Action of mapping class group on extended Bers slice

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1 Introduction

Let S be an oriented closed surface of genus $g \geq 2$. Put

$$V(S) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C}).$$

Let X be an element of Teichüller space $\mathrm{Teich}(S)$ of S and C_X be the subset of $V(S)$ consisting of function groups which uniformize X. We define the action of mapping class group $\mathrm{Mod}(S)$ on C_X and investigate the distribution of elements of C_X.

2 Preliminaries

A compact 3-manifold M is called *compression body* if it is constructed as follows: Let $S_1, \ldots, S_n$ be oriented closed surfaces of genus ≥ 1 (possibly $n = 0$). Let $I = [0, 1]$ be a closed interval. M is obtained from $S_1 \times I, \ldots, S_n \times I$ and a 3-ball B^3 by glueing a disk of $S_j \times \{0\}$ to a disk of ∂B^3 or a disk of ∂B^3 to a disk of ∂B^3 orientation reversingly. A component of ∂M which intersects ∂B^3 is denoted by $\partial_0 M$ and is called the *exterior boundary* of M.

A Kleinian group is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C}) = \mathrm{Isom}^+\mathbf{H}^3 = \mathrm{Aut}(\widehat{\mathbf{C}})$. We always assume that a Kleinian group is torsion-free and finitely generated. We denote by $\Omega(G)$ the region of giscontinuity of a Kleinian group G. For a Kleinian group G, $\mathbf{H}^3/G$ is a hyperbolic 3-manifold and each component of $\Omega(G)/G$ is a Riemann surface. $N_G := \mathbf{H}^3 \cup \Omega(G)/G$ is called a Kleinian manifold.

A Kleinian group G is called a *function group* if there is a G-invariant component $\Omega_0(G)$ of $\Omega(G)$. A function group G is called a *quasi-Fuchsian group* if there are two G-invariant component of $\Omega(G)$. A Kleinian group G is called *geometrically finite* if it has a finite sided convex polyhedron in $\mathbf{H}^3$.

Let S be a oriented closed surface of genus $g \geq 2$. Put

$$CB(S) = \{M | M \text{ is a compression body s.t. } \partial_0 M \cong S\}.$$

If G is a function group with invariant component $\Omega_0(G)$ such that $\Omega_0(G)/G \cong S$, then $\mathbf{H}^3/G$ is homeomorphic to the interior $\mathrm{int}M$ of some $M \in CB(S)$ (i.e. function group is topologically tame).

If G is a quasi-Fuchsian group such that each component of $\Omega(G)/G$ is homeomorphic to S, then $N_G = \mathbf{H}^3 \cup \Omega(G)/G \cong S \times I$.

Let $M \in CB(S)$. Let

$$V(M) = \mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C})$$

be the representation space equipped with algebraic topology. We denote the conjugacy class of $\rho : \pi_1(M) \to G \subset \mathrm{PSL}_2(\mathbf{C})$ by $[\rho, G]$ or $[\rho]$. Let

$$AH(M) = \{[\rho] \in V(M) | \rho \text{ is discrete and faithful}\}$$

and $MP(M) = \mathrm{int}AH(M)$. Any element $[\rho, G] \in MP(M)$ is geometrically finite and *minimally palabolic*, that is, any parabolic element $\gamma \in G$ is contained in $\rho(\pi_1(T))$ for some torus component T of ∂M.

Remark.

- It is conjectured that $\overline{MP(M)} = AH(M)$ (Bers-Thurston conjecture).
- If $M \in CB(S)$, $MP(M)$ is connected.

Put

$$V(S) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C}).$$

Then $MP(S \times I) \subset AH(S \times I) \subset V(S)$. For any $[\rho, G] \in MP(S \times I)$, G is a quasi-Fuchsian group. $MP(S \times I)$ is called the *quasi-Fuchsian space*.

Let $M \in CB(S)$. If an embedding $f : S \hookrightarrow M$ is homotopic to an orientation preserving homeomorphism $S \to \partial_0 M$, f is called an *admissible embedding*. For an admissible embedding $f : S \hookrightarrow M$, the map

$$f^* : V(M) \to V(S)$$

defined by $[\rho] \mapsto [\rho] \circ f_*$ is a proper embedding.

