Homological problems for cycle-finite algebras

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Outline

Basic concepts

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Preliminaries

Let A be an algebra, that is an artin K-algebra, where K is a commutative artin ring.

- By $\operatorname{mod} A$ we denote the category of finitely generated right A-modules, and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by all indecomposable modules.
- For a module X in $\operatorname{mod} A$, we denote by $\operatorname{pd}_A X$ (respectively, by $\operatorname{id}_A X$) the projective (respectively, injective) dimension of X.
- $\tau_A = D\operatorname{Tr}$ is the **Auslander-Reiten translation** ($D = \operatorname{Hom}_K(-, E)$ is the standard duality on $\operatorname{mod} A$, E is a minimal injective cogenerator in $\operatorname{mod} K$)
- We denote by Γ_A the **Auslander-Reiten quiver** of A.

Preliminaries

- rad_A is the **Jacobson radical** of $\mathrm{mod}\, A$, that is, the ideal in $\mathrm{mod}\, A$ generated by all nonisomorphisms in $\mathrm{ind}\, A$; $\mathrm{rad}_A^\infty = \bigcap_{i=1}^\infty \mathrm{rad}_A^i$ (the **infinite Jacobson radical**)

$\mathcal{C}$ is a component of $\Gamma_A \Rightarrow$

- $\mathcal{C}$ is **semiregular** iff $\mathcal{C}$ doesn't contain both a projective and an injective module.
- $\mathcal{C}$ is **generalized standard** iff $\mathrm{rad}_A^\infty(X, Y) = 0$, for all modules X and Y from $\mathcal{C}$.
- ${}_c\mathcal{C}$ is the **cyclic part** of $\mathcal{C}$ obtained from $\mathcal{C}$ by deleting vertices not lying on cycles (${}_c\Gamma_A$=**cyclic comoponents** of Γ_A). Recall $\mathcal{C}$ is **almost acyclic** iff ${}_c\mathcal{C}$ is finite
- $\mathcal{C}$ is a semiregular tube iff $\mathcal{C}$ is a **ray tube** (obtained from a stable tube by a finite number of ray insertions) or a **coray tube**

Classes of algebras

1) **Tilted algebras**

- A is called a tilted algebra, provided that $A \cong \mathrm{End}_H(T)$, where H is a hereditary algebra and T is a tilting module in mod H.
- $A = \mathrm{End}_H(T)$ is tilted $\Rightarrow$
  - there is a splitting torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ is mod A
  - Γ_A has a component $\mathcal{C}_T$ (called the **connecting component**) with a faithful section Δ such that the predecessors of Δ in $\mathcal{C}$ are in $\mathcal{Y}(T)$ and the proper successors of Δ (in $\mathcal{C}$) are in $\mathcal{X}(T)$
- We have the following characterization of tilted algebras **Theorem [Liu-Skowroński].** A is a tilted algebra iff Γ_A admits a generalized standard component with faithful section.
- If $A = \mathrm{End}_H(T)$ is a tilted algebra, then A is said to be of **Euclidean type** iff H is a hereditary algebra of Euclidean type. In particular, A is then a tame algebra.
- If A is a representation-infinite tilted algebra of Euclidean type, then

Classes of algebras

- one of the following holds:
  (1) $\Gamma_A = \mathcal{P}^A \cup \mathcal{T}^A \cup \mathcal{Q}^A$, where $\mathcal{P}^A$ is the postprojective component of Γ_A, $\mathcal{T}^A$ is an infinite family of pairwise orthogonal ray tubes, and $\mathcal{Q}^A$ is the preinjective component of Γ_A containing all injective A-modules.
  (2) $\Gamma_A = \mathcal{P}^A \cup \mathcal{T}^A \cup \mathcal{Q}^A$, where $\mathcal{P}^A$ is the postprojective component of Γ_A containing all projective A-modules, $\mathcal{T}^A$ is an infinite family of pairwise orthogonal coray tubes, and $\mathcal{Q}^A$ is the preinjective component of Γ_A.

