Periodicity of selfinjective algebras of polynomial growth

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joint work with **Karin Erdmann** and **Andrzej Skowroński**

K – algebraically closed field

algebra – finite dimensional K-algebra

A an algebra

mod A category of finitely dimensional right A-modules

M module in mod A, exact sequence in mod A

$$0 \longrightarrow \underbrace{\Omega_A(M)}_{\text{syzygy module of } M} \longrightarrow \underbrace{P_A(M)}_{\text{projective cover}} \longrightarrow M \longrightarrow 0$$

M is **periodic** if $\Omega_A^n(M) \cong M$ for some $n \geq 1$

A selfinjective and M periodic, then the Ext-algebra of M

$$\mathrm{Ext}_A^*(M, M) = \bigoplus_{i \geq 0} \mathrm{Ext}_A^i(M, M)$$

is a graded noetherian algebra (**Schultz**, **1986**)

$A^e = A^{\text{op}} \otimes_K A$ **enveloping algebra** of A

$\text{mod}\, A^e = \text{bimod}\, A$ category of finite dimensional A-A-bimodules

A projective in $\text{mod}\, A^e \iff A$ is a semisimple algebra

A is a **periodic algebra** if A is a periodic module in $\text{mod}\, A^e$ (periodic A-A-bimodule)

A periodic algebra $\Rightarrow A$ selfinjective algebra

A periodic algebra $\Rightarrow \text{mod}\, A$ is a **periodic category**
(every module in $\text{mod}\, A$ without projective direct summands is periodic)

In fact, we have

$\Omega^n_{A^e}(A) \cong A$ in $\text{mod}\, A^e \Rightarrow \Omega^n_A(M) \cong M$ for any module M in $\text{mod}\, A$ without projective direct summands

$HH^*(A) = \mathrm{Ext}^*_{A^e}(A, A)$ the **Hochschild cohomology algebra** of A

A selfinjective $\Rightarrow$ A^e selfinjective $\Rightarrow$ $\mathrm{Ext}^i_{A^e}(A, A) \cong \underline{\mathrm{Hom}}_{A^e}(\Omega^i_{A^e}(A), A)$ for $i \geq 0$

A periodic algebra $\Rightarrow$ $HH^*(A)$ a graded noetherian algebra

$D^b(\mathrm{mod}\, A)$ **derived category** of bounded complexes over mod A

A, Λ algebras

A and Λ are **derived equivalent** if $D^b(\mathrm{mod}\, A)$ and $D^b(\mathrm{mod}\, \Lambda)$ are equivalent as triangulated categories

A and Λ are derived equivalent $\xRightarrow{\textbf{Happel, Rickard}}$ $HH^*(A) \cong HH^*(\Lambda)$ as graded K-algebras

Theorem (Rickard)

Let A and Λ be derived equivalent algebras.

$$A \text{ is a periodic algebra} \iff \Lambda \text{ is a periodic algebra}$$

Moreover, if A and Λ are periodic, then their periods coincide.

PROBLEM

Determine the periodic algebras (up to Morita equivalence, derived equivalence)

PERIODICITY CONJECTURE

Assume A is an algebra for which all simple modules in $\operatorname{mod} A$ *are periodic. Then A is a periodic algebra.*

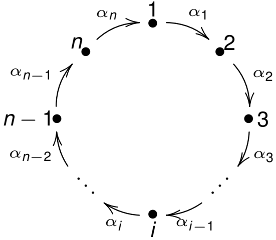
Theorem (Green–Snashall–Solberg, 2003)

Let A be an algebra such that every simple module in $\operatorname{mod} A$ *is periodic. Then A is a selfinjective algebra and $\Omega^d_{A^e}(A) \cong {}_1A_\sigma$ for a positive integer d and a K-algebra automorphism σ of A.*

Is then σ of finite order?

Selfinjective Nakayama algebras

$N_n^m(K) = K\Delta_n / I_{m,n}$, $m \geq 2$, $n \geq 1$



$K\Delta_n$ path algebra of Δ_n

$I_{m,n}$ ideal generated by all compositions of m consecutive arrows in Δ_n

$N_n^m(K)$ symmetric algebra $\iff$ $n | m+1$

$N_n^m(K)$ a periodic algebra (**Erdmann–Holm**, **1999**)

$$\text{period of } N_n^m(K) = \begin{cases} n & \text{, char } K = 2,\ m = 2,\ n \text{ odd} \\ \frac{2\operatorname{lcm}(m,n)}{m} & \text{, otherwise} \end{cases}$$

Hence every **Brauer tree algebra** is a periodic algebra

Brauer tree algebras are derived equivalent to symmetric Nakayama algebras (**Rickard**, **1989**)

Tame and wild algebras

Theorem (Drozd, 1979)

Every algebra A is either tame or wild, and not both.

