Polygonal construction of Riemann surfaces and Teichmüller spaces

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January 8, 2017
Riemann Surfaces and Discontinuous Groups
Sakura Hall, Tohoku University, Sendai, Japan

Contents

Note

本研究は科研費・挑戦的萌芽研究
研究課題名「新たなタイヒミュラー空間の構成とその解析」
（平成 27 年～平成 29 年，研究代表者：須川敏幸）の援助による．

Projective and affine structures on Riemann surfaces

Subgroups of analytic automorphism groups

Denote the Riemann sphere by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Recall:

$\mathrm{Aut}(\widehat{\mathbb{C}}) = \mathrm{PSL}(2,\mathbb{C}) = \mathrm{M\ddot{o}b}$: projective group
$\mathrm{Aut}(\mathbb{C}) = \mathrm{Aff}(1,\mathbb{C})$: affine group, and
$\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2,\mathbb{R})$,

where $\mathbb{H}$ is the upper half-plane. We will denote the group of orientation-preserving Euclidean motions by $\mathrm{EM}(2,\mathbb{R})$. Note that $\mathrm{EM}(2,\mathbb{R}) < \mathrm{Aff}(1,\mathbb{C}) < \mathrm{M\ddot{o}b}$ and

$$\dim_{\mathbb{R}} \mathrm{EM}(2,\mathbb{R}) = 3, \quad \dim_{\mathbb{R}} \mathrm{Aff}(1,\mathbb{C}) = 4, \quad \dim_{\mathbb{R}} \mathrm{M\ddot{o}b} = 6.$$

Projective, affine and Euclidean structures

Let us recall a definition of Riemann surfaces. Let $\mathfrak{U} = (U_\alpha, z_\alpha)_{\alpha \in A}$ be an atlas of a Hausdorff space X. Namely, $(U_\alpha)_{\alpha \in A}$ is an open covering of X and z_α is a homeomorphism of U_α onto an open set V_α in $\mathbb{C}$ for each $\alpha \in A$. The atlas $\mathfrak{U}$ is called holomorphic or complex analytic if the transition function $z_{\beta\alpha} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ is holomorphic whenever $U_\alpha \cap U_\beta$ is non-empty. Two atlases $\mathfrak{U}$ and $\mathfrak{U}'$ of X are said to be equivalent if the union of the two is again a holomorphic atlas of X. An equivalence class of holomorphic atlases is called a complex structure of X. The space X equipped with the complex structure $[\mathfrak{U}]$ is called a Riemann surface.
In the above, if we replace the terminology "holomorphic" by "projective", "affine" or "Euclidean", then the corresponding atlas or structure are called "projective", "affine" or 'Euclidean' respectively.

Euclidean $\Rightarrow$ affine $\Rightarrow$ projective $\Rightarrow$ complex ($\Rightarrow$ differentiable)

Canonical projective structure

By the uniformization theorem, a hyperbolic Riemann surface X has a (holomorphic) universal covering projection of the upper half-plane $\mathbb{H}$ onto X. Its local inverses gives a projective atlas of X, whose equivalence class will be called the canonical projective structure of X. In particular, we have seen

Theorem

Every hyperbolic Riemann surface admits a projective (more strongly, a $\mathrm{PSL}(2, \mathbb{R})$) structure.

Note that non-hyperbolic Riemann surfaces are $\widehat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ or complex tori only.

Existence of affine structures

The situation is different for affine structures. Obviously, complex tori have affine (indeed, Euclidean) structures. However, the other compact Riemann surfaces never have affine structures.

Theorem (cf. Gunning 1967)

If a compact Riemann surface admits an affine structure, then the genus must be 1.

On the other hand, for an open Riemann surface, we have the following result.

Theorem (Gunning 1967)

Every open Riemann surface admits an affine structure.

The space of projective structures

Fix a Riemann surface X and choose a reference projective atlas $(Y_\gamma, \zeta_\gamma)_{\gamma \in C}$ on X. For an arbitrary projective atlas $(U_\alpha, z_\alpha)_{\alpha \in A}$, We define a function $f_{\alpha,\gamma} = z_\alpha \circ \zeta_\gamma^{-1}$ on $\zeta_\gamma(U_\alpha \cap Y_\gamma)$ and take its Schwarzian derivative $\varphi_\gamma = S(f_{\alpha,\gamma})$, where

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 .$$

This $\varphi = \varphi_\gamma(\zeta_\gamma) d\zeta_\gamma^2$ does not depend on the choices of z_α and ζ_γ as a holomorphic quadratic differential on X. Even φ does not depend on the representative of an equivalence class of projective structures. As for this correspondence, the following fact is well known.