2) Quasitilted algebras

- A is a **quasitilted algebra** iff $\mathrm{gl.dim}(A) \leqslant 2$ and, for every module X in $\operatorname{ind} A$, we have $\mathrm{pd}_A X \leqslant 1$ or $\mathrm{id}_A X \leqslant 1$.
- Every tilted algebra is a quasitilted algebra. If A is a quasitilted algebra but not a tilted algebra, then A is, so called, quasitilted algebra of canonical type.

Classes of algebras

- A is quasitilted of canonical type $\Rightarrow$ there are two associated factor algebras $A^{(l)}$ and $A^{(r)}$ of A which essentaily determine the structure of Γ_A. Moreover, if A is tame, then both $A^{(l)}$ and $A^{(r)}$ are tilted of Euclidean type or tubular algebras.
- A is tame quasitilted of canonical type $\Rightarrow$ all components of Γ_A are semiregular and

$$\Gamma_A = \mathcal{P}^A \cup \mathcal{T}^A \cup \mathcal{Q}^A,$$

  - $\mathcal{T}^A$ is an infinite family of pairwise orthogonal semiregular tubes of Γ_A.
  - If $A^{(l)}$ is tilted of Euclidean type, then $\mathcal{P}^A = \mathcal{P}^{A^{(l)}}$. Otherwise

$$\mathcal{P}^A = \mathcal{P}_0^{A^{(l)}} \cup \mathcal{T}_0^{A^{(l)}} \cup \left(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^{A^{(l)}} \right).$$

Classes of algebras

- If $A^{(r)}$ is tilted of Euclidean type, then $\mathcal{Q}^A = \mathcal{Q}^{A^{(r)}}$. Otherwise

$$\mathcal{Q}^A = \left(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^{A^{(r)}} \right) \cup \mathcal{T}_\infty^{A^{(r)}} \cup \mathcal{Q}_\infty^{A^{(r)}}.$$

- The family $\mathcal{P}^A$ is a family of components of $\Gamma_{A^{(l)}}$.
- The family $\mathcal{Q}^A$ is a family of components of $\Gamma_{A^{(r)}}$.

Classes of algebras

3) **Generalized double tilted algebras**

Theorem[Reiten-Skowroński]. A is a generalized double tilted algebra iff Γ_A admits a generalized standard component with a faithful multisection.

Recall that

- A **multisection** Δ in a component $\mathcal{C}$ of Γ_A is a full valued subquiver of $\mathcal{C}$ satisfying the following conditions.
  (a) Δ is almost acyclic;
  (b) Δ is convex in $\mathcal{C}$;
  (c) for each τ_A orbit $\mathcal{O}$ in $\mathcal{C}$, we have $1 \leqslant |\Delta \cap \mathcal{O}| < \infty$;
  (d) for all but finitely many τ_A orbits $\mathcal{O}$ in $\mathcal{C}$, we have $|\Delta \cap \mathcal{O}| = 1$;
  (e) no proper full valued subquiver of Δ satisfies conditions (a)-(d).
- A component $\mathcal{C}$ of Γ_A admits a multisection iff $\mathcal{C}$ is almost acyclic (**Reiten and Skowroński**).

Classes of algebras

- Δ is a multisection in a component $\mathcal{C} \Rightarrow$ there are associated full valued subquivers Δ_l, Δ_c, and Δ_r such that

$$\mathcal{C} = \mathcal{C}_l \cup \Delta_c \cup \mathcal{C}_r,$$

  where $\mathcal{C}_l$ (respectively, $\mathcal{C}_r$) is the full translation subquiver of $\mathcal{C}$ formed by predecessors of Δ_l (respectively, by successors of Δ_r)

4) **Cycle-finite algebras**

- A **cycle** in mod A is a sequence

$$X = X_0 \xrightarrow{f_1} X_1 \longrightarrow \dots \xrightarrow{f_r} X_r = X$$

  of nonzero nonisomorphisms in ind A
- Such a cycle is called **finite**, provided that $f_1, \dots, f_r \notin \mathrm{rad}_A^{\infty}$. Following Assem and Skowroński, an algebra A is said to be **cycle-finite**, iff all cycles in mod A are finite.