A and algebra

A is **wild** if there is a $K\langle x, y\rangle$-bimodule M such that:

- M is a finite rank free left $K\langle x, y\rangle$-module
- the functor $- \otimes_{K\langle x,y\rangle} \operatorname{mod} K\langle x, y\rangle \to \operatorname{mod} A$ preserves indecomposability and respects isomorphism classes

A a wild algebra $\Rightarrow$ for any algebra Λ over K there is an exact functor $F : \operatorname{mod} \Lambda \to \operatorname{mod} A$ which preserves indecomposability and respects isomorphism classes

A is **tame** if, for any dimension d, there is a finite number of $K[x]$-A-bimodules M_i, $1 \leq i \leq n_d$, such that

- M_i, $1 \leq i \leq n_d$, are finite rank free left $K[x]$-modules,
- $K[x]/(x - \lambda) \otimes_{K[x]} M_i$, $\lambda \in K$, $i \in \{1, \ldots, n_d\}$, exhaust all but finitely many isomorphism classes of indecomposable modules of dimension d in $\operatorname{mod} A$

$\mu_A(d)$ the least number of $K[x]$-A-bimodules satisfying the above condition for d

Hierarchy of algebras

finite type

$\forall_{d \geq 1}\ \mu_A(d) = 0$

domestic type

$\exists_{m \geq 1}\ \forall_{d \geq 1}\ \mu_A(d) \leq m$

polynomial growth

$\exists_{m \geq 1}\ \forall_{d \geq 1}\ \mu_A(d) \leq d^m$

tame type

$\forall_{d \geq 1}\ \mu_A(d) < \infty$

wild **wild** **wild**

Preprojective algebras of Dynkin type

Δ Dynkin graph of type $\mathbb{A}_n (n \geq 1)$, $\mathbb{D}_n (n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$

Q_Δ double quiver of Δ

$P(\Delta) = KQ_\Delta / I_\Delta$ **preprojective algebra of type Δ**

KQ_Δ the path algebra of Q_Δ

I_Δ the ideal of KQ_Δ generated by the sums $\sum_{a, ia=v} a\bar{a}$ for all 2-cycles at the vertices v of Q_Δ

$P(\Delta)$ finite dimensional selfinjective K-algebra

$\Omega^6_{P(\Delta)^e}(P(\Delta)) \cong P(\Delta)$ for $\Delta \neq \mathbb{A}_1$

Schofield (1990), **Erdmann–Snashall (1998)**, **Brenner–Butler–King (2002)**,

For $\Delta \neq \mathbb{A}_n (n \leq 5)$ and $\mathbb{D}_4$, the algebras $P(\Delta)$ are **wild**

$Q_{\mathbb{A}_n}$: $(n \geq 1)$

$$\begin{tikzcd} 0 \arrow[r, shift left, "a_0"] & 1 \arrow[l, shift left, "\bar{a}_0"] \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "\bar{a}_1"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & n-2 \arrow[l, shift left] \arrow[r, shift left, "a_{n-2}"] & n-1 \arrow[l, shift left, "\bar{a}_{n-2}"] \end{tikzcd}$$

$Q_{\mathbb{D}_n}$: $(n \geq 4)$

$$\begin{tikzcd} 0 \arrow[dr, shift left, "a_0"] & & & & & \\ & 2 \arrow[ul, shift left, "\bar{a}_0"] \arrow[dl, shift left, "\bar{a}_1"] \arrow[r, shift left, "a_2"] & 3 \arrow[l, shift left, "\bar{a}_2"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & n-2 \arrow[l, shift left] \arrow[r, shift left, "a_{n-2}"] & n-1 \arrow[l, shift left, "\bar{a}_{n-2}"] \\ 1 \arrow[ur, shift left, "a_1"] & & & & & \end{tikzcd}$$

$Q_{\mathbb{E}_6}$:

$$\begin{tikzcd} & & 0 \arrow[d, shift left, "a_0"] & & \\ 1 \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "\bar{a}_1"] \arrow[r, shift left, "a_2"] & 3 \arrow[u, shift left, "\bar{a}_0"] \arrow[l, shift left, "\bar{a}_2"] \arrow[r, shift left, "a_3"] & 4 \arrow[l, shift left, "\bar{a}_3"] \arrow[r, shift left, "a_4"] & 5 \arrow[l, shift left, "\bar{a}_4"] \end{tikzcd}$$

$Q_{\mathbb{E}_7}$:

$$\begin{tikzcd} & & 0 \arrow[d, shift left, "a_0"] & & & \\ 1 \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "\bar{a}_1"] \arrow[r, shift left, "a_2"] & 3 \arrow[u, shift left, "\bar{a}_0"] \arrow[l, shift left, "\bar{a}_2"] \arrow[r, shift left, "a_3"] & 4 \arrow[l, shift left, "\bar{a}_3"] \arrow[r, shift left, "a_4"] & 5 \arrow[l, shift left, "\bar{a}_4"] \arrow[r, shift left, "a_5"] & 6 \arrow[l, shift left, "\bar{a}_5"] \end{tikzcd}$$

$Q_{\mathbb{E}_8}$:

$$\begin{tikzcd} & & 0 \arrow[d, shift left, "a_0"] & & & & \\ 1 \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "\bar{a}_1"] \arrow[r, shift left, "a_2"] & 3 \arrow[u, shift left, "\bar{a}_0"] \arrow[l, shift left, "\bar{a}_2"] \arrow[r, shift left, "a_3"] & 4 \arrow[l, shift left, "\bar{a}_3"] \arrow[r, shift left, "a_4"] & 5 \arrow[l, shift left, "\bar{a}_4"] \arrow[r, shift left, "a_5"] & 6 \arrow[l, shift left, "\bar{a}_5"] \arrow[r, shift left, "a_6"] & 7 \arrow[l, shift left, "\bar{a}_6"] \end{tikzcd}$$

Algebras of quaternion type

A is of **quaternion type** if

- A is indecomposable and symmetric
- A is tame of infinite type
- every indecomposable nonprojective module in mod A is periodic with period dividing 4
- the Cartan matrix C_A is nonsingular

Erdmann proved (1988) that every algebra of quaterion type is Morita equivalent to an algebra in 12 families of algebras listed bellow.

Moreover, A is of **pure quaternion type**, if A is not of polynomial growth.

