SPECIALIZATION ORDERS ON ATOM SPECTRA OF GROTHENDIECK CATEGORIES

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Abstract. This is a lecture note of my talk in Perspectives of Representation Theory of Algebras, Conference honoring Kunio Yamagata on the occasion of his 65th birthday (The 13th International Conference by Graduate School of Mathematics, Nagoya University) on November 11–15, 2013 in Nagoya University. The talk was based on [Kan13].

1. Introduction

Throughout this talk, let $\mathcal{A}$ be a Grothendieck category. It is known that the category $\text{Mod}\,A$ of right modules over a ring $A$ and the category $\text{QCoh}\,X$ of quasi-coherent sheaves on a scheme $X$ are Grothendieck categories (see [Con20, Lem 2.1.7]).

For a commutative ring $R$, we often consider the poset $(\text{Spec}\,R, \subseteq)$. Similarly, for a Grothendieck category $\mathcal{A}$, we construct a poset $(\text{ASpec}\,\mathcal{A}, \subseteq)$.

Theorem 1.1 (Hochster [Hoc69, Proposition 10] and Speed [Spe72, Corollary 1]). Let $P$ be a partially ordered set. Then $P$ is isomorphic to the prime spectrum of some commutative ring with the inclusion relation if and only if $P$ is an inverse limit of finite partially ordered sets in the category of partially ordered sets.

Theorem 1.2 ([Kan13, Theorem 7.27]). Any partially ordered set is isomorphic to the atom spectrum of some Grothendieck category as a partially ordered set.

2. Atom spectrum

Definition 2.1. An object $H$ in $\mathcal{A}$ is called monoform if for any nonzero subobject $L$ of $H$, there exists no common nonzero subobject of $H$ and $H/L$, that is, there does not exist a nonzero subobject of $H$ which is isomorphic to a subobject of $H/L$.

Definition 2.2. We say that monoform objects $H_1$ and $H_2$ in $\mathcal{A}$ are atom-equivalent if there exists a common nonzero subobject of $H_1$ and $H_2$.

Definition 2.3. Denote by $\text{ASpec}\,\mathcal{A}$ the quotient class of the class of monoform objects by the atom equivalence, and call it the atom spectrum of $\mathcal{A}$. We call an element of $\text{ASpec}\,\mathcal{A}$ an atom in $\mathcal{A}$. The equivalence class of a monoform object $H$ in $\mathcal{A}$ is denoted by $\overline{H}$.

Definition 2.4. Let $M$ be an object in $\mathcal{A}$. Define a subset $\text{ASupp}\,M$ of $\text{ASpec}\,\mathcal{A}$ by

$$\text{ASupp}\,M = \{ \alpha \in \text{ASpec}\,\mathcal{A} \mid \alpha = \overline{H} \text{ for a monoform subquotient } H \text{ of } M \},$$

and call it the atom support of $M$.

Proposition 2.5. Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\mathcal{A}$. Then we have $\text{ASupp}\,M = \text{ASupp}\,L \cup \text{ASupp}\,N$.

Definition 2.6. Let $\alpha, \beta \in \text{ASpec}\,\mathcal{A}$. We denote by $\alpha \leq \beta$ that any object $M$ in $\mathcal{A}$ satisfying $\alpha \in \text{ASupp}\,M$ also satisfies $\beta \in \text{ASupp}\,M$.

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Proposition 2.7. Let $R$ be a commutative ring. Then the map $\text{Spec } R \to \text{ASpec}(\text{Mod } R)$ defined by $p \mapsto R/p$ gives a poset isomorphism between $(\text{Spec } R, \subset)$ and $(\text{ASpec}(\text{Mod } R), \leq)$.

Proposition 2.8. Let $\Lambda$ be a right artinian ring. Then there exists a bijection between the set of isomorphism classes of simple $\Lambda$-modules and $\text{ASpec}(\text{Mod } \Lambda)$. The correspondence is given by $S \mapsto S$.

3. Construction of Grothendieck categories

Theorem 1.2 is proven by the following steps.

1. From a poset $P$, construct a colored quiver $\Gamma$ (technically).
2. From a colored quiver, associate a Grothendieck category $A$.
3. Take a quotient category $A = A/\mathcal{X}$ by a localizing subcategory $\mathcal{X}$ (if necessary).

In this talk, we mainly deal with the second step. We see key ideas in the first step by treating examples.

Definition 3.1. A sextuple $\Gamma = (Q_0, Q_1, C, s, t, u)$ is called a colored quiver if it satisfies the following conditions.