Classes of algebras

- The class of cycle-finite algebras is large and contains, for example: algebras of finite representation type, tame tilted algebras, tame generalized double tilted algebras, tubular algebras, tame quasitilted algebras, tame generalized multicoil algebras, and strongly simply connected algebras of polynomial growth.

Motivation

Recall that there are two important full subcategories of ind A, defined as follows:

- $\mathcal{L}_A$ is the full subcategory of ind A formed by all indecomposable modules X such that any predecessor Y of X in ind A satisfies $\operatorname{pd}_A Y \leqslant 1$;
- $\mathcal{R}_A$ is the full subcategory of ind A formed by all indecomposable modules X such that any successor Y of X in ind A satisfies $\operatorname{id}_A Y \leqslant 1$.

Motivation

The following theorem of Skowroński is the starting point of our considerations.

Theorem

For an algebra A, the following conditions are equivalent.

(i) *A is a generalized double tilted algebra or a quasitilted algebra.*

(ii) $\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ *is finite.*

(iii) *There are at most finitely many isomorphism classes of modules X in* $\operatorname{ind} A$ *lying on paths from an injective module to a projective module.*

Problems (1) and (2)

Note that, if one of the above statements (i)-(iii) hold, then the following two conditions are satisfied.

(H1) For all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$, we have $\operatorname{pd}_A X \leqslant 1$ or $\operatorname{id}_A X \leqslant 1$.

(H2) For all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$, we have $\operatorname{Hom}_A(D(A), X) = 0$ or $\operatorname{Hom}_A(X, A) = 0$.

We are interested in the following two problems, posed by Skowroński:

Problem (1). Let A be an algebra satisfying the condition (H1). Is then A a generalized double tilted algebra or a quasitilted algebra?

Problem (2). Let A be an algebra satisfying the condition (H2). Is then A a generalized double tilted algebra or a quasitilted algebra?

Theorem A

The following theorem provides the solution of the Problems (1) and (2), for cycle-finite algebras.

Theorem A

For a cycle-finite algebra, the following conditions are equivalent.

(i) *A is a generalized double tilted algebra or a quasitilted algebra.*

(ii) *For all but finitely many isomorphism classes of modules X in* $\operatorname{ind} A$, *we have* $\operatorname{pd}_A X \leqslant 1$ *or* $\operatorname{id}_A X \leqslant 1$.

(iii) *For all but finitely many isomorphism classes of modules X in* $\operatorname{ind} A$, *we have* $\operatorname{Hom}_A(D(A), X) = 0$ *or* $\operatorname{Hom}_A(X, A) = 0$.

A remark

There are weaker versions of homological conditions $(H1)$ and $(H2)$. Namely, we consider the following conditions:

$(H1^*)$ For all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$, we have $\operatorname{pd}_A X \leqslant 1$.

$(H1^{**})$ For all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$, we have $\operatorname{id}_A X \leqslant 1$.

$(H2^*)$ For all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$, we have $\operatorname{Hom}_A(D(A), X) = 0$.

$(H2^{**})$ For all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$, we have $\operatorname{Hom}_A(X, A) = 0$.

It has been proved by Skowroński that, if A is an algebra satisfying one of the above conditions $(H1^*)$, $(H1^{**})$, $(H2^*)$, $(H2^{**})$, then A is a generalized double tilted algebra.

Problem (3)

The origins of the third problem go back to the concept of a short chain, introduced and investigated by **Reiten, Smalø and Skowroński**. Recall that a **short chain** (in mod A) is a sequence $X \to M \to \tau_A X$ of nonzero homomorphisms in mod A with X being indecomposable, and M is then called the middle of this short chain. The following theorem due to **Jaworska, Malicki and Skowroński**, characterizes the class of tilted algebras in terms of short chains.

Theorem. Let A be an algebra. Then A is a tilted algebra if and only if mod A admits a faithful module which is not the middle of a short chain.

Problem (3)

The last (third) of the Skowroński's problems is formulated as follows.

Problem(3). A is a generalized double tilted algebra iff mod A admits a faithful module M which is the middle of at most finitely many short chains.

Theorem B

The following theorem provides the solution of Problem(3) for cycle-finite algebras.

