 1$, $a_2 \equiv 1$ in (1.4) has been studied in our former paper [17].

In this paper, we give symmetric functional equations of traditional type for the above double Dirichlet series (Theorem 2.1). In particular when $a_1 \equiv 1$, $a_2 \equiv 1$, this coincides with the functional equation for double zeta-functions given in [17] which is based on a pioneering result of the second-named author [21]. As a corollary of our present result, for primitive Dirichlet characters χ_1, χ_2 of conductor $f > 1$, we obtain a symmetric functional equation for the ordinary double Dirichlet L-function

$$L_2^{ш}(s_1, s_2; \chi_1, \chi_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi_1(m)\chi_2(n)}{m^{s_1}(m+n)^{s_2}} \tag{1.5}$$

(Corollary 2.3). On the other hand, if χ_1 or χ_2 is a principal character or a non-primitive character, then the function appearing in the counter-part of the functional equation is not necessarily a (double) Dirichlet L-function (see Remark 2.4). This is the reason why we work in a general framework of periodic coefficients.

In Section 3, using those functional equations, we evaluate double L-functions at non-positive integers. Indeed we determine certain *trivial zeros* of double L-functions (Theorem 3.2). Also we describe values of double L-functions at non-positive integers in terms of Dirichlet L-values (Theorem 3.4). This can be regarded as a χ-analogue of a known fact for double zeta-functions given by Akiyama-Egami-Tanigawa [1] and recovered by the authors [17].

This type of double L-functions is attractive from the viewpoint of number theory. In fact, we have already known that, for example,

$$L_2^{ш}(1,-1;\chi,\chi)=B_{1,\chi}^2-\frac{1}{2}L(1;\chi)B_{2,\chi} \tag{1.6}$$

(see [27, p.167, Remark]), where χ is a primitive Dirichlet character of conductor $f>1$ and $\{B_{n,\chi}\}$ are the generalized Bernoulli numbers defined by

$$\sum_{j=1}^{f}\frac{\chi(j)te^{jt}}{e^{ft}-1}=\sum_{n=0}^{\infty}B_{n,\chi}\frac{t^n}{n!} \tag{1.7}$$

(see [29, Chap. 4]). This implies that double L-functions have number theoretic information of abelian number fields.

In Section 4, based on the observation for values of double L-functions, we will construct p-adic double L-functions by using p-adic measures (see (4.4)). We will prove some functional relations between p-adic double L-functions and Kubota-Leopoldt p-adic L-functions (Theorem 4.2). More general investigation for p-adic multiple L-functions will be written in our forthcoming paper.

A part of the results in the present paper has been announced in [16].

2. The statement of functional equations

First we recall some results of finite Fourier expansions (see, for example, [3, Chapter 8]). Let $a_1,a_2\in\mathcal{R}_f$. We can write

$$a_j(m)=\sum_{\nu=1}^{f}\widehat{a}_j(\nu)e^{2\pi i\nu m/f}\quad (j=1,2), \tag{2.1}$$

where

$$\widehat{a}_j(m)=\frac{1}{f}\sum_{\nu=1}^{f}a_j(\nu)e^{-2\pi i\nu m/f}\quad (j=1,2). \tag{2.2}$$

In particular $\widehat{a}_j(0)=0$ if and only if $\sum_{1\le\nu\le f}a_j(\nu)=0$. Also we can see that $\lambda(\widehat{a}_j)=\lambda(a_j)$ $(j=1,2)$. When a_1 and a_2 are primitive Dirichlet characters ψ_1 and ψ_2 of conductor f, respectively, then we have

$$\widehat{\psi}_j(m)=\frac{\overline{\psi_j}(m)}{\tau(\overline{\psi_j})}, \tag{2.3}$$

where $\overline{\psi_j}$ is the complex conjugate of ψ_j and $\tau(\cdot)$ denotes the Gauss sum defined by

$$\tau(\chi)=\sum_{m=1}^{f}\chi(m)e^{2\pi im/f}. \tag{2.4}$$

Therefore

$$\psi_j(m)\tau(\overline{\psi_j}) = \sum_{\nu=1}^{f} \overline{\psi_j}(\nu) e^{2\pi i\nu m/f} \tag{2.5}$$

for a primitive character ψ_j $(j = 1, 2)$ of conductor f.

Now we state the first main result in this paper, that is, the functional equation for the double series (1.4). For $a \in \mathcal{R}_f$ we let

$$L(s, a) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}. \tag{2.6}$$

In particular when a is a Dirichlet character χ, $L(s, \chi)$ is a Dirichlet L-function.

Theorem 2.1. *For any $a_1, a_2 \in \mathcal{R}_f$, the double series $L_2^{ш}(s_1, s_2; a_1, a_2; \omega_1, \omega_2)$ can be continued meromorphically (as a function in s_1, s_2) to the whole space $\mathbb{C}^2$, and the functional equation*

$$\begin{aligned} &\left(\frac{2\pi i}{f\omega_1\omega_2}\right)^{\frac{1-s_1-s_2}{2}} \Bigg\{\Gamma(s_2) L_2^{ш}(s_1, s_2; a_1, a_2; \omega_1, \omega_2) \\ &\qquad - \frac{\omega_2^{1-s_1-s_2}}{f\omega_1} \Gamma(1-s_1)\Gamma(s_1+s_2-1) L(s_1+s_2-1, a_2) \sum_{\nu=1}^{f} a_1(\nu)\Bigg\} \\ &= \left(\frac{2\pi i}{f\omega_1\omega_2}\right)^{\frac{s_1+s_2-1}{2}} \Bigg\{\Gamma(1-s_1) L_2^{ш}(1-s_2, 1-s_1; \widehat{a}_2, \widehat{a}_1; \omega_1, \omega_2) \\ &\qquad - \frac{\omega_2^{s_1+s_2-1}}{f\omega_1} \Gamma(s_2)\Gamma(1-s_1-s_2) L(1-s_1-s_2, \widehat{a}_1) \sum_{\nu=1}^{f} \widehat{a}_2(\nu)\Bigg\} \end{aligned} \tag{2.7}$$

holds on the hyperplane $s_1 + s_2 = 2k + 1$ $(k \in \mathbb{Z})$ if $\lambda(a_1)\lambda(a_2) = 1$, and on the hyperplane $s_1 + s_2 = 2k$ $(k \in \mathbb{Z})$ if $\lambda(a_1)\lambda(a_2) = -1$.

Remark 2.2. Formula (2.7) is valid for any $k \in \mathbb{Z}$, but double L-functions may be singular on some of the above hyperplanes. This point will be discussed at the end of the present paper.

In particular when a_1, a_2 are primitive Dirichlet characters χ_1, χ_2 of conductor $f > 1$, it is well-known that

$$\sum_{\nu=1}^{f} \chi_1(\nu) = 0, \ \sum_{\nu=1}^{f} \widehat{\chi}_2(\nu) = 0, \tag{2.8}$$

because of (2.3). Therefore we obtain the following corollary, which is clearly the "double-analogue" of the functional equation for Dirichlet L-functions. It is an interesting point that the following formula expresses the duality not only between s_1 and s_2, but also between χ_1 and χ_2.

Corollary 2.3. *If χ_1, χ_2 are primitive Dirichlet characters of conductor $f > 1$, then the functional equation*

$$
\begin{aligned}
&\left(\frac{2\pi i}{f}\right)^{\frac{1-s_1-s_2}{2}} \frac{\Gamma(s_2)}{\tau(\chi_1)} L_2^{ш}(s_1, s_2; \chi_1, \chi_2) \\
&\quad = \left(\frac{2\pi i}{f}\right)^{\frac{s_1+s_2-1}{2}} \frac{\Gamma(1-s_1)}{\tau(\overline{\chi}_2)} L_2^{ш}(1-s_2, 1-s_1; \overline{\chi}_2, \overline{\chi}_1)
\end{aligned} \tag{2.9}
$$

holds on the hyperplane $s_1 + s_2 = 2k + 1$ $(k \in \mathbb{Z})$ if $\chi_1(-1)\chi_2(-1) = 1$, and on the hyperplane $s_1 + s_2 = 2k$ $(k \in \mathbb{Z})$ if $\chi_1(-1)\chi_2(-1) = -1$.