Proposition

The totality of projective structures on X corresponds bijectively to the set $Q(X)$ of holomorphic quadratic differentials on X.

Integrable projective structures

In the Teichmüller theory, an important subset of $Q(X)$ is the set of integrable ones; namely,

$$Q_1(X) = \{\varphi \in Q(X) : \iint_X |\varphi(z)||dz|^2 < +\infty\}.$$

When X is finite conformal type (g, n) with $2g - 2 + n > 0$,

$$\dim_{\mathbb{C}} Q_1(X) = 3g - 3 + n.$$

The space of affine structures

The set $Q(X)$ has a natural linear structure. When X is compact with genus g, it is also well known that $\dim_{\mathbb{C}} Q(X) = 3g - 3$ for $g > 1$. On the other hand, $\dim_{\mathbb{C}} Q(X) = \infty$ for non-compact X. For affine structures, a similar correspondence to holomorphic 1-forms (Abelian differentials) can be constructed for an open Riemann surface X by replacing the Schwarzian by the pre-Schwarzian derivative: $T(f) = \dfrac{f''}{f'}$.

Proposition

The totality of affine structures on an open Riemann surface X corresponds bijectively to the set $\Omega(X)$ of holomorphic 1-forms on X.

Note that $\dim_{\mathbb{C}} \Omega(X) = \infty$ for an open X whereas $\dim_{\mathbb{C}} \Omega(X) = g$ for a compact Riemann surface X of genus g.

Allowable affine structures

The natural counterpart of $Q_1(X)$ for affine structures would be the set $\Omega_2(X)$ of square integrable holomorphic 1-forms on X. However, such a 1-form does not have any pole at punctures. Namely, $\dim_{\mathbb{C}} \Omega_2(X) = g$ for X of type (g, n).
We will therefore say that a holomorphic 1-form $\varphi = \varphi(z)dz$ on X is allowable if φ extends meromorphically to each puncture of X in such a way that the order of the pole is at most one at the puncture. Let $\Omega_1(X)$ denote the set of allowable holomorphic 1-forms on X. By the Riemann-Roch theorem and the Serre duality, we compute

$$\dim_{\mathbb{C}} \Omega_1(X) = g + n - 1 \quad \text{for } n \geq 1.$$

Recall that $\dim_{\mathbb{C}} \Omega_1(X) = \dim_{\mathbb{C}} \Omega(X) = g$ when $n = 0$.

Cone metrics and Teichmüller spaces

Standard metrics

- **Elliptic case.** For the Rieman sphere $\widehat{\mathbb{C}}$, the spherical metric
$$\frac{|dz|}{1+|z|^2}$$
has constant curvature $+4$.
- **Parabolic case.** For the complex plane $\mathbb{C}$, the Euclidean metric
$$|dz|$$
has constant curvature 0.
- **Hyperbolic case.** For the unit disk $\mathbb{D} = \{z : |z| < 1\}$, the hyperbolic (or Poincaré) metric
$$\frac{|dz|}{1-|z|^2}$$
has constant curvature -4 and it is invariant under the pull-back action of $\mathrm{Aut}(\mathbb{D}) = \mathrm{PSU}(1,1) \cong \mathrm{PSL}(2,\mathbb{R})$.

Standard metric on Riemann surfaces

Let X be an arbitrary Riemann surface and let $p : \hat{X} \to X$ be a holomorphic universal covering. By the uniformization theorem, $\hat{X}$ is biholomorphically equivalent to $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$. Thus we may assume that $\hat{X}$ is one of them. Since the standard metric is invariant under the pull-back action of the covering transformation group $\Gamma = \{\gamma \in \mathrm{Aut}(\hat{X}) : p \circ \gamma = p\}$, the standard metric descends to a conformal metric on X. This will be called the standard metric of X and denoted by $\lambda_X = \lambda_X(z)|dz|$. Note that λ_X has Gaussian curvature $+4, 0$ or -4 and is often called spherical, flat (Euclidean), or hyperbolic according to the cases described in the previous slide. Note that the standard metric λ_X is complete.