Theorem B

Let A be a cycle-finite algebra. The following conditions are equivalent.

(i) *A is a generalized double tilted algebra.*

(ii) $\operatorname{mod} A$ *admits a faithful module being the middle of at most finitely many short chains.*

Proof of Theorem A: Semiregular case

Theorem A.1. Let A be a cycle-finite algebra such that all components of Γ_A are semiregular. Then tfcae

(i) A is a quasitilted algebra of cannonical type.

(ii) For all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$, we have $\operatorname{pd}_A X \leqslant 1$ or $\operatorname{id}_A X \leqslant 1$.

(iii) For all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$, we have $\operatorname{Hom}_A(D(A), X) = 0$ or $\operatorname{Hom}_A(X, A) = 0$.

Proof of Theorem A: Semiregular case

Obviously, implications (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii) hold. We assume that one of the conditions (ii) or (iii) holds. We need the following theorem.

Theorem[Białkowski, Skowroński, -, Wiśniewski] Let A be a cycle-finite algebra such that all components of Γ_A are semiregular. Then there is a sequence $\mathbb{B} = (B_1, \ldots, B_n)$ of tame quasitilted algebras of canonical type such that

(1) $B_1^{(l)}$ and $B_n^{(r)}$ are tilted algebras of Euclidean type and $B_i^{(r)} = B_{i+1}^{(l)}$ is a tubular algebra, for any $i \in \{1, \ldots, n-1\}$.

(2) Γ_A has the following form

$$\Gamma_A = \mathcal{P}^{\mathbb{B}} \cup \left(\bigcup_{q \in \mathbb{Q} \cap [1,n]} \mathcal{T}_q^{\mathbb{B}} \right) \cup \mathcal{Q}^{\mathbb{B}},$$

where

Proof of Theorem A: Semiregular case

- $\mathcal{P}^{\mathbb{B}} = \mathcal{P}^{B_1^{(l)}}$
- $\mathcal{Q}^{\mathbb{B}} = \mathcal{Q}^{B_n^{(r)}}$
- for each $i \in \{1, \dots, n\}$, $\mathcal{T}_i^{\mathbb{B}} = \mathcal{T}^{B_i}$
- for each rational number $q \in [1, n] \setminus \{1, \dots, n\}$, $\mathcal{T}_q^{\mathbb{B}}$ is an infinite family of pairwise orthogonal stable tubes of Γ_A

(3) A is isomorphic to the following pushout algebra:

$$A(\mathbb{B}) = B_1 \underset{B_1^{(r)}}{\sqcup} B_2 \underset{B_2^{(r)}}{\sqcup} \dots \underset{B_{n-2}^{(r)}}{\sqcup} B_{n-1} \underset{B_{n-1}^{(r)}}{\sqcup} B_n$$

Corrolary. Let $A = A(\mathbb{B})$ be a cycle-finite algebra with all components of Γ_A semiregular, $\mathbb{B} = (B_1, \dots, B_n)$. Then, if A is not a quasitilted algebra of canonical type, then $n \geqslant 2$ and there is $i \in \{1, \dots, n-1\}$ such that $\mathcal{T}_i^{\mathbb{B}}$ has a coray tube containing an injective module and $\mathcal{T}_{i+1}^{\mathbb{B}}$ has a ray tube containing a projective module.

Proof of Theorem A: Semiregular case

The above corrolary implies theorem A.1. Indeed, it follows that, if A is a cycle-finite algebra of semiregular type which is not a quasitilted algebra of canonical type, then there exists a stable tube $\mathcal{T}^{\mathbb{B}}_{q,\lambda}$ of $\mathcal{T}^{\mathbb{B}}_{q}$, $q \in (i, i+1)$, such that

$$\mathrm{Hom}_A(D(A), \mathcal{T}^{\mathbb{B}}_{q,\lambda}) \neq 0 \text{ and } \mathrm{Hom}_A(\mathcal{T}^{\mathbb{B}}_{q,\lambda}, A) \neq 0,$$

which leads to a contradiction with (ii) and (iii).

Proof of Theorem A: Non-semiregular case

The remaining statement is formulated as follows.