Remark 2.4. When χ_1 or χ_2 is a principal character or a non-primitive Dirichlet character, (2.8) does not hold. Therefore, in this case, (2.9) does not hold while (2.7) holds. In fact, we will treat this case in Theorem 3.4.

The proof of Theorem 2.1 will be given in the last section of the present paper.

3. Values at non-positive integers

In this section, using the results in the preceding section, we compute the values $L_2^{ш}(-p, -q; a_1, a_2)$ $(p, q \in \mathbb{N}_0)$ under some assumptions for a_1, a_2.

First we assume that $a_1, a_2 \in \mathfrak{R}_f$ with $f > 1$ and that $\widehat{a}_1(0) = \widehat{a}_2(0) = 0$. Let

$$
G(t_1, t_2; a_1, a_2) = \sum_{\nu=1}^{f} \sum_{\rho=1}^{f} \widehat{a}_1(\nu)\widehat{a}_2(\rho) \frac{e^{2\pi i\nu/f}}{e^{t_1+t_2} - e^{2\pi i\nu/f}} \frac{e^{2\pi i\rho/f}}{e^{t_2} - e^{2\pi i\rho/f}}, \tag{3.1}
$$

and define $\beta_{M,N}(a_1, a_2)$ by the expansion

$$
G(t_1, t_2; a_1, a_2) = \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \beta_{M,N}(a_1, a_2) \frac{t_1^M t_2^N}{M!N!}. \tag{3.2}
$$

Theorem 3.1. *Assume that $a_1, a_2 \in \mathfrak{R}_f$ with $f > 1$, and $\widehat{a}_1(0) = \widehat{a}_2(0) = 0$. Then $L_2^{ш}(s_1, s_2; a_1, a_2)$ can be analytically continued to the whole space $\mathbb{C}^2$ with no singularities. For $p, q \in \mathbb{N}_0$,*

$$
L_2^{ш}(-p, -q; a_1, a_2) = (-1)^{p+q} \beta_{p,q}(a_1, a_2). \tag{3.3}
$$

Proof. We use the method of contour integrals (see, for example, [29, Proof of Theorem 4.2]). We define Υ which consists of the positive real axis $[\varepsilon, \infty)$ for a small positive ε (top side), a circle C_ε around 0 counterclockwise of radius ε, and the positive real axis $[\varepsilon, \infty)$ (bottom side). Note that we interpret t^s to mean $\exp(s \log t)$, where the imaginary part of $\log t$ varies from 0 (on the top side of the real axis) to 2π (on the bottom side). Let

$$
H(s_1, s_2; a_1, a_2) = \int_\Upsilon \int_\Upsilon G(t_1, t_2; a_1, a_2) t_1^{s_1-1} t_2^{s_2-1} dt_1 dt_2. \tag{3.4}
$$

We can easily see that $H(s_1,s_2;a_1,a_2)$ is holomorphic for any $(s_1,s_2)\in\mathbb{C}^2$, and

$$\begin{aligned}H(s_1,s_2;a_1,a_2)=&\left(e^{2\pi is_1}-1\right)\left(e^{2\pi is_2}-1\right)\int_\varepsilon^\infty\int_\varepsilon^\infty G(t_1,t_2;a_1,a_2)t_1^{s_1-1}t_2^{s_2-1}dt_1dt_2\\&+\left(e^{2\pi is_2}-1\right)\int_\varepsilon^\infty\int_{C_\varepsilon} G(t_1,t_2;a_1,a_2)t_1^{s_1-1}t_2^{s_2-1}dt_1dt_2\\&+\left(e^{2\pi is_1}-1\right)\int_{C_\varepsilon}\int_\varepsilon^\infty G(t_1,t_2;a_1,a_2)t_1^{s_1-1}t_2^{s_2-1}dt_1dt_2\\&+\int_{C_\varepsilon}\int_{C_\varepsilon} G(t_1,t_2;a_1,a_2)t_1^{s_1-1}t_2^{s_2-1}dt_1dt_2.\end{aligned}$$

When $\Re s_1>1$ and $\Re s_2>1$, we see that $\int_{C_\varepsilon}\to 0$ as $\varepsilon\to 0$ (because $\hat{a}_1(f)=\hat{a}_2(f)=0$ by the assumption, while if $\nu\le f-1$, $\rho\le f-1$ then the denominator on the right-hand side of (3.1) does not tend to 0 when $t_1,t_2\to 0$). Therefore, in the usual way, we have

$$H(s_1,s_2;a_1,a_2)=\left(e^{2\pi is_1}-1\right)\left(e^{2\pi is_2}-1\right)\Gamma(s_1)\Gamma(s_2)L_2^{ш}(s_1,s_2;a_1,a_2). \qquad (3.5)$$

This implies that $L_2^{ш}(s_1,s_2;a_1,a_2)$ can be continued meromorphically to the whole space $\mathbb{C}^2$.

The zeros of $e^{2\pi is_j}-1$ $(j=1,2)$, that is $s_j=l_j\in\mathbb{Z}$, are candidates of singularities of $L_2^{ш}(s_1,s_2;a_1,a_2)$. However they are cancelled by the gamma factors if $l_j\le 0$. When $l_1\ge 1$ or $l_2\ge 2$, they should not be singularities because of (1.3). To prove that the remaining case $s_2=1$ is also not singular, we show that the series (1.2) (in that order of summation) is convergent uniformly in the region $\Re s_2>0$, $\Re(s_1+s_2)>1$. In fact, $\sum_{n\le x}a_2(n)$ is bounded uniformly in x, because $\widehat{a}_2(0)=0$. Therefore by partial summation we have

$$\begin{aligned}\sum_{n=1}^\infty\frac{a_2(n)}{(m+n)^{s_2}}&=s_2\int_1^\infty\left(\sum_{n\le x}a_2(n)\right)\frac{dx}{(m+x)^{s_2+1}}\\&\ll\int_1^\infty\frac{dx}{(m+x)^{\sigma_2+1}}\ll\frac{1}{m^{\sigma_2}},\end{aligned}$$

so the double series (1.2) is

$$\ll\sum_{m=1}^\infty\frac{|a_1(m)|}{m^{\sigma_1+\sigma_2}}.$$

This is convergent when $\sigma_1+\sigma_2>1$, which implies the assertion. Therefore $L_2^{ш}(s_1,s_2;a_1,a_2)$ is entire.

Lastly, putting $s_1=-p$, $s_2=-q$ in (3.4), substituting (3.2) into the right-hand side, and using the well-known fact

$$\lim_{s\to -k}(e^{2\pi is}-1)\Gamma(s)=(2\pi i)\frac{(-1)^k}{k!}\quad(k\in\mathbb{N}_0),$$

we can obtain (3.3). □

In particular when a_1, a_2 are primitive Dirichlet characters χ_1, χ_2 of conductor $f > 1$, it was already shown that $L_2^{ш}(s_1, s_2; \chi_1, \chi_2)$ is entire ([23,24]). In this case, we proved that (2.9) holds (Corollary 2.3). From this, we can explicitly determine the 'trivial zeros' of $L_2^{ш}(s_1, s_2; \chi_1, \chi_2)$ as follows.