Local model of conical singularity

As we saw, any Riemann surface has its own standard metric. In particular, all Riemann surfaces have hyperbolic metric except for $\widehat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*$ and complex tori. However, if we allow a sort of singularities, this "rigid" situation can be relaxed.
Let $\lambda(z) = 1/(1+\varepsilon|z|^2)$ for $\varepsilon = +1, 0, -1$ according to the three cases as before. We consider the (truncated) sector $\{re^{i\theta} : 0 \le r < \delta, 0 \le \theta \le 2\pi\alpha\}$ in the z-plane and glue the two sides by identifying r with $re^{2\pi\alpha i}$. The uniformizing parameter is now given as $w = z^{1/\alpha}$. The pull-back of the metric $\lambda(z)|dz|$ is given by

$$\lambda_\alpha(w)|dw| := \lambda(w^\alpha)|dw^\alpha| = \frac{\alpha|w|^{\alpha-1}}{1+\varepsilon|w|^{2\alpha}}|dw|.$$

By this formula, the parameter α is allowed to be in $(0, +\infty)$. This is a model of the metric of constant curvature 4ε with conical singularity at $w = 0$. Note that the Gaussian curvature of λ_α is 4ε for $w \ne 0$.

Conical singularity

If a conformal metric on a punctured neighbourhood of a point $x_0 \in X$ is isometric to the above model through a biholomorphic map near x_0 then we say that the metric has a conical singularity at z_0 of angle $2\pi\alpha$ or of order $\sigma = 1 - \alpha$. When $\alpha = 1/k$, a conical singularity is nothing but a branch singularity (or a gyration point) of order k.
A spherical (Euclidean, hyperbolic) cone metric on X will mean a complete conformal metric ρ of constant curvature $+4$ $(0, -4$ resp.) on X with conical singularities at a discrete subset of X. We denote by $\mathrm{Sing}(\rho)$ the singular locus of ρ and by $\alpha(x_0)$ and $\sigma(x_0)$ the angle/2π and the order at a point $x_0 \in \mathrm{Sing}(\rho)$.
For existence of such a metric, the following relation is necessary to hold by Gauss-Bonnet formula:

$$\chi(X) - \sum_{x \in \mathrm{Sing}(\rho)} \sigma(x) = \frac{4\varepsilon}{2\pi} \iint_X \rho^2,$$

where $\chi(X)$ denotes the Euler characteristic of X.

Teichmüller spaces

We recall a definition of Teichmüller spaces briefly. Let $f_j : X \to Y_j$ $(j = 1, 2)$ be quasiconformal homeomorphisms of a Riemann surface X onto another Y_j. These two maps are said to be Teichmüller equivalent if there is a biholomorphic map $g : Y_1 \to Y_2$ such that $f_2^{-1} \circ g \circ f_1$ is homotopic to the identity map with a homotopy keeping the ideal boundary fixed pointwise. The Teichmüller space of X is the set of Teichmüller equivalence classes $[f : X \to Y]$ of quasiconformal maps $f : X \to Y$ and denoted by $\mathrm{Teich}(X)$. It is well known that Teichmüller spaces carry natural complex structures. In particular, for a conformally finite surface X of type (g, n) we know if $2g - 2 + n > 0$:

$$\dim_{\mathbb{C}} \mathrm{Teich}(X) = 3g - 3 + n.$$

Teichmüller spaces: another model

When X is of finite conformal type, a classical description of the Teichmüller space is available and more convenient in some cases. For a Riemann surface X of type (g,n) with a basepoint x we can choose generators a_j, b_j, c_k $(j=1,\dots,g, k=1,\dots,n)$ of the fundamental group $\pi_1(X,x)$ with the relation

$$[a_1,b_1]\cdots[a_g,b_g]c_1\cdots c_n=1.$$

We will call it a standard generator system of $\pi_1(X,x)$ and denote by $\mathscr{G}=(a_1,\dots,c_n)$. Two pairs $(Y_j,\mathscr{G}_j)$, $j=1,2$, are said to be equivalent if there exists a conformal homeomorphism $g:Y_1\to Y_2$ such that $g_*(\mathscr{G}_1)=\mathscr{G}_2$ up to basepoint move. The set of equivalent classes $[Y,\mathscr{G}]$, where Y are Riemann surfaces of finite conformal type (g,n), is the Teichmüller space $T_{g,n}$ of type (g,n). We have a natural identification of $T_{g,n}$ with $\mathrm{Teich}(X)$.

