Stable categories of Cohen-Macaulay modules and cluster categories

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(joint work with Claire Amiot, Idun Reiten)

Let \(\mathcal{T}\) be a \(k\)-linear triangulated category with the suspension functor \([1]\) over a field \(k\). For an integer \(n\), we say that \(\mathcal{T}\) is \(n\)-Calabi-Yau (\(n\)-CY) if there exists a functorial isomorphism \(\text{Hom}_\mathcal{T}(X,Y) \simeq D\text{Hom}_\mathcal{T}(Y,X[n])\) for any \(X,Y \in \mathcal{T}\), where \(D = \text{Hom}_k(-,k)\) is the \(k\)-dual. In representation theory, there are two important classes of \(n\)-CY triangulated categories. One is the generalized \(n\)-cluster categories [BMRRT, Am, G] appearing in study of Fomin-Zelevinsky cluster algebras. The other is the stable categories of Cohen-Macaulay modules over Gorenstein isolated singularities [Au1]. The aim of this paper is to compare these two classes of categories. We will show that the stable categories of Cohen-Macaulay modules over certain Gorenstein isolated singularities are triangle equivalent to generalized \(n\)-cluster categories (Theorem 1).

1. Preliminaries

Let \(n \geq 1\). A key notion in \(n\)-CY triangulated categories \(\mathcal{T}\) is \(n\)-cluster tilting objects \(M \in \mathcal{T}\) defined by \(\text{add} M = \{X \in \mathcal{T} \mid \text{Hom}_\mathcal{T}(M,X[i]) = 0 \ (0 < i < n)\}\). They are certain analogue of tilting objects, and 1-cluster tilting objects are nothing but additive generators of \(\mathcal{T}\).

1.1. Cluster categories. Let \(n \geq 2\), and let \(A\) be a finite dimensional \(k\)-algebra with \(\text{gl.dim} A \leq n\). We denote by \(\mathcal{D}_A\) the bounded derived category of the category \(\text{mod} A\) of finitely generated \(A\)-modules, and by \(\nu := - \otimes_A \mathcal{D}_A : \mathcal{D}_A \to \mathcal{D}_A\) the Nakayama functor. We have Auslander-Reiten-Serre duality \(\text{Hom}_{\mathcal{D}_A}(X,Y) \simeq D\text{Hom}_{\mathcal{D}_A}(Y,\nu X)\) for any \(X,Y \in \mathcal{D}_A\) [Ha]. Let \(\nu_n := \nu \circ [-n] : \mathcal{D}_A \to \mathcal{D}_A\). If \(\text{gl.dim} A \leq 1\), then the orbit category \(\mathcal{C}_A^{(n)} := \mathcal{D}_A/\nu_n\) forms an \(n\)-CY triangulated category called the \(n\)-cluster category [BMRRT, K1]. This is not the case for \(\text{gl.dim} A \geq 2\), and the generalized \(n\)-cluster category \(\mathcal{C}_A^{(n)}\) is defined in [K1, Am, G] as a ‘triangulated hull’ of the orbit category \(\mathcal{D}_A/\nu_n\) under the assumption that the functor \(R^0(\nu_n) : \text{mod} A \to \text{mod} A\) is nilpotent. This is an \(n\)-CY triangulated category with a triangle functor \(\pi : \mathcal{D}_A \to \mathcal{C}_A^{(n)}\) satisfying a certain universal property and has an \(n\)-cluster tilting object \(\pi A \in \mathcal{C}_A^{(n)}\).

1.2. Stable categories. Let \(R\) be a complete local Gorenstein ring of Krull dimension \(d\). We denote by \(\text{CM}(R) := \{X \in \text{mod} R \mid \text{Ext}^1_R(X,R) = 0 \ (0 < i)\}\) the category of maximal Cohen-Macaulay \(R\)-modules, and by \(\mathcal{CM}(R)\) its stable category. It is known that \(\text{CM}(R)\) forms a triangulated category [Ha], and is triangle equivalent to \(\mathcal{D}_R/\text{per} R\) [B]. Assume that \(R\) is an isolated singularity. Then \(\mathcal{CM}(R)\) forms a \((d-1)\)-CY triangulated category by a classical result due to Auslander [Au1]. If \(M \in \mathcal{CM}(R)\) is \((d-1)\)-cluster tilting, then \(\Gamma := \text{End}_R(R \oplus M)\) satisfies \(\text{gl.dim} \Gamma = d\) and \(\Gamma \in \mathcal{CM}(R)\) [12]. In particular \(\Gamma\) is a non-commutative crepant resolution in the sense of Van den Bergh [V]. The existence of a \((d-1)\)-cluster
tilting object in $\text{CM}(R)$ is closely related to the geometry of resolutions of the singularity $\text{Spec}R$.