Let $M_1, M_2 \in CB(S)$ and $f_j : S \hookrightarrow M_j (j = 1, 2)$ be admissible embeddings. Then the following holds:

- $\ker(f_1)_* = \ker(f_2)_* \Leftrightarrow (f_1)^*(AH(M_1)) = (f_2)^*(AH(M_2))$,
- $\ker(f_1)_* \neq \ker(f_2)_* \Leftrightarrow (f_1)^*(AH(M_1)) \cap (f_2)^*(AH(M_2)) = \emptyset$.

Let $M \in CB(S)$. Put

$$\begin{array}{ll} \mathcal{AH}(M) & = \bigcup_f f^*(AH(M)) \subset V(S) \\ \cup & \\ \mathcal{MP}(M) & = \bigcup_f f^*(MP(M)) \subset V(S), \end{array}$$

where the union is taken over all admissible embeddings $f : S \hookrightarrow M$.

Remark. In general, $\mathcal{MP}(M)$ consists of infinitely many connected components. On the other hand, $\mathcal{MP}(S \times I) = MP(S \times I)$ is connected.

Let $\mathrm{Teich}(S)$ be the Teichmüller space of S. Then

$$\mathcal{MP}(S \times I) = \mathrm{Teich}(S) \times \mathrm{Teich}(S).$$

We always fix $X \in \mathrm{Teich}(S)$ in the following. Let

$$C_X = \{[\rho, G] | G \text{ is a function group s.t. } \Omega_0(G)/G \cong X\}.$$

More precisely, if $\rho : \pi_1(S) \to G \cong \pi_1(N_G)$ is induced by $S \to X \cong \Omega_0(G)/G \hookrightarrow N_G$ for some function group G, then $[\rho, G]$ is an element of C_X. C_X is called an *extended Bers slice*.

Lemma 1. C_X is compact.

Put

$$\begin{array}{cl} \mathcal{AH}_X(M) & := \mathcal{AH}(M) \cap C_X \\ \cup & \\ \mathcal{MP}_X(M) & := \mathcal{MP}(M) \cap C_X. \end{array}$$

$B_X := \mathcal{MP}_X(S \times I) = \mathcal{MP}(S \times I) \cap C_X$ is called a *Bers slice*. Obviously

$$C_X = \bigcup_{M \in CB(S)} \mathcal{AH}_X(M).$$

3 Action of $\mathrm{Mod}(S)$ on C_X

Let $\mathrm{Mod}(S)$ denote the mapping class group of S. Let $[\rho, G] \in C_X$. Let $\mathrm{Belt}(X)_1$ denote the set of Beltrami differentials $\mu = \mu(z)\overline{dz}/dz$ on X such that $||\mu||_\infty < 1$.

$$\begin{array}{ccc} \mathrm{Belt}(X)_1 & \xrightarrow{\cong} & \mathrm{Belt}(\Omega_0(G)/G)_1 \\ \downarrow & & \downarrow \\ \mathrm{Teich}(S) & \xrightarrow[\Psi_\rho]{} & QC_0(\rho). \end{array}$$

$QC_0(\rho)$ consists of the qc-deformations of $[\rho, G]$ whose Beltrami differentials are supported on $\Omega_0(G)$.

The action of $\sigma \in \mathrm{Mod}(S)$ on C_X is defined by

$$[\rho] \mapsto [\rho]^\sigma := \Psi_\rho(\sigma^{-1}X) \circ \sigma_*^{-1},$$

where σ_* is the group automorphism of $\pi_1(S)$ induced by σ.

4 Continuity of the action

Theorem 2. Let $[\rho, G] \in C_X$. If all components of $\Omega(G)/G$ except for $X = \Omega_0(G)/G$ are thrice-punctured spheres, then the action of $\mathrm{Mod}(S)$ is continuous at $[\rho]$; that is, if $[\rho_n] \to [\rho]$ in C_X then $[\rho_n]^\sigma \to [\rho]^\sigma$ for all $\sigma \in \mathrm{Mod}(S)$.