Theorem A.2. Let A be a cycle-finite algebra such that Γ_A admits a non-semiregular component. Then tfcae

(i) A is a generalized double tilted algebra.

(ii) For all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$, we have $\operatorname{pd}_A X \leqslant 1$ or $\operatorname{id}_A X \leqslant 1$.

(iii) For all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$, we have $\operatorname{Hom}_A(D(A), X) = 0$ or $\operatorname{Hom}_A(X, A) = 0$.

Proof of Theorem A: Non-semiregular case

As before, implications (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii) of theorem A.2 hold. Assume that one of the conditions (ii) or (iii) holds. The following proposition is playing a prominent role in this part of the proof of theorem A.

Proposition 1.
Let A be a cycle-finite algebra such that one of the conditions (H1) or (H2) is satisfied. Then every infinite cyclic component of Γ_A is the cyclic part ${}_c\mathcal{C}$ of a semiregular tube $\mathcal{C}$ of Γ_A.

Let $\mathcal{C}$ be a non-semiregular component of Γ_A. Proposition 1. implies that $\mathcal{C}$ is an almost acyclic component of Γ_A ($\Rightarrow \mathcal{C}$ admits a multisection).

Proof of Theorem A: Non-semiregular case

We prove that $\mathcal{C}$ is faithful and generalized standard component.
We investigate the structure of Γ_A.

- $\mathcal{C} = \mathcal{C}_l \cup \Delta_c \cup \mathcal{C}_r$
- $\mathcal{C}_l = \mathcal{C}_l^{(1)} \cup \cdots \cup \mathcal{C}_l^{(p)}$
- assume, for simplicity, that $\mathcal{C}_l^{(i)}$ is infinite, for all i's
- for all $i \in \{1, \ldots, p\}$, $\mathcal{C}_l^{(i)}$ admits a left stable acyclic full translation subquiver $\mathcal{D}^{(i)}$, closed under predecessors.

  We use the following theorem due to Malicki, de la Pẽna, and Skowroński

  Theorem. Let A be a cycle-finite algebra, $\mathcal{C}$ a component of Γ_A. Then, for every left stable acyclic full translation subquiver $\mathcal{D}$ of $\mathcal{C}$, closed under predecessors in $\mathcal{C}$, there exists a tilted algebra $B = \mathrm{End}_H(T)$ of Euclidean type such that $\mathcal{Y}(T) \cap \mathcal{C}_T$ is a full translation subquiver of $\mathcal{D}$ closed under predecessors.

Proof of Theorem A: Non-semiregular case

The above theorem implies that, for every $i \in \{1, \ldots, p\}$, there is a factor tilted algebra $B_i = \mathrm{End}_{H_i}(T_i)$ of A such that

- B_i is of Euclidean type
- $\mathcal{C}_{T_i} \cap \mathcal{Y}(T_i)$ is a full translation subquiver of $\mathcal{D}^{(i)}$ closed under predecessors in $\mathcal{C}$
- $B = B_1 \times \cdots \times B_p$ is a factor tilted algebra of A
- $\Gamma_{B_i} = \mathcal{P}^{B_i} \cup \mathcal{T}^{B_i} \cup \mathcal{Q}^{B_i}$, where $\mathcal{T}^{B_i}$ is a family of ray tubes, and $\mathcal{P}^{B_i}$ is the postprojective component, and $\mathcal{Q}^{B_i} = \mathcal{C}_{T_i}$ contains all injective modules in ind B_i,

- A is cycle-finite + Proposition 1. $\Rightarrow$ for every ray tube $\mathcal{T}_\lambda^B$ of $\mathcal{T}^B = \mathcal{T}^{B_1} \cup \cdots \cup \mathcal{T}^{B_p}$, there is a semiregular tube $\mathcal{T}_\lambda^A$ of Γ_A, containing all modules from ${}_c\mathcal{T}_\lambda^B$.