Theorem 3.2. *For primitive Dirichlet characters χ_1, χ_2 of conductor $f > 1$, and $p, q, k \in \mathbb{N}$ with $p \le 2k$ and $q \le 2k-1$,*

$$L_2^{ш}(p-2k, 1-p; \chi_1, \chi_2) = 0 \quad (\text{if } \chi_1(-1)\chi_2(-1) = 1), \tag{3.6}$$
$$L_2^{ш}(q+1-2k, 1-q; \chi_1, \chi_2) = 0 \quad (\text{if } \chi_1(-1)\chi_2(-1) = -1). \tag{3.7}$$

Proof. For $(s_1, s_2) = (l_1, l_2) \in \mathbb{N}^2$, we see (as in the proof of Theorem 3.1) that $L_2^{ш}(l_1, l_2; \chi_1, \chi_2)$ converges when $f > 1$. Hence, from (2.9), we see that if $\chi_1(-1)\chi_2(-1) = 1$ then $L_2^{ш}(1-l_2, 1-l_1; \overline{\chi_2}, \overline{\chi_1}) = 0$ with $l_1 + l_2 = 2k+1$, because of the Γ-factor. Putting $p = l_1$ and replacing $(\overline{\chi_2}, \overline{\chi_1})$ by (χ_1, χ_2), we obtain (3.6). Similarly we can obtain (3.7). □

Remark 3.3. Unlike the Dirichlet L-function, it is unknown whether the double L-function $L_2^{ш}(s_1, s_2; a_1, a_2)$ has an Euler product or not. Hence we cannot prove that, for example, $L_2^{ш}(s_1, s_2; a_1, a_2) \neq 0$ when $\Re s_1 > 1$ and $\Re s_2 > 1$. Therefore there might exist other 'trivial zeros' of $L_2^{ш}(s_1, s_2; a_1, a_2)$ besides (3.6) and (3.7).

Next, when a_1 is a primitive Dirichlet character χ of conductor $f > 1$, and a_2 is the principal character χ_0, that is, $\chi_0(m) = 1$ $(m \in \mathbb{Z})$. Then we can see that $\chi_0 \in \mathcal{R}_f$ and

$$\widehat{\chi_0}(m) = \begin{cases} 1 & (f \mid m) \\ 0 & (f \nmid m). \end{cases}$$

It is known that (see [29, Chapter 4])

$$L(1-k, \chi) = -\frac{B_{k,\chi}}{k} \quad (k \in \mathbb{N}),$$
$$L(k, \chi) = \frac{(-1)^{k-1}}{2}\frac{\tau(\chi)}{k!}\left(\frac{2\pi i}{f}\right)^k B_{k,\overline{\chi}} \quad (k \in \mathbb{N}),$$

where $\{B_{n,\chi}\}$ are the generalized Bernoulli numbers defined by (1.7). Hence, by putting $(s_1, s_2) = (-p, -q)$ with $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ in (2.7), we obtain the following result, which can be regarded as a χ-analogue of the known fact for double zeta-functions given by Akiyama-Egami-Tanigawa [1, Equation (8)] and recovered by the authors [17, Corollary 2.4].

Theorem 3.4. *Let χ be a primitive Dirichlet character χ of conductor $f(>1)$. For $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ with the condition that $p+q$ is odd (resp. even) if $\chi(-1) = 1$ (resp.*

$= -1$),

$$\begin{aligned}L_2^{ш}(-p,-q;\chi,\chi_0) &= -\frac{f^{p+q}(p+q)!}{(2\pi i)^{p+q+1}}L(p+q+1,\widehat{\chi}) \\ &= \frac{(-1)^{p+q+1}}{2f}\frac{B_{p+q+1,\chi}}{p+q+1} = \frac{(-1)^{p+q}}{2f}L(-p-q,\chi).\end{aligned} \tag{3.8}$$

Example 3.5. It is known that (see, for example, [25,26,27])

$$L_2^{ш}(1,2;\psi_3,\psi_3) = \frac{3}{2}L(1,\psi_3)L(2,\psi_3) - \frac{1}{2}L(3,\psi_3^2), \tag{3.9}$$

where ψ_3 is the odd quadratic Dirichlet character of conductor 3. Putting $(s_1,s_2) = (s,3-s)$ in (2.9) and letting $s \to 1$ we have

$$\begin{aligned}&\left(\frac{2\pi i}{3}\right)^{-1}\Gamma(2)L_2^{ш}(1,2;\psi_3,\psi_3) \\ &= \left(\frac{2\pi i}{3}\right)\lim_{s\to1}(s-1)\Gamma(1-s)\cdot\lim_{s\to1}\frac{L_2^{ш}(s-2,1-s;\psi_3,\psi_3)}{s-1}.\end{aligned}$$

Hence we have

$$\frac{d}{dt}L_2^{ш}(t-1,-t;\psi_3,\psi_3)\bigg|_{t=0} = \frac{9}{8\pi^2}\left\{3L(1,\psi_3)L(2,\psi_3) - L(3,\psi_3^2)\right\}.$$

4. *p*-adic double *L*-functions

Theorem 3.4 in the previous section suggests that values of the double L-function at non-positive integers are closely connected with those of the Dirichlet L-function. Hence, from the viewpoint of p-adic interpolations, it is natural to expect that there exists the notion of p-adic double L-functions, or more generally, p-adic multiple L-functions, which is closely connected with the well-known Kubota-Leopoldt p-adic L-functions [19]. In this section, we construct a certain p-adic double L-function associated with the Dirichlet character of conductor p, and prove a functional relation for this function and the Kubota-Leopoldt p-adic L-function. More general investigation for p-adic multiple L-functions will be written in our forthcoming paper.

We prepare the notation and quote some known results (see, for example, [15, 29]). Let p be an odd prime number, and let $\mathbb{Z}_p$, $\mathbb{Z}_p^\times$ and $\mathbb{Q}_p$ be the ring of p-adic integers, its unit group and the field of p-adic rational numbers, respectively. For any $a \in \mathbb{Z}_p^\times$, let $\omega(a) \in \mathbb{Z}_p^\times$ be a $(p-1)$th root of unity with $a \equiv \omega(a) \pmod{p}$. Note that ω is often called the Teichmüller character (see [29, §5.1]). Further we define $\langle a\rangle = a/\omega(a)$ for $a \in \mathbb{Z}_p^\times$, namely, $a = \omega(a)\langle a\rangle$, which corresponds to the decomposition $\mathbb{Z}_p^\times = W_{p-1}\times(1+p\mathbb{Z}_p)$, where W_{p-1} is the set of $(p-1)$th roots of unity in $\mathbb{Z}_p^\times$.

Coresponding to the disjoint union $\mathbb{Z}_p = \cup_{0\le a<p^N}(a+p^N\mathbb{Z}_p)$ for any $N \in \mathbb{N}$, we consider the p-adic measure μ on $\mathbb{Z}_p$ defined by

$$\mu(a+p^N\mathbb{Z}_p) = \frac{(-1)^a}{2},$$

which is a special type of the Koblitz measure (see [15, Chapter 2]). It is known that the Kubota-Leopoldt p-adic L-function associated with ω^k can be expressed by the following p-adic integral (see [15, Chapter 2, (4.5)]):

$$\begin{aligned} L_p(s;\omega^k) &= \frac{1}{\langle 2\rangle^{1-s}\omega^k(2)-1}\int_{\mathbb{Z}_p^\times}\langle x\rangle^{-s}\omega^{k-1}(x)d\mu(x) \\ &= \frac{1}{\langle 2\rangle^{1-s}\omega^k(2)-1}\lim_{N\to\infty}\sum_{\substack{a=1\\ p\nmid a}}^{p^N-1}\langle a\rangle^{-s}\omega^{k-1}(a)\frac{(-1)^a}{2}. \end{aligned} \tag{4.1}$$

This is the p-adic interpolation of the p-adic integral expression of non-positive values of L-functions, typically

$$b_m := -\frac{2^{m+1}-1}{m+1}B_{m+1} = \int_{\mathbb{Z}_p} x^m d\mu(x) = \lim_{N\to\infty}\sum_{a=0}^{p^N-1} a^m\frac{(-1)^a}{2} \tag{4.2}$$

for any $m \in \mathbb{N}_0$ (see [15, Chapter 2, (3.1)]), where $\{B_m\}$ are Bernoulli numbers defined by

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty} B_m\frac{t^m}{m!}. \tag{4.3}$$

Now we define p-adic double L-functions as follows. Let $\eta \in p\mathbb{Z}_p$ and $k, l \in \mathbb{N}$. For $s_1, s_2 \in \mathbb{Z}_p$, we define

$$\begin{aligned} L_{p,2}(s_1,s_2;\omega^k,\omega^l;\eta) &= \int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p}\langle x\rangle^{-s_1}\langle x+\eta y\rangle^{-s_2}\omega^{k+l}(x)d\mu(x)d\mu(y) \\ &= \lim_{N\to\infty}\sum_{\substack{a=1\\ p\nmid a}}^{p^N-1}\sum_{b=0}^{p^N-1}\langle a\rangle^{-s_1}\langle a+\eta b\rangle^{-s_2}\omega^{k+l}(a)\frac{(-1)^{a+b}}{4}. \end{aligned} \tag{4.4}$$

By [29, Proposition 5.8], we see that $L_{p,2}(s_1,s_2;\omega^k,\omega^l;\eta)$ is continuous on $\mathbb{Z}_p\times\mathbb{Z}_p$.