Teichmüller space of cone metric?

For a conformally finite Riemann surface X, the Teichmüller space $\mathrm{Teich}(X)$ can also be understood as hyperbolic metrics on X (as a differentiable manifold) modulo isotopies.
Let X be a compact (orientable) surface minus finitely many punctures, namely, X is of type (g, n) for some g, n. For a given m-tuple $\alpha = (\alpha_1, \dots, \alpha_m) \in \big((0, +\infty) \setminus \{1\}\big)^m$ and $\varepsilon = +1, 0, -1$, we may consider the Teichmüller space of standard cone metrics of constant curvature 4ε on X with m (varying) singularities at $x_1, \dots, x_m$ of angles $2\pi\alpha_1, \dots, 2\pi\alpha_m$, respectively. This Teichmüller space will be denoted by $\mathrm{Teich}(X, \alpha)$. Though we have no intrinsic realization of $\mathrm{Teich}(X, \alpha)$ except for the orbifold cases, we can think of $\mathrm{Teich}(X \setminus \{x_1, \dots, x_m\})$ as its model. Hence, generically, we have

$$\dim_{\mathbb{C}} \mathrm{Teich}(X, \alpha) = 3g - 3 + n + m.$$

Polygonal construction of Riemann surfaces

A polygonal construction

Let P be a closed polygonal Jordan domain with $2N$ vertices $z_1, z_2, \dots, z_{2N}$ in counter clockwise order. In what follows, the indices will be understood modulo $2N$; namely, the indices are elements in $\mathbb{Z}(2N) := \mathbb{Z}/2N\mathbb{Z}$. We denote by s_j the side $[z_{j-1}, z_j]$ with positive orientation. A pairing of the sides will mean an involution $\sigma : \mathbb{Z}(2N) \to \mathbb{Z}(2N)$ without fixed points. Let A_j be a unique affine map sending s_j to $s_{\sigma j}$ with orientation being reversed. Hence, $A_j(z_{j-1}) = z_{\sigma j}$ and $A_j(z_j) = z_{\sigma j - 1}$. Note that $A_{\sigma j} = A_j^{-1}$. By gluing the side s_j with $s_{\sigma j}$ by A_j for each j, we obtain a compact Riemann surface, say, X. We denote by $\pi : P \to X$ the natural projection. The image of the vertex set under the projection π plays a special role in X. We set $X^\circ = X \setminus \pi(\{z_1, \dots, z_{2N}\})$. Let m be the number of the punctures of X°. The Riemann-Hurwitz relation tells us that the genus g of X can be computed by

$$2g = N - m + 1.$$

Computation of m

This kind of construction is not new. For instance, see "Complex Analysis 2" by E. Freitag. This was also studied by my (former) master students Yamashita and Gouke.

As we saw, $A_j(z_j) = z_{\sigma j-1}$. Thus we define a map $\tau : \mathbb{Z}(2N) \to \mathbb{Z}(2N)$ by $\tau j = \sigma j - 1$. Then $\pi(z_j) = \pi(z_k)$ iff j and k are in the same orbit under the action of τ. Thus m is the number of the orbits in $\mathbb{Z}(2N)$ under τ.

Mapping into the Teichmüller space

Fix $N \geq 2$ and a pairing $\sigma : \mathbb{Z}(2N) \to \mathbb{Z}(2N)$. We will normalize polygons so that $z_{2N}(= z_0) = 0$ and $z_1 = 1$. Let V be the set of $(2N-2)$-tuples $(z_2, \ldots, z_{2N-1}) \in \mathbb{C}^{2N-2}$ for which $(1, z_2, \ldots, z_{2N-1}, 0)$ forms a $2N$-gon, say P_v, in positive order. Here, we allow $0 < \theta < 2\pi$ (including $\theta = \pi$) as the interior angle at each vertex. Then V is a domain in $\mathbb{C}^{2N-2}$.
For each $v \in V$, we take a basepoint x_v in the interior of the polygon P_v. Then any element of the fundamental group $\pi_1(X_v^\circ, x_v)$ can be encoded by labeling the sides through which the (representing) curve passes in the order of occurrence. Therefore, upon fixing a marking (a generator system of the fundamental group), we can construct a mapping from V into the Teichmüller space $T_{g,m}$. We denote it by $F : V \to T_{g,m}$.