Theorem (Erdmann-Skowroński, 2006)

Let A be an algebra of pure quaternion type. Then A is a periodic algebra of period 4.

The derived equivalence clasification of algebras of pure quaternion type established by **Holm (1999)** is applied.

The algebras of pure quaternion type are **tame**.

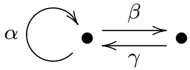
$Q^k(c)$:

$$\alpha^2 = (\beta\alpha)^{k-1}\beta + c(\alpha\beta)^k$$
$$\beta^2 = (\alpha\beta)^{k-1}\alpha$$
$$(\alpha\beta)^k = (\beta\alpha)^k,\ (\alpha\beta)^k\alpha = 0$$
$$k \geq 2$$

$Q^k(c,d)$:

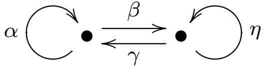
$$\text{char } K = 2$$
$$\alpha^2 = (\beta\alpha)^{k-1}\beta + c(\alpha\beta)^k$$
$$\beta^2 = (\alpha\beta)^{k-1}\alpha + d(\alpha\beta)^k$$
$$(\alpha\beta)^k = (\beta\alpha)^k,\ (\alpha\beta)^k\alpha = 0$$
$$(\beta\alpha)^k\beta = 0$$
$$k \geq 2,\ c, d \in K,\ (c,d) \neq (0,0)$$

$Q(2\mathcal{A})^k(c)$:



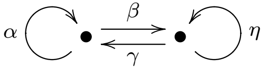
$$\gamma\beta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha$$
$$\beta\gamma\beta = (\alpha\beta\gamma)^{k-1}\alpha\beta$$
$$\alpha^2 = (\beta\gamma\alpha)^{k-1}\beta\gamma + c(\beta\gamma\alpha)^k$$
$$\alpha^2\beta = 0$$
$$k \geq 2,\ c \in K$$

$Q(2\mathcal{B})_1^{k,s}(a,c)$:



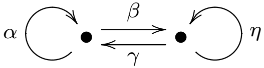
$$\gamma\beta = \eta^{s-1},$$
$$\beta\eta = (\alpha\beta\gamma)^{k-1}\alpha\beta$$
$$\eta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha$$
$$\alpha^2 =$$
$$a(\beta\gamma\alpha)^{k-1}\beta\gamma + c(\beta\gamma\alpha)^k$$
$$\alpha^2\beta = 0,\ \gamma\alpha^2 = 0$$
$$k \geq 1,\ s \geq 3,\ a \in K^*,\ c \in K$$

$Q(2\mathcal{B})_2^s(a,c)$:



$$\alpha\beta = \beta\eta,\ \eta\gamma = \gamma\alpha,\ \beta\gamma = \alpha^2$$
$$\gamma\beta = \eta^2 + a\eta^{s-1} + c\eta^s$$
$$\alpha^{s+1} = 0,\ \eta^{s+1} = 0$$
$$\gamma\alpha^{s-1} = 0,\ \alpha^{s-1}\beta = 0$$
$$s \geq 4,\ a \in K^*,\ c \in K$$

$Q(2\mathcal{B})_3^t(a,c)$:



$$\alpha\beta = \beta\eta,\ \eta\gamma = \gamma\alpha,\ \beta\gamma = \alpha^2$$
$$\gamma\beta = a\eta^{t-1} + c\eta^t$$
$$\alpha^4 = 0,\ \eta^{t+1} = 0,\ \gamma\alpha^2 = 0$$
$$\alpha^2\beta = 0$$
$$t \geq 3,\ a \in K^*,\ c \in K$$
$$(t = 3 \Rightarrow a \neq 1,\ t > 3 \Rightarrow a = 1)$$

$Q(3\mathcal{A})_1^{k,s}(d)$:

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\beta"] & \bullet \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\delta"] & \bullet \arrow[l, shift left, "\eta"] \end{tikzcd}$$

$$\beta\delta\eta = (\beta\gamma)^{k-1}\beta$$
$$\delta\eta\gamma = (\gamma\beta)^{k-1}\gamma$$
$$\eta\gamma\beta = d(\eta\delta)^{s-1}\eta$$
$$\gamma\beta\delta = d(\delta\eta)^{s-1}\delta$$
$$\beta\delta\eta\delta = 0,\ \eta\gamma\beta\gamma = 0$$
$$k, s \geq 2,\ d \in K^*$$
$$(k = s = 2 \Rightarrow d \neq 1, \text{ else } d = 1)$$

$Q(3\mathcal{A})_2^{k}$:

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\beta"] & \bullet \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\delta"] & \bullet \arrow[l, shift left, "\eta"] \end{tikzcd}$$

$$\beta\gamma\beta = (\beta\delta\eta\gamma)^{k-1}\beta\delta\eta$$
$$\gamma\beta\gamma = (\delta\eta\gamma\beta)^{k-1}\delta\eta\gamma$$
$$\eta\delta\eta = (\eta\gamma\beta\delta)^{k-1}\eta\gamma\beta$$
$$\delta\eta\delta = (\gamma\beta\delta\eta)^{k-1}\gamma\beta\delta$$
$$\beta\gamma\beta\delta = 0,\ \eta\delta\eta\gamma = 0$$
$$k \geq 2$$