In the rest of this section, we will prove a certain functional relation between this p-adic double L-function and the Kubota-Leopoldt p-adic L-function. Let

$$G(t) := \frac{1}{e^t+1} = \frac{1}{e^t-1} - \frac{2}{e^{2t}-1} = \sum_{m=0}^{\infty} b_m\frac{t^m}{m!}. \tag{4.5}$$

Then, by (4.2), we have

$$G(t) = \sum_{m=0}^{\infty}\int_{\mathbb{Z}_p} x^m d\mu(x)\frac{t^m}{m!} = \int_{\mathbb{Z}_p} e^{xt}d\mu(x). \tag{4.6}$$

Note that it is possible to change the order of summation and integration because this double series convergent absolutely with respect to the p-adic metric (see [15, p.13]).

From the definition of $G(t)$, we can easily see that $G(t)+G(-t)=1$. Hence $G(t)-1/2$ is an odd function, so is $G(\gamma t)-1/2$ for any $\gamma \in \mathbb{Z}_p$. Furthermore, for

$k \in \mathbb{N}$, we see that $(d^k/dt^k)G(t) = (d^k/dt^k)(G(t) - 1/2)$ is an odd (resp. even) function if k is even (resp. odd). For $\gamma \in \mathbb{Z}_p$, let

$$H_k(t;\gamma) := \left(\frac{d^k}{dt^k}G(t)\right) \cdot \left(G(\gamma t) - \frac{1}{2}\right). \tag{4.7}$$

From (4.6), we have

$$\begin{aligned} H_k(t;\gamma) &= \int_{\mathbb{Z}_p} x^k e^{xt} d\mu(x) \left\{ \int_{\mathbb{Z}_p} e^{\gamma y t} d\mu(y) - \frac{1}{2} \right\} \\ &= \int_{\mathbb{Z}_p \times \mathbb{Z}_p} x^k e^{(x+\gamma y)t} d\mu(x) d\mu(y) - \frac{1}{2} \int_{\mathbb{Z}_p} x^k e^{xt} d\mu(x) \\ &= \sum_{n=0}^{\infty} \left\{ \int_{\mathbb{Z}_p \times \mathbb{Z}_p} x^k (x+\gamma y)^n d\mu(x) d\mu(y) - \frac{1}{2} \int_{\mathbb{Z}_p} x^{k+n} d\mu(x) \right\} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} h_{k,n}(\gamma) \frac{t^n}{n!}, \end{aligned} \tag{4.8}$$

say. From (4.5) and (4.7), noting $b_0 = -B_1 = 1/2$, we have

$$h_{k,n}(\gamma) = \sum_{j=1}^{n} \binom{n}{j} b_{k+n-j} b_j \gamma^j. \tag{4.9}$$

We see that $H_k(t;\gamma)$ is an odd (resp. even) function if k is odd (resp. even). Therefore we have the following.

Proposition 4.1. *For $k, l \in \mathbb{N}$ with $k \not\equiv l \pmod 2$ and $\gamma \in \mathbb{Z}_p$, we have $h_{k,l}(\gamma) = 0$.*

Using this fact, we will prove the following functional relation.

Theorem 4.2. *Let $k, l \in \mathbb{N}$ with $k \not\equiv l \pmod 2$. For $s_1, s_2 \in \mathbb{Z}_p$ and $\eta \in p\mathbb{Z}_p$,*

$$L_{p,2}(s_1, s_2; \omega^k, \omega^l; \eta) = \frac{1}{2} \left(\langle 2 \rangle^{1-s_1-s_2} \omega^{k+l+1}(2) - 1 \right) L_p(s_1 + s_2; \omega^{k+l+1}). \tag{4.10}$$

Proof. Let $m, n \in \mathbb{N}$ with $m \equiv k \pmod{(p-1)}$ and $n \equiv l \pmod{(p-1)}$. Set

$$I = L_{p,2}(-m, -n; \omega^k, \omega^l; \eta) - \frac{1}{2} \int_{\mathbb{Z}_p^\times} x^{m+n} d\mu(x). \tag{4.11}$$

Then, by (4.4) and from the choice of m, n, we have

$$\begin{aligned}
I &= \lim_{N\to\infty} \sum_{\substack{a=1\\ p\nmid a}}^{p^N-1} \sum_{b=0}^{p^N-1} a^m (a+\eta b)^n \frac{(-1)^{a+b}}{4} - \frac{1}{2} \lim_{N\to\infty} \sum_{\substack{a=1\\ p\nmid a}}^{p^N-1} a^{m+n} \frac{(-1)^a}{2} \\
&= \lim_{N\to\infty} \left\{ \sum_{a=0}^{p^N-1} \sum_{b=0}^{p^N-1} a^m (a+\eta b)^n \frac{(-1)^{a+b}}{4} - \sum_{c=0}^{p^{N-1}-1} \sum_{b=0}^{p^N-1} (pc)^m (pc+\eta b)^n \frac{(-1)^{pc+b}}{4} \right\} \\
&\quad - \frac{1}{2} \lim_{N\to\infty} \left\{ \sum_{a=0}^{p^N-1} a^{m+n} \frac{(-1)^a}{2} - \sum_{c=0}^{p^{N-1}-1} (pc)^{m+n} \frac{(-1)^{pc}}{2} \right\} \\
&= \int_{\mathbb{Z}_p\times\mathbb{Z}_p} x^m (x+\eta y)^n d\mu(x) d\mu(y) - p^{m+n} \int_{\mathbb{Z}_p\times\mathbb{Z}_p} x^m (x+(\eta/p)y)^n d\mu(x) d\mu(y) \\
&\quad - \frac{1}{2} \left\{ \int_{\mathbb{Z}_p} x^{m+n} d\mu(x) - p^{m+n} \int_{\mathbb{Z}_p} x^{m+n} d\mu(x) \right\} \\
&= h_{m,n}(\eta) - p^{m+n} h_{m,n}(\eta/p). \qquad (4.12)
\end{aligned}$$

When $k \not\equiv l \pmod 2$, namely $m \not\equiv n \pmod 2$, we see that the right-hand side of (4.12) is equal to 0 by using Proposition 4.1 with $\gamma = \eta$ and $\eta/p \in \mathbb{Z}_p$. On the other hand, by (4.1), the second term on the right-hand side of (4.11) is equal to

$$\frac{1}{2} \left(\langle 2 \rangle^{m+n+1} \omega^{k+l+1}(2) - 1 \right) L_p(-m-n; \omega^{k+l+1}),$$

because $\omega^{m+n+1} = \omega^{k+l+1}$. Therefore (4.10) holds for $(s_1, s_2) = (-m, -n)$ when $(m, n) \in (k + (p-1)\mathbb{N}) \times (l + (p-1)\mathbb{N})$. Since $(k + (p-1)\mathbb{N}) \times (l + (p-1)\mathbb{N})$ is p-adically dense in $\mathbb{Z}_p \times \mathbb{Z}_p$, we complete the proof. □

From the above proof we can see that the key underlying fact is that $G(t) - 1/2$ is an odd function, namely $B_{2m+1} = 0$ $(m \in \mathbb{N})$. In other words, we can say that Proposition 4.1 and Theorem 4.2 are consequences of the existence of trivial zeros of the Riemann zeta-function.