Let $S := k[[x_1, \ldots, x_d]]$ be the formal power series ring over a field $k$ of characteristic zero, and let $G$ be a finite subgroup of $\text{SL}_d(k)$. If the quotient singularity $R := S^G$ is isolated, then $S \in \text{CM}(R)$ is $(d - 1)$-cluster tilting [I1]. In particular, if $d = 2$, we have $\text{CM}(R) = \text{add}S$ and so $R$ is representation-finite [Au2, He].

2. Main results

Let $k$ be a field of characteristic zero. Let $G = \frac{1}{n}(a_1, \ldots, a_d)$ be a cyclic subgroup of $\text{SL}_d(k)$ generated by a diagonal matrix $g = \text{diag}(\zeta^{a_1}, \ldots, \zeta^{a_d})$ with a primitive $n$-th root $\zeta$ of unity and integers $a_i$ satisfying $0 < a_i < n$, $(n, a_i) = 1$ and $\sum_{i=1}^d a_i = n$. Let $S = k[x_1, \ldots, x_d]$ be a polynomial algebra of $d$ variables. Then $S$ has a $\frac{1}{n}$-graded algebra structure $S = \bigoplus_{i \geq 0} S_i$ defined by $\deg x_i := \frac{i}{n}$.

The invariant subring $R := S^G = \bigoplus_{i \geq 0} S_i$ is a Gorenstein isolated singularity. For $0 \leq j < n$, we define a $\mathbb{Z}$-graded $R$-module $T^j := \bigoplus_{i \geq 0} (T^j)^i$ by $(T^j)^i := S_{i + j}$. Let $T := \bigoplus_{j=0}^{n-1} T^j$. Then $B := \text{End}_R(T)$ has a $\mathbb{Z}$-graded algebra structure $B = \bigoplus_{i \geq 0} B_i$ with the degree zero part $A := B_0 = \text{End}_R(T)$. Let $e$ be the idempotent of $A$ corresponding to the direct summand $T^0$ of $T$, and $\A := A/e$. Our main result is the following [AIR]:

**Theorem 1** We have a triangle equivalence $\text{CM}(R) \simeq \mathcal{C}^{[d-1]}_A$.

**Remark 2** (a) $B$ is isomorphic to the skew group algebra $S*G$ [Au2], whose quiver is given by the McKay quiver of $G$. The relations are given by higher derivative of a potential [BSW].

(b) A related result is given in [DV].

(c) Theorem 1 is an analogue of Ueda’s equivalence $\text{CM}^2(R) \simeq \mathcal{D}_A$ [U].

**Example 3** Let $G = \frac{1}{3}(1, 1, 1)$. The algebras $B$, $A$ and $\A$ are presented by quivers

- $B : \begin{array}{c} 0 \\ 2 \hline 1 \end{array}$
- $A : \begin{array}{c} 0 \\ 2 \hline 1 \end{array}$
- $\A : \begin{array}{c} 2 \hline 1 \end{array}$

Thus $\text{CM}(R)$ is triangle equivalent to the cluster category of $\begin{array}{c} 2 \hline 1 \end{array}$, and we recover a result by Keller and Reiten [KR].

Theorem 1 is a special case of the following result:

Let $B = \bigoplus_{i \geq 0} B_i$ be a graded $k$-algebra such that $\dim_k B_i < \infty$.

- $B$ is a bimodule $d$-Calabi-Yau algebra of Gorenstein parameter 1, i.e. $B \in \text{per}B^p$ and $\text{RHom}_{B^p}(B, B^p)[d] \simeq B(1)$.
- $A := B_0$ has an idempotent $e$ such that $eA(1 - e) = 0$. $B$ is noetherian and $B := B/e$ is a finite dimensional $k$-algebra.
- $C := eBe$ satisfies $\text{End}_{C}(Be) = B$ and $\text{End}_{C^p}(eB) = B$. 


**Theorem 4** We have a triangle equivalence $F$ and the commutative diagram:

$$
\begin{array}{ccc}
D_A & \xrightarrow{F} & D_C \\
\downarrow & & \downarrow \\
C & \xrightarrow{F} & CM(C)
\end{array}
$$

The key observation is the following.

**Lemma 5** There exists a triangle in $D\text{-mod}(A^{op} \otimes_k B))$:

$$
A[-1] \rightarrow R\text{Hom}_A(A, A') \rightarrow L B(-1)[d-1] \rightarrow B \rightarrow A
$$

As an application of Lemma 5, the derived $d$-preprojective DG algebra $[K2]$ of $A$ is $B$. In particular $A$ is $(d-1)$-representation-infinite in the sense of [IO] or a quasi $(d-1)$-Fano algebra in the sense of [MM].

**References**