$Q(3\mathcal{B})^{k,s}$:

$$\begin{tikzcd} \bullet \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & \bullet \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\delta"] & \bullet \arrow[l, shift left, "\eta"] \end{tikzcd}$$

$$\beta\gamma = \alpha^{s-1}$$
$$\alpha\beta = (\beta\delta\eta\gamma)^{k-1}\beta\delta\eta$$
$$\gamma\alpha = (\delta\eta\gamma\beta)^{k-1}\delta\eta\gamma$$
$$\eta\delta\eta = (\eta\gamma\beta\delta)^{k-1}\eta\gamma\beta$$
$$\delta\eta\delta = (\gamma\beta\delta\eta)^{k-1}\gamma\beta\delta$$
$$\alpha^2\beta = 0,\ \beta\delta\eta\delta = 0$$
$$k \geq 1,\ s \geq 3$$

$Q(3\mathcal{C})^{k,s}$:

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\beta"] & \bullet \arrow[loop above, "\varrho"] \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\delta"] & \bullet \arrow[l, shift left, "\eta"] \end{tikzcd}$$

$$\beta\varrho = 0,\ \varrho\gamma = 0,\ \eta\varrho^2 = 0$$
$$\varrho^2\delta = 0$$
$$\delta\eta - \gamma\beta = \varrho^{s-1},\ \eta\varrho = (\eta\delta)^{k-1}\eta$$
$$\varrho\delta = (\delta\eta)^{k-1}\delta,\ (\beta\gamma)^{k-1}\beta\delta = 0$$
$$(\eta\delta)^{k-1}\eta\gamma = 0$$
$$k \geq 2,\ s \geq 3$$

$Q(3\mathcal{D})^{k,s,t}$:

$$\begin{tikzcd} \bullet \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & \bullet \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\delta"] & \bullet \arrow[l, shift left, "\eta"] \arrow[loop right, "\xi"] \end{tikzcd}$$

$$\beta\gamma = \alpha^{s-1}$$
$$\gamma\alpha = (\delta\eta\gamma\beta)^{k-1}\delta\eta\gamma$$
$$\alpha\beta = (\beta\delta\eta\gamma)^{k-1}\beta\delta\eta$$
$$\eta\delta = \xi^{t-1}$$
$$\delta\xi = (\gamma\beta\delta\eta)^{k-1}\gamma\beta\delta$$
$$\xi\eta = (\eta\gamma\beta\delta)^{k-1}\eta\gamma\beta$$
$$\alpha^2\beta = 0,\ \delta\eta\delta = 0$$
$$k \geq 1,\ s, t \geq 3$$

$Q(3\mathcal{K})^{a,b,c}$:

$$\begin{tikzcd} \bullet \arrow[rr, shift left, "\beta"] \arrow[dr, shift left, "\kappa"] & & \bullet \arrow[ll, shift left, "\gamma"] \arrow[dl, shift left, "\delta"] \\ & \bullet \arrow[ul, shift left, "\lambda"] \arrow[ur, shift left, "\eta"] & \end{tikzcd}$$

$$\beta\delta = (\kappa\lambda)^{a-1}\kappa,\ \eta\gamma = (\lambda\kappa)^{a-1}\lambda$$
$$\delta\lambda = (\gamma\beta)^{b-1}\gamma,\ \kappa\eta = (\beta\gamma)^{b-1}\beta$$
$$\lambda\beta = (\eta\delta)^{c-1}\eta,\ \gamma\kappa = (\delta\eta)^{c-1}\delta$$
$$\gamma\beta\delta = 0,\ \delta\eta\gamma = 0,\ \lambda\kappa\eta = 0$$
$$a, b, c \geq 1 \text{ (at most one equal 1)}$$

Theorem (Dugas, 2010)

Let A be a nonsimple, indecomposable, selfinjective algebra of finite type. Then A is a periodic algebra.

Special cases:

- Erdmann-Holm-Snashall (1999, 2002) Dynkin type $\mathbb{A}_n$
- Brenner-Butler-King (2002): the trivial extension algebras $\mathrm{T}(B) = B \ltimes D(B)$ of titled algebras B of Dynkin type
- Erdmann-Skowroński (2008): almost all standard selfinjective algebras of finite type

Basic, nonsimple, indecomposable, selfinjective algebras of finite type

→ **standard algebras** (admit simply connected Galois coverings)

Riedtmann, Waschbüsch (1980–1983)

→ **nonstandard algebras** (occur only in characteristic 2)

Periodicity of standard selfinjective algebras of finite type follows from the periodicity of preprojective algebras of Dynkin type (**Brenner-Butler-King**, **Dugas**)

Periodicity of nonstandard selfinjective algebras of finite type follows from the periodicity of a family of Brauer tree algebras (**Dugas**)

Λ basic, indecomposable algebra

$1_\Lambda = e_1 + \cdots + e_n$,
$e_1, \dots, e_n$ pairwise orthogonal primitive idempotents of Λ

G a finite subgroup of $\mathrm{Aut}_K(\Lambda)$ acting freely on the chosen set $e_1, \dots, e_n$ of primitive idempotents

Then we have a finite Galois covering $\Lambda \to \Lambda/G$ where Λ/G is the **orbit algebra** of Λ with respect to G

In fact, we have an isomorphism of K-algebras
$\Lambda/G \cong \Lambda^G = \{\lambda \in \Lambda \mid g(\lambda) = \lambda \text{ for all } g \in G\}$ **invariant algebra** (**Auslander-Reiten-Smalø, 1989**)

Theorem (Dugas, 2010)

Let A and Λ be basic indecomposable algebras related by a finite Galois covering $\Lambda \to A = \Lambda/G$. Then

$$\Lambda \text{ is a periodic algebra} \iff A \text{ is a periodic algebra}$$

Selfinjective algebras of polynomial growth

Theorem (Skowroński, 1989, 2006)