It should be noted that (4.10) does not hold when $k+l$ is even. In fact, if $k+l$ is even then $L_p(s; \omega^{k+l+1})$ is the zero function (see [29, Remark after Theorem 5.11]). On the other hand, we can see that $L_{p,2}(s_1, s_2; \omega^k, \omega^l; \eta)$ is not the zero function (see below). This phenomenon implies that $L_{p,2}(s_1, s_2; \omega^k, \omega^l; \eta)$ contains information outside the scope of the theory of the Kubota-Leopoldt p-adic L-functions.

In order to confirm this fact, we again consider I (defined by (4.11)) in the case when $k + l$ is even, where the second term vanishes because $L_p(s; \omega^{k+l+1})$ is the zero function. Hence we have $I = L_{p,2}(-m, -n; \omega^k, \omega^l; \eta)$ by (4.11). Therefore, combining (4.2), (4.9) and (4.12), we obtain the following.

Proposition 4.3. *Let $k, l \in \mathbb{N}$ with $k \equiv l \pmod 2$. For $m, n \in \mathbb{N}$ with $m \equiv k$*

(mod $(p-1)$) *and* $n \equiv l$ (mod $(p-1)$),

$$L_{p,2}(-m,-n;\omega^k,\omega^l;\eta) \\ = \sum_{j=1}^{n}\binom{n}{j}\frac{\left(2^{m+n+1-j}-1\right)\left(2^{j+1}-1\right)}{(m+n+1-j)(j+1)}B_{m+n+1-j}B_{j+1}\eta^j\left(1-p^{m+n-j}\right). \tag{4.13}$$

For example, we have

$$L_{p,2}(-1,-1;\omega,\omega;\eta) = \frac{9(1-p)}{4}B_2^2\eta, \tag{4.14}$$

$$L_{p,2}(-2,-2;\omega^2,\omega^2;\eta) = \frac{45(1-p^3)}{8}B_2B_4\eta \tag{4.15}$$

which do not vanish.

Remark 4.4. In [10,11], Imai considered a certain multiple analogue of the p-adic L-function. His theory shares some common feature with ours. In fact, he constructed a certain multiple analogue of the Kubota-Leopoldt p-adic L-function by using the p-adic measure and the theory of Γ-transforms (see [15, Chapter 2], [29, Chapter 12]).

For this decade, certain p-adic multiple zeta values have been investigated by Furusho, Jafari, and so on (see, for example, [7,8,9]). Our p-adic double L-functions are constructed in a totally different way from their work. Hence it is unclear whether there is some connection between their theory and ours.

5. Double Hurwitz-Lerch zeta-functions

Define the double zeta-function of Hurwitz-Lerch type by

$$\zeta_2(s_1,s_2;\alpha,\beta;\omega_1,\omega_2) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{e^{2\pi in\beta}}{((\alpha+m)\omega_1)^{s_1}((\alpha+m)\omega_1+n\omega_2)^{s_2}} \tag{5.1}$$

for $0<\alpha\le 1$, $0\le\beta\le 1$, and $\omega_1,\omega_2\in\mathbb{C}$ with $\Re\omega_1>0$, $\omega_2>0$. We write $\theta_j = \arg\omega_j$ $(j=1,2)$. When $\omega_1=1$ and $\omega_2>0$, this double zeta-function was first introduced in [21]. In this section we prove the functional equation for $\zeta_2(s_1,s_2;\alpha,\beta;\omega_1,\omega_2)$, which is necessary for the proof of Theorem 2.1.

Let $\Psi(b,c;x)$ be the confluent hypergeometric function defined by

$$\Psi(b,c;x) = \frac{1}{\Gamma(b)}\int_0^{e^{i\theta}\infty} e^{-xy}y^{b-1}(1+y)^{c-b-1}dy, \tag{5.2}$$

where $\Re b>0$, $-\pi<\theta<\pi$, $|\theta+\arg x|<\pi/2$ (see Erdélyi et al. [6, formula 6.5 (3)]), and

$$\sigma_{s_1+s_2-1}(k;\alpha,\beta) = \sum_{d|k} e^{2\pi id\alpha}e^{2\pi i(k/d)\beta}d^{s_1+s_2-1}$$

for $k \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$. Define

$$F_{\pm}(s_1, s_2; \alpha, \beta; \omega) = \sum_{k=1}^{\infty} \sigma_{s_1+s_2-1}(k; \alpha, \beta) \Psi(s_2, s_1 + s_2; 2\pi k \omega e^{\pm \pi i/2}) \tag{5.3}$$

for $\omega \in \mathbb{C}$, $-\pi < \arg \omega < \pi$. This function is periodic of period 1 with respect to both α and β. Let $\phi(s, \beta) = \sum_{m=1}^{\infty} e^{2\pi i m \beta} m^{-s}$ be the Lerch zeta-function, which is known to be continued meromorphically to the whole of $\mathbb{C}$, and is entire unless $\beta \in \mathbb{Z}$.

Lemma 5.1. *The function $F_{\pm}(s_1, s_2; \alpha, \beta; \omega)$ can be continued meromorphically (as a function in s_1, s_2) to the whole space $\mathbb{C}^2$, and the following functional equation holds:*

$$F_{\pm}(1 - s_2, 1 - s_1; \beta, \alpha; \omega) = (\pm 2\pi i \omega)^{s_1+s_2-1} F_{\pm}(s_1, s_2; \alpha, \beta; \omega). \tag{5.4}$$

Lemma 5.2. *The function $\zeta_2(s_1, s_2; \alpha, \beta; \omega_1, \omega_2)$ can be continued meromorphically (as a function in s_1, s_2) to the whole space $\mathbb{C}^2$, and the following functional equation holds:*

$$\begin{aligned}
&\zeta_2(s_1, s_2; \alpha, \beta; \omega_1, \omega_2) \\
&= \frac{\Gamma(1 - s_1)}{\Gamma(s_2)} \Gamma(s_1 + s_2 - 1) \phi(s_1 + s_2 - 1, \beta) \omega_1^{-1} \omega_2^{1-s_1-s_2} \\
&\quad + \Gamma(1 - s_1) \omega_1^{-1} \omega_2^{1-s_1-s_2} \\
&\quad \times \left\{ F_{+}\left(1 - s_2, 1 - s_1; \beta, \alpha, \omega_2/\omega_1\right) + F_{-}\left(1 - s_2, 1 - s_1; \beta, -\alpha, \omega_2/\omega_1\right) \right\}.
\end{aligned} \tag{5.5}$$

When $\omega_1 = 1$ and $\omega_2 > 0$, these two lemmas are essentially included in [21]. Indeed, Lemma 5.1 in this case is Proposition 2 of [21]. Lemma 5.2 in this case is not explicitly stated in [21], but it can be immediately shown from Propositions 1 and 2 of [21]. The form of Lemma 5.2 was first stated in [22]; see also [17, (1.8)].

The following proof of these two lemmas is a generalization of that in [20] [21], so we omit some parts of the proof. However we describe the details of some other parts carefully, where the existence of complex parameters causes new complication.

Let us start the proof. The series (5.1) is absolutely convergent in the region (1.3). We first show that, in the subregion $\sigma_1 > 0, \quad \sigma_2 > 1, \quad \sigma_1 + \sigma_2 > 2$, the following integral expression holds:

$$\begin{aligned}
&\zeta_2(s_1, s_2; \alpha, \beta; \omega_1, \omega_2) \\
&= \frac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^{\infty} \frac{y^{s_2-1}}{e^{\omega_2 y - 2\pi i \beta} - 1} \int_0^{\infty} \frac{e^{(1-\alpha)\omega_1(x+y)} x^{s_1-1}}{e^{\omega_1(x+y)} - 1} dx dy.
\end{aligned} \tag{5.6}$$

In fact, (5.6) is a generalization of [20, (3.4)], and the proof is quite similar to that described there. (On line 16, p.391 of [20], the term $+w$ in the exponent is to be deleted.) A small difference is that, when one tries to mimic the argument on line 10, p.391 of [20], one encounters the path of integration which is the half-line from the origin to $e^{i\theta_1}\infty$. But we can rotate this half-line to the positive real axis, because $|\theta_1| < \pi/2$.