1. $(Q_0, Q_1, s, t)$ is a quiver.
2. $C$ is a set.
3. $u: Q_1 \to C$ is a map.
4. For any $v \in Q_0$ and $c \in C$, there are only finitely many $r \in Q_1$ satisfying $s(r) = v$ and $u(r) = c$.

Example 3.2. The diagram

$$
\begin{array}{ccc}
  v & 1 & w \\
  \downarrow & & \downarrow \\
  & & 2
\end{array}
$$

represents a colored quiver.

We fix a field $K$.

Definition 3.3. Let $\Gamma = (Q_0, Q_1, C, s, t, u)$ be a colored quiver. Denote the free $K$-algebra on $C$ by $F_C = K \langle f_c | c \in C \rangle$. Define a $K$-vector space $M_\Gamma$ by $M_\Gamma = \bigoplus_{v \in Q_0} x_v K$, where $x_v K$ is a one-dimensional $K$-vector space generated by an element $x_v$. Regard $M_\Gamma$ as a right $F_C$-module by defining the action of $f_c \in F_C$ as follows: for each vertex $v$ in $Q$,

$$
x_v \cdot f_c = \sum_{\substack{r \in Q_1 \\
 s(r) = v \\
 u(r) = c}} x_{t(r)},
$$

Denote by $A_\Gamma$ the smallest full subcategory of $\text{Mod } F_C$ closed under subobject, quotient object, and arbitrary direct sums, and containing $M_\Gamma$.

Lemma 3.4. The poset $A \text{Spec } A_\Gamma$ is isomorphic to the subset $A \text{Supp } M_\Gamma$ of $A \text{Spec}(\text{Mod } F_C)$.

Example 3.5. Let $\Gamma$ be the colored quiver

$$
\begin{array}{ccc}
  v & 1 & w \\
  \downarrow & & \downarrow \\
  & &
\end{array}
$$

Then we have $F_C = K \langle f_1 \rangle = K[f_1]$ and $M_\Gamma = x_v K \oplus x_w K$, where $x_v f_1 = x_w$ and $x_w f_1 = 0$. $M_\Gamma$ has a $F_C$-submodule $S$ of the form $0 \oplus x_w K$. Then we have the exact sequence

$$
0 \to S \to M_\Gamma \to S \to 0.
$$

Hence we have

$$
A \text{Spec } A_\Gamma = A \text{Supp } M_\Gamma = A \text{Supp } S = \{S\}.
$$
Example 3.6. Let $\Gamma$ be the colored quiver

$$
\begin{array}{c}
\bullet \\
\downarrow^1 \\
\bullet
\end{array}
$$

Then $\text{ASpec} \mathcal{A}_\Gamma$ consists of two elements, and every element is maximal.

Example 3.7. Let $\Gamma$ be the colored quiver

$$
\bullet \quad \bullet \quad \bullet \quad \bullet \\
1 \quad 2 \quad 3 \quad \ldots
$$

Then $\text{ASpec} \mathcal{A}_\Gamma = \{ \overline{M_\Gamma}, \overline{K} \}$, and we have $\overline{M_\Gamma} < \overline{K}$.

We introduce a notation of \textit{bold arrow}. The precise definition is given in \cite{Kan13}, Notation 7.20. It is explained by using an example here.

Example 3.8. Let $\Gamma$ be the colored quiver

$$
\begin{array}{c}
v \\
\downarrow^1 \\
w
\end{array}
$$

Then the diagram

$$
\Gamma \quad \Longrightarrow \quad \Gamma \quad \Longrightarrow \quad \Gamma
$$

represents the colored quiver

$$
\begin{array}{c}
v_a \quad v_b \quad v_c \\
\downarrow_{1} \quad \downarrow_{1} \quad \downarrow_{1} \\
w_a \quad w_b \quad w_c
\end{array}
\quad \begin{array}{c}
v_a \quad v_b \quad v_c \\
\downarrow_{1} \quad \downarrow_{1} \quad \downarrow_{1} \\
w_a \quad w_b \quad w_c
\end{array}
$$

Lemma 3.9. Let $\Gamma$ be a colored quiver. Let $\tilde{\Gamma}$ a colored quiver

$$
\Gamma \quad \Longrightarrow \quad \Gamma \quad \Longrightarrow \quad \Gamma \quad \Longrightarrow \quad \ldots
$$

Then $\text{ASpec} \mathcal{A}_{\tilde{\Gamma}} = \text{ASpec} \mathcal{A}_\Gamma \cup \{ \overline{M_\Gamma} \}$, where $\overline{M_\Gamma}$ is smaller than any element in $\text{ASpec} \mathcal{A}_\Gamma$.

References

\cite{Spe72} T. P. Speed, On the order of prime ideals, \textit{Algebra Universalis} \textbf{2} (1972), 85-87.

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