Let A be a basic, indecomposable, selfinjective algebra over an algebraically closed field K. Then

1. *A is representation-infinite domestic $\iff$ A is socle equivalent to an orbit algebra $\widehat{B}/G$, where B is a tilted algebra of Euclidean type and G is an admissible infinite cyclic group of automorphisms of $\widehat{B}$.*
2. *A is nondomestic of polynomial growth $\iff$ A is socle equivalent to an orbit algebra $\widehat{B}/G$, where B is a tubular algebra and G is an admissible infinite cyclic group of automorphisms of $\widehat{B}$.*

Two selfinjective algebras A and Λ are **socle equivalent** if the quotient algebras $A/\operatorname{soc}(A)$ and $\Lambda/\operatorname{soc}(\Lambda)$ are isomorphic

Riedtmann's classification of selfinjective algebras of finite type can be presented as follows

Theorem

Let A be a basic, nonsimple, indecomposable, selfinjective algebra over an algebraically closed field. Then A is of finite type $\iff$ A is socle equivalent to an orbit algebra $\widehat{B}/G$, where B is a tilted algebra of Dynkin type and G is an admissible infinite cyclic group of automorphisms of $\widehat{B}$.

THEOREM (Białkowski-Erdmann-Skowroński, 2013)

Let A be a basic, indecomposable, representation-infinite selfinjective algebra of polynomial growth. The following statements are equivalent.

1. *All simple modules in* $\operatorname{mod} A$ *are periodic.*
2. *A is a periodic algebra.*
3. *A is socle equivalent to an orbit algebra $\widehat{B}/G$, where B is a tubular algebra and G is an admissible infinite cyclic group of automorphisms of the repetitive category $\widehat{B}$ of B.*

(2) $\Rightarrow$ (1) Known

(1) $\Rightarrow$ (3) A socle equivalent to $\widehat{B}/G$, B tilted algebra of Euclidean type, then Γ_A admits an acyclic component $\mathcal{C}$ (with the stable part $\mathcal{C}^s = \mathbb{Z}\Delta$ for a Euclidean quiver Δ) containing a simple module S. But then S is not periodic, because for simple modules the periodicity is equivalent to τ_A-periodicity.

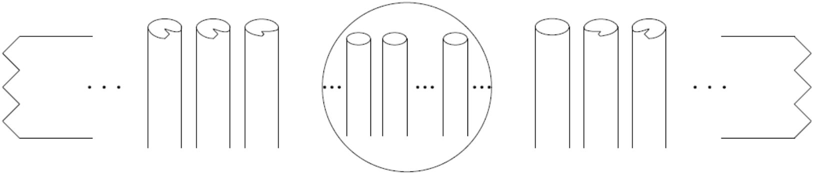
(3) $\Rightarrow$ (2) New (difficult) part.

We will present the main ingredients of our proof of this implication.

Tubular algebras

A **tubular algebra** B is a tubular (branch) extension of a tame concealed algebra of one of the tubular types $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$ (**Ringel**, **1984**)

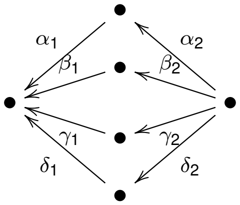
- $\operatorname{gl.dim} B = 2$
- $\operatorname{rk} K_0(B) = 6$, 8, 9, or 10
- B is triangular nondomestic of polynomial growth
- The Auslander-Reiten quiver Γ_B of B is of the form



$\mathcal{P}^B$	$\mathcal{T}_0^B$	$\bigvee_{q\in\mathbb{Q}^+} \mathcal{T}_q^B$	$\mathcal{T}_\infty^B$	$\mathcal{Q}^B$
preprojective component of Euclidean type	$\mathbb{P}_1(K)$-family of ray tubes	$\mathbb{P}_1(K)$-family of stable tubes	$\mathbb{P}_1(K)$-family of coray tubes	preinjective component of Euclidean type

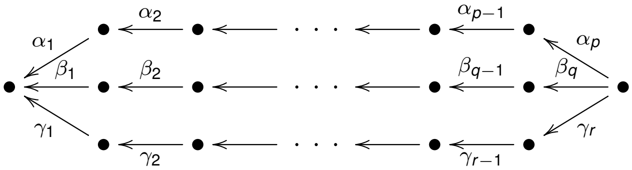
Canonical tubular algebras

$\Lambda(2,2,2,2,\lambda)$, $\lambda \in K \setminus \{0,1\}$, given by the quiver



and the relations $\alpha_2\alpha_1 + \beta_2\beta_1 + \gamma_2\gamma_1 = 0$, $\alpha_2\alpha_1 + \lambda\beta_2\beta_1 + \delta_2\delta_1 = 0$.

$\Lambda(p,q,r)$, $(p,q,r) \in \{(3,3,3),(2,4,4),(2,3,6)\}$, given by the quiver



and the relation $\alpha_p \dots \alpha_2\alpha_1 + \beta_q \dots \beta_2\beta_1 + \gamma_r \dots \gamma_2\gamma_1 = 0$.