Next, let

$$h(z,\alpha) = \frac{e^{(1-\alpha)\omega_1 z}}{e^{\omega_1 z}-1} - \frac{1}{\omega_1 z},$$

and divide the right-hand side of (5.6) as

$$\begin{aligned}&\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^\infty \frac{y^{s_2-1}}{e^{\omega_2 y-2\pi i\beta}-1}\int_0^\infty h(x+y,\alpha)x^{s_1-1}dxdy\\ &+\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^\infty \frac{y^{s_2-1}}{e^{\omega_2 y-2\pi i\beta}-1}\int_0^\infty \frac{x^{s_1-1}}{\omega_1(x+y)}dxdy\\ &= g_1+g_2,\end{aligned} \tag{5.7}$$

say. Since $h(x+y,\alpha) = O(|x+y|^{-1})$ if $|x+y|$ is large and $h(x+y,\alpha) = O(1)$ if $|x+y|$ is small, we see that g_1 is absolutely convergent when $0 < \sigma_1 < 1$, $\sigma_2 > 1$. On the other hand, if $y > 0$ and $0 < \sigma_1 < 1$, then from the beta integral formula we can easily deduce

$$\int_0^\infty \frac{x^{s_1-1}}{x+y}dx = y^{s_1-1}\Gamma(s_1)\Gamma(1-s_1).$$

Hence g_2 is convergent when $0 < \sigma_1 < 1$, $\sigma_1+\sigma_2 > 2$, and in this region we have

$$\begin{aligned}g_2 &= \frac{\Gamma(1-s_1)}{\omega_1\Gamma(s_2)}\int_0^\infty \frac{y^{s_1+s_2-2}}{e^{\omega_2 y-2\pi i\beta}-1}dy\\ &= \frac{\Gamma(1-s_1)}{\Gamma(s_2)}\omega_1^{-1}\omega_2^{1-s_1-s_2}\Gamma(s_1+s_2-1)\phi(s_1+s_2-1,\beta),\end{aligned} \tag{5.8}$$

where the last equality is proved by an argument again similarly to that on line 10, p.391 of [20]. Therefore the decomposition (5.7) is now established in the region $0 < \sigma_1 < 1$, $\sigma_1+\sigma_2 > 2$.

Let Υ be the contour as in Section 3. Then we have

$$\begin{aligned}g_1 = &\frac{1}{\Gamma(s_1)\Gamma(s_2)\left(e^{2\pi i s_1}-1\right)\left(e^{2\pi i s_2}-1\right)}\\ &\times\int_\Upsilon \frac{y^{s_2-1}}{e^{\omega_2 y-2\pi i\beta}-1}\int_\Upsilon h(x+y,\alpha)x^{s_1-1}dxdy.\end{aligned} \tag{5.9}$$

The right-hand side of (5.9) is convergent for $\sigma_1 < 1$ and any $s_2 \in \mathbb{C}$. Combining this fact with (5.7) and (5.8), we find that $\zeta_2(s_1,s_2;\alpha,\beta;\omega_1,\omega_2)$ can be continued meromorphically to $\Re s_1 < 1$ and any $s_2 \in \mathbb{C}$.

Lemma 5.3. *For $\Re s_1 < 0$ and $\Re s_2 > 1$,*

$$\begin{aligned}g_1 = &(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)\omega_1^{-s_1-s_2}\\ &\times\left\{e^{\pi i(1-s_1-s_2)/2}F_-\left(s_1,s_2;-\alpha,\beta;\frac{\omega_2}{\omega_1}\right)+e^{\pi i(s_1+s_2-1)/2}F_+\left(s_1,s_2;\alpha,\beta;\frac{\omega_2}{\omega_1}\right)\right\}.\end{aligned} \tag{5.10}$$

Proof. First we replace the contour Υ of the inner integral on the right-hand side of (5.9) by

$$\Upsilon_R = \{x = -y + 2\pi(R + 1/2)|\omega_1|^{-1}e^{i\phi} \mid 0 \le \phi < 2\pi\} \quad (R \in \mathbb{N}),$$

and let $R \to \infty$. A crucial point is that $h(x + y, \alpha) = O(1)$ for $x \in \Upsilon_R$. This can be proved similarly to [20, (5.2)], by considering two cases $\cos(\theta_1 + \phi) \le 0$ and $\cos(\theta_1 + \phi) > 0$ separately. Therefore, since $\sigma_1 < 0$, by residue calculus similar to the proof of [20, (5.3)], we obtain

$$g_1 = \frac{-2\pi i\omega_1^{-1}}{\Gamma(s_1)\Gamma(s_2)\,(e^{2\pi i s_1} - 1)\,(e^{2\pi i s_2} - 1)} \sum_{n\neq 0} e^{-2\pi i n\alpha} I_n, \tag{5.11}$$

if the summation and the integration can be interchanged, where

$$I_n = \int_\Upsilon \frac{y^{s_2-1}}{e^{\omega_2 y - 2\pi i\beta} - 1} \left(-y + \frac{2\pi i n}{\omega_1}\right)^{s_1-1} dy. \tag{5.12}$$

Consider the case $n > 0$. Since $\Re\omega_1 > 0$, the point $-y + 2\pi i n\omega_1^{-1}$ is in the upper half-plane, so we may write $-y + 2\pi i n\omega_1^{-1} = e^{\pi i}y + 2\pi n e^{\pi i/2}\omega_1^{-1}$. Replacing the contour Υ on the right-hand side of (5.12) by the half-line $[0, \infty)$ (which can be done because $\sigma_2 > 1$), and putting $y = 2\pi n\omega_1^{-1}e^{-i\pi/2}z$, we obtain

$$\begin{aligned} I_n &= (e^{2\pi i s_2} - 1)(2\pi n\omega_1^{-1})^{s_1+s_2-1} e^{\pi i(s_1-s_2-1)/2} \\ &\quad \times \int_0^{e^{i(\theta_1+\pi/2)}\infty} \frac{z^{s_2-1}(1 + z)^{s_1-1}}{\exp(2\pi n\omega_1^{-1}\omega_2 e^{-\pi i/2}z - 2\pi i\beta) - 1} dz \\ &= (e^{2\pi i s_2} - 1)(2\pi n\omega_1^{-1})^{s_1+s_2-1} e^{\pi i(s_1-s_2-1)/2} \\ &\quad \times \sum_{r=1}^{\infty} e^{2\pi i r\beta} \int_0^{e^{i(\theta_1+\pi/2)}\infty} z^{s_2-1}(1 + z)^{s_1-1} \exp\left(-2\pi n r\omega_1^{-1}\omega_2 e^{-\pi i/2}z\right) dz. \end{aligned} \tag{5.13}$$

To show the second equality of the above, we used the expansion

$$\frac{1}{1 - \exp(2\pi i\beta - 2\pi n\omega_1^{-1}\omega_2 e^{-i\pi/2}z)} = \sum_{r=0}^{\infty} e^{2\pi i r\beta} \exp\left(-2\pi n r\omega_1^{-1}\omega_2 e^{-i\pi/2}z\right),$$

which is valid because $\arg(2\pi n r\omega_1^{-1}\omega_2 e^{-i\pi/2}z) = \theta_2$ and $|\theta_2| < \pi/2$, and then changed the summation and integration. Summing up with respect to $n(> 0)$ and putting $nr = k$, we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} e^{-2\pi i n\alpha} I_n \\ &= \left(e^{2\pi i s_2} - 1\right) e^{\pi i(s_1-s_2-1)/2} \left(\frac{2\pi}{\omega_1}\right)^{s_1+s_2-1} \sum_{k=1}^{\infty} \sigma_{s_1+s_2-1}(k; -\alpha, \beta) \\ &\times \int_0^{e^{i(\theta_1+\pi/2)}\infty} z^{s_2-1}(1 + z)^{s_1-1} \exp\left(-2\pi k\omega_1^{-1}\omega_2 e^{-\pi i/2}z\right) dz. \end{aligned} \tag{5.14}$$

The above interchanges (two times) of summation and integration can be justified by checking the absolute convergence. This kind of argument of getting formulas similar to (5.14) by residue calculus goes back to Section 5 of Katsurada and Matsumoto [14].