Proof of Theorem A: Non-semiregular case

- Next we prove that $\mathcal{T}^A = (\mathcal{T}^A_\lambda)$ has no coray tubes with injective modules.
  Assume that this is not the case. Then
  - there is a module V in $\operatorname{ind} B_i$, lying on the mouth of a stable tube of Γ_{B_i} and an epimorphism $I \to V$ with I an indecomposable injective A-module
  - there is a module R in $\mathcal{D}^{(i)}$ and a monomorphism $R \to P$ with P a projective module (in $\mathcal{C}$)

  But this leads to a contradiction with (i) and (ii), because the following lemma is satisfied.

Proof of Theorem A: Non-semiregular case

Lemma 1. Let B be a tilted algebra of Euclidean type with infinite preinjective connecting component. Assume that V and R are modules in $\operatorname{ind} B$ such that V lies on the mouth of a stable tube of Γ_B and R is contained in the preinjective component of Γ_B. Then the following holds.

(1) There are infinitely many pairwise nonisomorphic indecomposable modules Z_k in $\mathcal{Q}^B$, $k \geqslant 0$, such that

$$\operatorname{Hom}_B(V, \tau_B Z_k) \neq 0 \text{ and } \operatorname{Hom}_B(\tau_B^{-1} Z_k, R) \neq 0,$$

for all $k \geqslant 0$.

(2) There are infinitely many pairwise nonisomorphic indecomposable modules Z_k in $\mathcal{Q}^B$, $k \geqslant 0$, such that

$$\operatorname{Hom}_B(V, Z_k) \neq 0 \text{ and } \operatorname{Hom}_B(Z_k, R) \neq 0,$$

for all $k \geqslant 0$.

A remark

The proof of Lemma 1 is based on the tilting Theorem of **Brenner and Butler** and the tables of composition vectors of mouth modules over hereditary algebras of Euclidean type from Memoirs by **Dlab and Ringel**.

Proof of Theorem A: Non-semiregular case

- Further, we prove that $\mathcal{T}^A = \mathcal{T}^B$
- Dually, we prove that there exists a factor tilted algebra $B' = B'_1 \times \cdots \times B'_q$ of A such that the family $\mathcal{T}^{B'}$ of all coray tubes of $\Gamma_{B'}$ is a family of components of Γ_A.
- Finally, we deduce that $\Gamma_A = \mathcal{P}^B \cup \mathcal{T}^B \cup \mathcal{C} \cup \mathcal{T}^{B'} \cup \mathcal{Q}^{B'}$, where
  - $\mathcal{P}^B = \mathcal{P}^{B_1} \cup \cdots \cup \mathcal{P}^{B_p}$
  - $\mathcal{Q}^{B'} = \mathcal{Q}^{B'_1} \cup \cdots \cup \mathcal{Q}^{B'_q}$

Using obtained information on Γ_A, we deduce that $\mathcal{C}$ is a faithful and generalized standard component of Γ_A. Summarizing, $\mathcal{C}$ is a generalized standard component with a faithful multisection Δ. $\square$

Proof of Theorem B

The proof of Theorem B is very similar to the proof of Theorem A.2. Observe first that (i) implies (ii), because, if A is a generalized double tilted algebra, then mod A admits a faithful module M which is the middle of at most finitely many short chains (for example, M may be defined as the direct sum of all modules lying on Δ).

Now, assume that the condition (ii) is satisfied. We prove that A is a generalized double tilted algebra in the same manner as in the proof of Theorem A.2. Instead of the Proposition 1, we use the following proposition.

Proposition 2. Let A be a cycle-finite algebra such that the condition (ii) is satisfied. Then every infinite cyclic component of Γ_A is the cyclic part ${}_c\mathcal{C}$ of a semiregular tube $\mathcal{C}$ of Γ_A.

Proof of Theorem B

Analogue of Lemma 1 is stated as follows.

Lemma 2. Let B, V, and R be as in Lemma 1. Then the following holds.

(1) There are infinitely many pairwise nonisomorphic indecomposable modules Z_k in $\mathcal{Q}^B$, $k \geqslant 0$, such that

$$\mathrm{Hom}_B(V, \tau_B Z_k) \neq 0 \text{ and } \mathrm{Hom}_B(Z_k, R) \neq 0,$$

for all $k \geqslant 0$.

Thank you for your attention !!!