Theorem (Ringel, 1984)

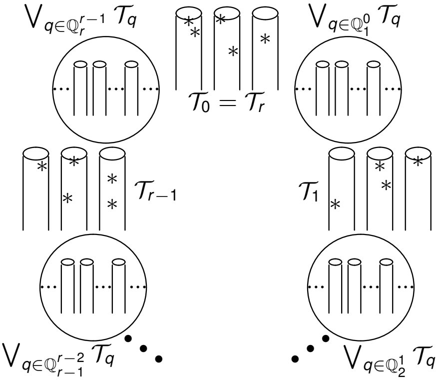
Let B be a basic, indecomposable algebra. Then

$$\boxed{\textit{B is a tubular algebra}} \iff \boxed{\textit{$B = \mathrm{End}_\Lambda(T)$ for a canonical tubular algebra Λ and a (multiplicity-free) tilting module T in the additive category of $\mathcal{P}_0^\Lambda \cup \mathcal{T}_0^\Lambda \cup \left(\bigcup_{q\in\mathbb{Q}^+} \mathcal{T}_q^\Lambda\right)$}}$$

Hence, any two tubular algebras B and C of the same tubular type $(p, q, r) \in \{(3, 3, 3), (2, 4, 4), (2, 3, 6)\}$ are derived equivalent.

Similarly, every two tubular algebras B and C of tubular type $(2, 2, 2, 2)$ given by the same canonical tubular algebra $\Lambda(2, 2, 2, 2, \lambda)$ are derived equivalent.

B tubular algebra, G admissible infinite cyclic group of automorphisms of $\widehat{B}$, $A = \widehat{B}/G$
Then $\Gamma_A = \Gamma_{\widehat{B}}/G$ and has the following clock structure



where $*$ denote projective modules, $r \geq 3$, $\mathbb{Q}_i^{i-1} = \mathbb{Q} \cap (i-1, i)$ for any $i \in \{1, \dots, r\}$, and

1. for each $i \in \{0, 1, \dots, r-1\}$, $\mathcal{T}_i^A$ is a $\mathbb{P}_1(K)$-family of quasi-tubes (the stable parts are stable tubes);
2. for each $q \in \mathbb{Q}_i^{i-1}$, $i \in \{1, \dots, r\}$, $\mathcal{T}_q^A$ is a $\mathbb{P}_1(K)$-family of stable tubes;
3. all $\mathbb{P}_1(K)$-families $\mathcal{T}_q^A$, $q \in \mathbb{Q} \cap [0, r]$, have the same tubular type $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$.

For an algebra B and a positive integer r, we have the **r-fold trivial extension algebra** of B

$$\mathrm{T}(B)^{(r)} = \widehat{B}/(\nu_{\widehat{B}}^{r}) = \left\{ \begin{array}{c} \begin{bmatrix} b_1 & 0 & 0 & & & & \\ f_2 & b_2 & 0 & & & & \\ 0 & f_3 & b_3 & & & & \\ & & \ddots & \ddots & & & \\ & 0 & & f_{r-1} & b_{r-1} & 0 \\ & & & 0 & f_1 & b_1 \end{bmatrix} \\ b_1, \dots, b_{r-1} \in B,\ f_1, \dots, f_{r-1} \in D(B) \end{array} \right\}$$

$\mathrm{T}(B)^{(1)} \cong \mathrm{T}(B) = B \ltimes D(B)$ the **trivial extension algebra** of B by the injective cogenerator $D(B) = \mathrm{Hom}_B(B, K)$

$\mathrm{T}(B)^{(r)}$ is a symmetric algebra $\iff r = 1$

Standard selfinjective algebra of tubular type: a selfinjective algebra of the form $A = \widehat{B}/H$, where B is a tubular algebra and H is an admissible infinite cyclic group of automorphisms of $\widehat{B}$

Then A admits a **simply connected Galois covering** $\widehat{B} \to \widehat{B}/H = A$

Theorem

Let A be a basic, indecomposable algebra. The following statements are equivalent.

1. *A is a standard selfinjective algebra of tubular type.*
2. *A is isomorphic to an orbit algebra $\mathrm{T}(B)^{(r)}/G$, where B is a tubular algebra, r a positive integer, and G an admissible finite automorphism group of $\mathrm{T}(B)^{(r)}$.*

Białkowski-Skowroński (2002): tubular types $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$

Lenzing-Skowroński (2000): tubular type $(2,3,6)$

Note that $\mathrm{T}(B)^{(r)}/G \cong \left(\mathrm{T}(B)^{(r)}\right)^G$ invariant algebra

Theorem (Rickard, 1989)

Let B and C be derived equivalent algebras. Then the trivial extension algebras $\mathrm{T}(B)$ and $\mathrm{T}(C)$ are derived equivalent.

We may consider the following scheme of finite Galois coverings

$$\begin{tikzcd}
 & \mathrm{T}(B)^{(r)} \arrow[dl] \arrow[dr] & & \mathrm{T}(C)^{(s)} \arrow[dl] \arrow[dr] & \\
\mathrm{T}(B)^{(r)}/G & & \mathrm{T}(B) \underset{\text{der}}{\sim} \mathrm{T}(C) & & \mathrm{T}(C)^{(s)}/H
\end{tikzcd}$$

where B and C are derived equivalent tubular algebras, r, s positive integers, G, H admissible finite automorphism groups of $\mathrm{T}(B)^{(r)}$ and $\mathrm{T}(C)^{(s)}$, respectively.