The above integral can be written as $\Gamma(s_2)\Psi(s_2, s_1 + s_2; 2\pi k\omega_1^{-1}\omega_2 e^{-\pi i/2})$, because $|(\theta_1 + \pi/2) + \arg(2\pi k\omega_1^{-1}\omega_2 e^{-\pi i/2})| = |\theta_2| < \pi/2$. Therefore we find that

$$\begin{aligned}&\sum_{n=1}^{\infty} e^{-2\pi i n\alpha} I_n \\ &= \left(e^{2\pi i s_2} - 1\right) e^{\pi i(s_1 - s_2 - 1)/2} \left(\frac{2\pi}{\omega_1}\right)^{s_1+s_2-1} \Gamma(s_2) F_- \left(s_1, s_2; -\alpha, \beta; \frac{\omega_2}{\omega_1}\right).\end{aligned} \tag{5.15}$$

If $n < 0$, then $-y + 2\pi i n\omega_1^{-1} = e^{\pi i} y + 2\pi|n|e^{3\pi i/2}\omega_1^{-1}$. Putting $y = 2\pi|n|\omega_1^{-1}e^{\pi i/2}z$ and proceeding similarly as above, we obtain

$$\begin{aligned}&\sum_{n=-\infty}^{-1} e^{-2\pi i n\alpha} I_n \\ &= \left(e^{2\pi i s_2} - 1\right) e^{\pi i(3s_1 + s_2 - 3)/2} \left(\frac{2\pi}{\omega_1}\right)^{s_1+s_2-1} \Gamma(s_2) F_+ \left(s_1, s_2; \alpha, \beta; \frac{\omega_2}{\omega_1}\right).\end{aligned} \tag{5.16}$$

Substituting (5.15) and (5.16) into the right-hand side of (5.11), and noting

$$\frac{1}{\Gamma(s_1)\left(e^{2\pi i s_1} - 1\right)} = \frac{\Gamma(1 - s_1)}{2\pi i e^{\pi i s_1}},$$

we obtain the proof of this lemma. □

Now we prove Lemmas 5.1 and 5.2. Properties of $F_\pm(s_1, s_2; \alpha, \beta; \omega)$ have already been studied in [21] when $\omega > 0$. To prove Lemma 5.1 we have to consider the case $\omega = \omega_2/\omega_1$, which is not necessarily real, but the argument is almost the same. Therefore we omit the details. We only quote here formula [21, (3.7)], which is valid in our present situation, and is necessary in the next section:

$$\begin{aligned}&F_\pm(s_1, s_2; \alpha, \beta; \omega) \\ &= \sum_{j=1}^{N-1} \frac{(-1)^j}{j!}(1 - s_1)_j (s_2)_j (2\pi e^{\pm i\pi/2}\omega)^{-s_2-j}\phi(j - s_1 + 1, \alpha)\phi(j + s_2, \beta) \\ &\quad + (2\pi e^{\pm i\pi/2}\omega)^{1-s_1-s_2} \sum_{k=1}^{\infty} \sigma_{s_1+s_2-1}(k; \alpha, \beta) k^{1-s_1-s_2} \\ &\qquad\quad \times \rho_N(1 - s_1, 2 - s_1 - s_2; 2\pi e^{\pm i\pi/2} k\omega),\end{aligned} \tag{5.17}$$

where $N \in \mathbb{N}$, $(a)_j = \Gamma(a + j)/\Gamma(a)$, ρ_N is given by [21, (3.3)], and the infinite sum on the right-hand side is absolutely convergent, and is holomorphic in s_1 and s_2, in the region $\sigma_1 < N$ and $\sigma_2 > -N + 1$.

The meromorphic continuation of $F_{\pm}(s_1, s_2; \alpha, \beta; \omega)$ implies, via Lemma 5.3, the meromorphic continuation of $\zeta_2(s_1, s_2; \alpha, \beta; \omega_1, \omega_2)$. Lastly, applying (5.4) to the right-hand side of (5.10), we obtain

$$\begin{aligned} g_1 &= \Gamma(1-s_1)\omega_1^{-1}\omega_2^{1-s_1-s_2} \\ &\quad \times \left\{ F_+ \left(1-s_2, 1-s_1; \beta, \alpha; \frac{\omega_2}{\omega_1}\right) + F_- \left(1-s_2, 1-s_1; \beta, -\alpha; \frac{\omega_2}{\omega_1}\right)\right\}. \end{aligned} \tag{5.18}$$

Formula (5.5) immediately follows from this and (5.6), (5.7), (5.8).

6. Proof of Theorem 2.1

Now we give the proof of Theorem 2.1. We first show the fact that our $L_2^{\text{ш}}(s_1, s_2; a_1, a_2; \omega_1, \omega_2)$ can be written as a linear combination of $\zeta_2(s_1, s_2; \alpha, \beta; \omega_1 . \omega_2)$.

Lemma 6.1.

$$L_2^{\text{ш}}(s_1, s_2; a_1, a_2; \omega_1, \omega_2) = \frac{1}{f^{s_1+s_2}} \sum_{\nu=1}^{f} \sum_{\rho=1}^{f} a_1(\nu)\widehat{a}_2(\rho)\zeta_2\Big(s_1, s_2; \frac{\nu}{f}, \frac{\rho}{f}; \omega_1, \frac{\omega_2}{f}\Big). \tag{6.1}$$

Proof. Putting $m = \nu + fj$ $(1 \le \nu \le f,\ j \ge 0)$ in (1.2), we have

$$\begin{aligned} &L_2^{\text{ш}}(s_1, s_2; a_1, a_2; \omega_1, \omega_2) \\ &= \sum_{\nu=1}^{f} \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} \frac{a_1(\nu)a_2(n)}{((\nu+fj)\omega_1)^{s_1}((\nu+fj)\omega_1 + n\omega_2)^{s_2}} \\ &= \frac{1}{f^{s_1+s_2}} \sum_{\nu=1}^{f} \sum_{\rho=1}^{f} \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} \frac{a_1(\nu)\widehat{a}_2(\rho)e^{2\pi i\rho n/f}}{((\nu/f+j)\omega_1)^{s_1}((\nu/f+j)\omega_1 + n\omega_2/f)^{s_2}} \\ &= \frac{1}{f^{s_1+s_2}} \sum_{\nu=1}^{f} \sum_{\rho=1}^{f} a_1(\nu)\widehat{a}_2(\rho)\zeta_2\Big(s_1, s_2; \frac{\nu}{f}, \frac{\rho}{f}; \omega_1, \frac{\omega_2}{f}\Big), \end{aligned}$$

where we used (2.1) for the second equality. □

From Lemmas 5.2 and 6.1, the assertion on the meromorphic continuation in Theorem 2.1 immediately follows. To prove the functional equation, we use (6.1) and (5.5) to obtain

$$\begin{aligned}
&L_2^{ш}(s_1,s_2;a_1,a_2;\omega_1,\omega_2)\\
&=\frac{\omega_2^{1-s_1-s_2}}{f\omega_1}\sum_{\nu=1}^{f}\sum_{\rho=1}^{f}a_1(\nu)\widehat{a}_2(\rho)\frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)\phi\left(s_1+s_2-1,\frac{\rho}{f}\right)\\
&\quad+\frac{\omega_2^{1-s_1-s_2}}{f\omega_1}\sum_{\nu=1}^{f}\sum_{\rho=1}^{f}a_1(\nu)\widehat{a}_2(\rho)\Gamma(1-s_1)\\
&\quad\times\left\{F_+\left(1-s_2,1-s_1;\frac{\rho}{f},\frac{\nu}{f};\frac{\omega_2}{f\omega_1}\right)+F_-\left(1-s_2,1-s_1;\frac{\rho}{f},-\frac{\nu}{f};\frac{\omega_2}{f\omega_1}\right)\right\}\\
&=\Sigma_1+\Sigma_2,
\end{aligned}\tag{6.2}$$

say. From (2.1) we see that

$$\sum_{\rho=1}^{f}\widehat{a}_2(\rho)\phi\left(s,\frac{\rho}{f}\right)=\sum_{m=1}^{\infty}\frac{1}{m^s}\sum_{\rho=1}^{f}\widehat{a}_2(\rho)e^{2\pi im\rho/f}=L(s,a_2)\tag{6.3}$$