Remark. In general, the action of $\mathrm{Mod}(S)$ is not continuous at $\partial B_X = \overline{B_X} - B_X$ (Kerckhoff-Thurston).

5 Maximal cusps

Put $\partial\mathcal{MP}_X(M) = \overline{\mathcal{MP}_X(M)} - \mathcal{MP}_X(M)$.

Definition. An element $[\rho, G] \in \partial\mathcal{MP}_X(M)$ is called a maximal cusp if G is geometrically finite and all components of $\Omega(G)/G$ except for $X = \Omega_0(G)/G$ are thrice-punctured spheres.

Theorem 3 (McMullen). The set of maximal cusps is dense in ∂B_X.

Proposition 4. For any $M \in CB(S)$, the set of maximal cusps is dense in $\partial\mathcal{MP}_X(M)$.

The set of maximal cusps in $\partial\mathcal{MP}_X(M)$ decomposes into finitely many orbit. The following theorem implies that "each" orbit is dense in $\partial\mathcal{MP}_X(M)$.

Theorem 5. For any maximal cusp $[\rho] \in \partial\mathcal{MP}_X(M)$, its orbit $\{[\rho]^\sigma\}_{\sigma\in\mathrm{Mod}(S)}$ is dense in $\partial\mathcal{MP}_X(M)$.

6 Statement of main thorem

Let $M_1, M_2 \in CB(S)$. An embedding $f : M_1 \hookrightarrow M_2$ is seid to be *admissible* if f is homotopic to an embedding $g : M_1 \hookrightarrow M_2$ such that $g|\partial M_1 : \partial M_1 \hookrightarrow M_2$ is a homeomorphism.

Theorem 6. Let $M \in CB(S)$ and $\{M_n\} \subset CB(S)$. If $\{[\rho_n] \in \mathcal{AH}_X(M_n)\}$ converges algebraically to $[\rho_\infty] \in \mathcal{AH}_X(M)$, then for large enough n there exist admissible embeddings $f_n : M \hookrightarrow M_n$.

This can be easily seen from the fact that $\ker\rho_n \supseteq \ker\rho_\infty$ for large enough n.

Lemma 7. Let $M_1, M_2 \in CB(S)$ and $[\rho] \in AH(M_2)$. If there is a sequence $\{\sigma_n\}$ of $\mathrm{Mod}(S)$ such that $[\rho]^{\sigma_n}$ converges algebraically to $[\rho_\infty] \in \mathcal{AH}_X(M_1)$, then there exist an admissible embedding $f : M_1 \hookrightarrow M_2$.

Conversely, the following holds.

Theorem 8. Let $M_1, M_2 \in CB(S)$. Suppose that there exists an admissible embedding $f : M_1 \hookrightarrow M_2$. Then for any geometrically finite element $[\rho] \in \mathcal{AH}_X(M_2)$, the set of accumulation points of $\{[\rho]^\sigma\}_{\sigma \in \mathrm{Mod}(S)}$ contains $\partial\mathcal{MP}_X(M_1)$.

Recall that S is a closed surface of genus $g \geq 2$. Let H_g be a handle body of genus g. Note that for any $M \in CB(S)$, there are embeddings

$$S \times I \hookrightarrow M, M \hookrightarrow H_g$$

which preserve the exterior bounbaries.

Corollary 9. (1) For any $[\rho] \in \mathcal{AH}_X(H_g)$, the set of accumulation points of $\{[\rho]^\sigma\}_{\sigma \in \mathrm{Mod}(S)}$ contains $\bigcup_{M \in CB(S)} \partial\mathcal{MP}_X(M)$.

(2) For any $M \in CB(S)$ and any geometrically finite $[\rho] \in \mathcal{AH}_X(M)$, the set of accumulation points of $\{[\rho]^\sigma\}_{\sigma \in \mathrm{Mod}(S)}$ contains $\partial B_X = \partial\mathcal{MP}_X(S \times I)$.

Remark (Hejhal,Matsuzaki). Let $[\rho] \in C_X$. $[\rho] \in \mathcal{AH}_X(H_g)$ if and only if $[\rho]$ is geometrically finite and isolated in C_X.