Then

$$\mathrm{T}(B)^{(r)}/G \text{ is a periodic algebra} \iff \mathrm{T}(C)^{(s)}/H \text{ is a periodic algebra}$$

Nonstandard nondomestic selfinjective algebras of polynomial growth (socle deformations of standard selfinjective algebras of tubular type)

Occur only in characteristic 2 and 3

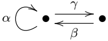
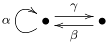
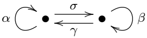
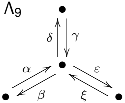
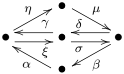
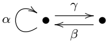
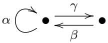
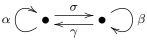
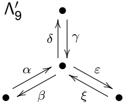
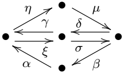
Λ nonstandard nondomestic selfinjective algebra of polynomial growth

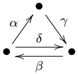
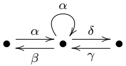
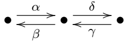
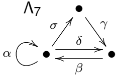
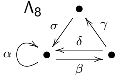
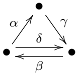
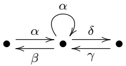
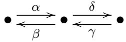
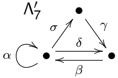
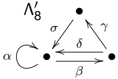
Then there exists a unique standard selfinjective algebra Λ' of tubular type such that

1. $\dim_K \Lambda = \dim_K \Lambda'$
2. Λ an Λ' are socle equivalent (but $\Lambda \not\cong \Lambda'$)
3. Λ' is a geometric degeneration of an Λ (belongs to the closure $\overline{\mathrm{GL}_K(d)\Lambda}$ in the affine variety of K-algebras of dimension $d = \dim_K \Lambda = \dim_K \Lambda'$)

Λ' the **standard form** of Λ

The pairs Λ an Λ' are described by the tables (**Białkowski-Skowroński, 2004**)

characteristic	3		2		
tubular type	(3,3,3)		(2,2,2,2)	(3,3,3)	(2,3,6)
nonstandard algebras	Λ_1  $\alpha^2 = \gamma\beta$, $\beta\alpha\gamma = \beta\alpha^2\gamma$, $\beta\alpha^2 = 0$, $\alpha^2\gamma = 0$	Λ_2  $\alpha^2\gamma = 0$, $\beta\alpha^2 = 0$, $\gamma\beta\gamma = 0$, $\beta\gamma\beta = 0$, $\beta\gamma = \beta\alpha\gamma$, $\alpha^3 = \gamma\beta$	$\Lambda_3(\lambda)$, $\lambda \in K \setminus \{0,1\}$  $\alpha^4 = 0$, $\gamma\alpha^2 = 0$, $\alpha^2\sigma = 0$, $\alpha^2 = \sigma\gamma + \alpha^3$, $\lambda\beta^2 = \gamma\sigma$, $\gamma\alpha = \beta\gamma$, $\sigma\beta = \alpha\sigma$	Λ_9  $\beta\alpha + \varrho\gamma + \varepsilon\xi = 0$, $\xi\varepsilon = 0$, $\gamma\varrho = 0$, $\alpha\beta\alpha = 0$, $\beta\alpha\beta = 0$, $\alpha\beta = \alpha\varrho\gamma\beta$	Λ_{10}  $\mu\beta = 0$, $\alpha\eta = 0$, $\beta\alpha = \delta\gamma$, $\xi\sigma = \eta\mu$, $\sigma\delta = \gamma\xi + \sigma\delta\sigma\delta$, $\delta\sigma\delta\sigma\delta = 0$, $\sigma\delta\sigma\delta\sigma = 0$, $\xi\sigma\delta\sigma\delta = 0$, $\sigma\delta\sigma\delta\gamma = 0$
standard algebras	Λ'_1  $\alpha^2 = \gamma\beta$, $\beta\alpha\gamma = 0$	Λ'_2  $\alpha^2\gamma = 0$, $\beta\alpha^2 = 0$, $\beta\gamma = 0$, $\alpha^3 = \gamma\beta$	$\Lambda'_3(\lambda)$, $\lambda \in K \setminus \{0,1\}$  $\alpha^2 = \sigma\gamma$, $\lambda\beta^2 = \gamma\sigma$, $\gamma\alpha = \beta\gamma$, $\sigma\beta = \alpha\sigma$	Λ'_9  $\beta\alpha + \varrho\gamma + \varepsilon\xi = 0$, $\alpha\beta = 0$, $\xi\varepsilon = 0$, $\gamma\varrho = 0$	Λ'_{10}  $\mu\beta = 0$, $\alpha\eta = 0$, $\beta\alpha = \delta\gamma$, $\xi\sigma = \eta\mu$, $\sigma\delta = \gamma\xi$