(this is Corollary 1 in [12, Section 1]), by which we find that

$$\Sigma_1=\frac{\omega_2^{1-s_1-s_2}}{f\omega_1}\frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)L(s_1+s_2-1,a_2)\sum_{\nu=1}^{f}a_1(\nu).\tag{6.4}$$

By Lemma 5.1, we see that

$$\begin{aligned}
\Sigma_2&=\frac{\omega_2^{1-s_1-s_2}}{f\omega_1}\Gamma(1-s_1)\sum_{\nu=1}^{f}\sum_{\rho=1}^{f}a_1(\nu)\widehat{a}_2(\rho)\left(\frac{2\pi i\omega_2}{f\omega_1}\right)^{s_1+s_2-1}F_+\left(s_1,s_2;\frac{\nu}{f},\frac{\rho}{f};\frac{\omega_2}{f\omega_1}\right)\\
&+\frac{\omega_2^{1-s_1-s_2}}{f\omega_1}\Gamma(1-s_1)\sum_{\nu=1}^{f}\sum_{\rho=1}^{f}a_1(\nu)\widehat{a}_2(\rho)\left(-\frac{2\pi i\omega_2}{f\omega_1}\right)^{s_1+s_2-1}F_-\left(s_1,s_2;-\frac{\nu}{f},\frac{\rho}{f};\frac{\omega_2}{f\omega_1}\right).
\end{aligned}\tag{6.5}$$

In the second double sum of (6.5), we replace indices (ν,ρ) by $(f-\nu,f-\rho)$. Noting the periodicity of a_1, $\widehat{a}_2$ and $F_\pm$ (with respect to α, β) and the fact $\lambda(a_2)=\lambda(\widehat{a}_2)$, we find that the second term of (6.5) is

$$\begin{aligned}
&\frac{\omega_2^{1-s_1-s_2}}{f\omega_1}\Gamma(1-s_1)\sum_{\nu=1}^{f}\left(-\frac{2\pi i\omega_2}{f\omega_1}\right)^{s_1+s_2-1}\\
&\times\sum_{\rho=1}^{f}a_1(\nu)\widehat{a}_2(\rho)\lambda(a_1)\lambda(a_2)F_-\left(s_1,s_2;\frac{\nu}{f},-\frac{\rho}{f};\frac{\omega_2}{f\omega_1}\right).
\end{aligned}$$

Now we consider the situation when (s_1,s_2) is on the hyperplanes $s_1+s_2=2k+1$ ($k\in\mathbb{Z}$) if $\lambda(a_1)\lambda(a_2)=1$, or on $s_1+s_2=2k$ ($k\in\mathbb{Z}$) if $\lambda(a_1)\lambda(a_2)=-1$. Then

$$\lambda(a_1)\lambda(a_2)(-1)^{s_1+s_2-1}=1,$$

and so (6.5) can be rewritten to

$$\begin{aligned}&\frac{1}{(f\omega_1)^{s_1+s_2}}\Gamma(1-s_1)\,(2\pi i)^{s_1+s_2-1}\sum_{\nu=1}^{f}\sum_{\rho=1}^{f}a_1(\nu)\widehat{a}_2(\rho)\\ &\times\left\{F_+\left(s_1,s_2;\frac{\nu}{f},\frac{\rho}{f};\frac{\omega_2}{f\omega_1}\right)+F_-\left(s_1,s_2;\frac{\nu}{f},-\frac{\rho}{f};\frac{\omega_2}{f\omega_1}\right)\right\}.\end{aligned}\tag{6.6}$$

We again apply (5.5) with replacing (s_1,s_2) by $(1-s_2,1-s_1)$ to (6.6). Then (6.6) can be rewritten to

$$\begin{aligned}&(2\pi i)^{s_1+s_2-1}\left\{\frac{\Gamma(1-s_1)}{f\Gamma(s_2)}\omega_2^{1-s_1-s_2}\sum_{\nu=1}^{f}\sum_{\rho=1}^{f}a_1(\nu)\widehat{a}_2(\rho)\zeta_2\left(1-s_2,1-s_1;\frac{\rho}{f},\frac{\nu}{f},\frac{\omega_2}{f\omega_1}\right)\right.\\ &\left.-\frac{\omega_1^{1-s_1-s_2}}{f^{s_1+s_2}}\Gamma(1-s_1-s_2)\sum_{\nu=1}^{f}\sum_{\rho=1}^{f}a_1(\nu)\widehat{a}_2(\rho)\phi\left(1-s_1-s_2,\frac{\nu}{f}\right)\right\}\\ &=(2\pi i)^{s_1+s_2-1}\left\{(f\omega_2)^{1-s_1-s_2}\frac{\Gamma(1-s_1)}{\Gamma(s_2)}L_2^{ш}(1-s_2,1-s_1;\widehat{a}_2,\widehat{a}_1)\right.\\ &\left.-\frac{\omega_1^{1-s_1-s_2}}{f^{s_1+s_2}}\Gamma(1-s_1-s_2)L(1-s_1-s_2,\widehat{a}_1)\sum_{\rho=1}^{f}\widehat{a}_2(\rho)\right\}.\end{aligned}\tag{6.7}$$

Substituting (6.4) and (6.7) into (6.2), we obtain formula (2.7). This completes the proof of Theorem 2.1.

Remark 6.2. Finally we discuss whether hyperplanes $s_1+s_2=2k+1$, $s_1+s_2=2k$ are singular or not. It is clear from (5.17) that a hyperplane of the form $s_1+s_2=l$ $(l\in\mathbb{Z})$ is not a singular locus of Σ_2. Formula (2.7) is actually a relation between two Σ_2s, so in this sense (2.7) is valid for any $k\in\mathbb{Z}$.

However double L-functions themselves may be singular on those hyperplanes. In fact, (6.4) shows that Σ_1 may have a singular locus of the form $s_1+s_2=l$. If $\widehat{a}_1(0)=0$, then $\Sigma_1\equiv 0$, hence there is no singular locus. (This is also seen from Theorem 3.1.) Consider the case $\widehat{a}_1(0)\neq 0$. Formula (6.3) implies that $L(s,a_2)$ can be continued meromorphically to the whole of $\mathbb{C}$. It is entire if $\widehat{a}_2(f)=\widehat{a}_2(0)=0$. If not, then $L(s,a_2)$ has a pole of order 1 at $s=1$ with the residue $\widehat{a}_2(0)$. Therefore, if $\widehat{a}_2(0)\neq 0$, then the hyperplane $s_1+s_2=2$ is a singular locus of Σ_1. Also, the gamma factor $\Gamma(s_1+s_2-1)$ is singular (of order 1) when $s_1+s_2=1-m$, $m\in\mathbb{N}_0$. If $L(-m,a_2)=0$, then this singular locus is cancelled, but if not, this is also a singular locus of $L_2^{ш}(s_1,s_2;\alpha,\beta;\omega_1,\omega_2)$.

Remark 6.3. It is desirable to generalize our result in the present paper to the case of more general multiple sums. Recently we proved a certain functional equation for Barnes multiple zeta-functions (see [18]). The method there is also based on the residue calculus, but it seems impossible to use this method in order to prove functional equations for multiple sums in the present context. Also, the method suggested at the end of [21] has not been successful yet.

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