characteristic	2				
tubular type	(2,4,4)				
nonstandard algebras	Λ_4  $\delta\beta\delta = \alpha\gamma$, $(\beta\delta)^3\beta = 0$, $\gamma\beta\alpha\gamma = 0$, $\alpha\gamma\beta\alpha = 0$, $\gamma\beta\alpha = \gamma\beta\delta\beta\alpha$	Λ_5  $\alpha^2 = \gamma\beta$, $\alpha^3 = \delta\sigma$, $\beta\delta = 0$, $\sigma\gamma = 0$, $\alpha\delta = 0$, $\sigma\alpha = 0$, $\gamma\beta\gamma = 0$, $\beta\gamma\beta = 0$, $\beta\gamma = \beta\alpha\gamma$	Λ_6  $\alpha\delta\gamma\delta = 0$, $\gamma\delta\gamma\beta = 0$, $\alpha\beta\alpha = 0$, $\beta\alpha\beta = 0$, $\alpha\beta = \alpha\delta\gamma\beta$, $\beta\alpha = \delta\gamma\delta\gamma$	Λ_7  $\beta\delta = \beta\alpha\delta$, $\alpha\sigma = 0$, $\alpha\delta = \sigma\gamma$, $\gamma\beta\alpha = 0$, $\alpha^2 = \delta\beta$, $\gamma\beta\delta = 0$, $\beta\delta\beta = 0$, $\delta\beta\delta = 0$	Λ_8  $\delta\beta = \delta\alpha\beta$, $\sigma\alpha = 0$, $\delta\alpha = \gamma\sigma$, $\alpha\beta\gamma = 0$, $\alpha^2 = \beta\delta$, $\delta\beta\gamma = 0$, $\beta\delta\beta = 0$, $\delta\beta\delta = 0$
standard algebras	Λ'_4  $\delta\beta\delta = \alpha\gamma$, $(\beta\delta)^3\beta = 0$, $\gamma\beta\alpha = 0$	Λ'_5  $\alpha^2 = \gamma\beta$, $\alpha^3 = \delta\sigma$, $\beta\delta = 0$, $\sigma\gamma = 0$, $\alpha\delta = 0$, $\sigma\alpha = 0$, $\beta\gamma = 0$	Λ'_6  $\alpha\delta\gamma\delta = 0$, $\gamma\delta\gamma\beta = 0$, $\alpha\beta = 0$, $\beta\alpha = \delta\gamma\delta\gamma$	Λ'_7  $\beta\delta = 0$, $\alpha\sigma = 0$, $\alpha\delta = \sigma\gamma$, $\gamma\beta\alpha = 0$, $\alpha^2 = \delta\beta$	Λ'_8  $\delta\beta = 0$, $\sigma\alpha = 0$, $\delta\alpha = \gamma\sigma$, $\alpha\beta\gamma = 0$, $\alpha^2 = \beta\delta$

Theorem (Białkowski-Holm-Skowroński, 2003)

1. Λ_1 *and* Λ_2 *are derived equivalent* (char $K = 3$)
2. Λ'_1 *and* Λ'_2 *are derived equivalent* (char K *arbitrary)*
3. $\Lambda_4, \Lambda_5, \Lambda_6, \Lambda_7, \Lambda_8$ *are derived equivalent* (char $K = 2$)
4. $\Lambda'_4, \Lambda'_5, \Lambda'_6, \Lambda'_7, \Lambda'_8$ *are derived equivalent* (char K *arbitrary)*

- Λ_i and Λ'_i, $i \in \{1, \dots, 8\}$ are symmetric algebras
- Λ_9 (char $K = 2$) is weakly symmetric but not symmetric
- Λ'_9 (char $K = 2$) is symmetric
- Λ'_9 (char $K \neq 2$) is weakly symmetric but not symmetric
- Λ_{10} and Λ'_{10} are not weakly symmetric

THEOREM

- Λ_1 (char $K = 3$) *periodic algebra of period* 6
- Λ_1' (char K *arbitrary) periodic algebra of period* 6
- $\Lambda_3(\lambda)$ (char $K = 2$) *periodic algebra of period* 4
- $\Lambda_3'(\lambda)$ (char K *arbitrary) periodic algebra of period* 4
- Λ_6 (char $K = 2$) *periodic algebra of period* 8
- Λ_6' (char K *arbitrary) periodic algebra of period* 8
- Λ_9 (char $K = 2$) *periodic algebra of period* 6
- Λ_9' *periodic algebra of period* $= \begin{cases} 3 & \text{char } K = 2 \\ 6 & \text{char } K \neq 2 \end{cases}$
- Λ_{10} (char $K = 2$) *periodic algebra of period* 6
- Λ_{10}' *periodic algebra of period* $= \begin{cases} 3 & \text{char } K = 2 \\ 6 & \text{char } K \neq 2 \end{cases}$

Hard work

Λ basic, indecomposable, finite dimensional algebra, K algebraically closed

$1_A = e_1 + \cdots + e_n$, $e_1, \dots, e_n$ pairwise orthogonal primitive idempotents of A

$e_i \otimes e_j$, $i, j \in \{1, \dots, n\}$, pairwise orthogonal primitive idempotents of $A^e = A^{\mathrm{op}} \otimes_K A$

$1_{A^e} = \sum_{1 \leq i,j \leq n} e_i \otimes e_j$

$P(i,j) = (e_i \otimes e_j)A^e = Ae_i \otimes e_j A$, $i, j \in \{1, \dots, n\}$, complete set of pairwise nonisomorphic indecomposable projective modules in $\operatorname{mod} A^e = \operatorname{bimod} A$

$S_i = e_i A / e_i \operatorname{rad} A$, $i \in \{1, \dots, n\}$, complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$

The following theorem describes the terms of a minimal projective bimodule resolution of A.

Theorem (Happel, 1989)

A admits a minimal projective resolution in $\operatorname{mod} A^e$ *of the form*

$$\cdots \longrightarrow \mathbb{P}_r \xrightarrow{d_r} \mathbb{P}_{r-1} \longrightarrow \cdots \longrightarrow \mathbb{P}_1 \xrightarrow{d_1} \mathbb{P}_0 \xrightarrow{d_0} A \longrightarrow 0,$$

where

$$\mathbb{P}_r = \bigoplus_{0 \leq i,j \leq n} P(i,j)^{\dim_K \operatorname{Ext}_A^r(S_i, S_j)}.$$

Theorem (Białkowski–Erdmann–Skowroński, 2013)

Let A and Λ be representation-infinite periodic algebras of polynomial growth such that Λ is a nonstandard algebra and A a standard algebra.
Then A and Λ are not derived equivalent.

Symmetric algebras case **Holm-Skowroński (2011)**, using Külshammer ideals

A similar result holds for representation-finite selfinjective algebras **Asashiba (1999)**

Holm-Skowroński (2006): different proof using Külshammer ideals