A simple observation

It is an interesting problem to describe the image $F(V)$ in $T_{g,m}$, where m is the number of equivalence classes of the vertices of P_v and $g = (N - m + 1)/2$. Note that

$$\begin{aligned} & \dim_{\mathbb{C}} V - \dim_{\mathbb{C}} T_{g,m} \\ = \ & (2N - 2) - (3g - 3 + m) \\ = \ & \frac{1}{2}(N + m - 1) = g + m - 1. \end{aligned}$$

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This is the same as $\dim_{\mathbb{C}} \Omega_1(X_v)$!

Mapping into the bundle of allowable affine structures

It is thus reasonable to consider the holomorphic vector bundle of allowable affine structures over the Teichmüller space:

$$\mathscr{B} = \bigsqcup_{[X,\mathscr{G}]\in T_{g,m}} \Omega_1(X).$$

We denote by $\Pi : \mathscr{B} \to T_{g,n}$ the canonical projection. Then each fibre $\Pi^{-1}(X)$ is the vector space $\Omega_1(X)$. For each $v \in V$, we can associate an affine structure on the surface $X_v = \pi(P_v)$ in a natural way. If we have a holomorphic family of allowable affine structure on $T_{g,m}$ (maybe locally only). Then the above natural affine structure is realized as a holomorphic 1-form φ_v on X_v.

Theorem

The mapping $v \mapsto (\varphi_v, X_v)$ defines a holomorphic map $\hat{F} : V \to \mathscr{B}$ with $F = \Pi \circ \hat{F}$.

We can ask if $\hat{F}$ is locally biholomorphic.

Outline of the proof

Let P_0 be a reference polygon with vertices $1, \zeta_2, \dots, \zeta_{2N-1}, 0$ in this order and let X_0 be the corresponding Riemann surface. (For instance, P_0 may be a regular $2N$-gon.) The reference polygon can be triangulated with $2N-2$ triangles. We fix this triangulation. For any $v \in V$, we can construct a triangulation corresponding to that of P_0. Let $\tilde{f} : P_0 \to P_v$ be the piecewise affine mapping which preserves the triangulations. Then $\tilde{f}$ induces a quasiconformal mapping $f : X_0 \to X = \pi(P_v)$. Since the Beltrami coefficient $\mu_f = \bar{\partial} f / \partial f$ depends holomorphically on v, this induces a holomorphic mapping $F : V \to \mathrm{Teich}(X_0^\circ)$. Holomorphy of φ_v needs more investigations.

Relation with singular Euclidean structures

Singular Euclidean structure

Let X be a Riemann surface of finite conformal type (g, n) with $n \geq 1$. A Euclidean structure on X is called singular if (by abuse of language) we can choose a finite triangulation of X as a representative of the Euclidean structure. For details, a nice survey is recommended: "On the moduli space of singular euclidean surfaces" by M. Troyanov, in "Handbook of Teichmüller Theory Vol.I".
In particular, he showed that every punctured surface admits a singular Euclidean structure.

Euclidean isometry case

In the polygonal construction, we consider the special situation that paired sides have the same length for each. In other words, the side pairing transformation A_j is a Euclidean motion for each j. Then the resulting surface X_v has a singular Euclidean structure and, indeed, the Euclidean metric on P_v induces a Euclidean metric on X_v with conical singularities at the image of the vertices. Let $x_1, \ldots, x_m$ be the distinct images of the vertices and denote by $2\pi\alpha_j$ be the cone angle of the metric. Since the total interior angle of the polygon P_v is $(2N-2)\pi$, we have the relation

$$\alpha_1 + \cdots + \alpha_m = N - 1.$$

We remark that this can also be obtained by the Gauss-Bonnet formula.

Moduli space of singular Euclidean surfaces

Let V_0 denote the subset of V for which the above conditions are satisfied. We now compute the real dimension of V_0. The number of constraints is N intuitively. Therefore,

$$\dim_{\mathbb{R}} V_0 = \dim_{\mathbb{R}} V - N = 3N - 4 = 6g + 3m - 7.$$

Let $\tilde{T}_{g,m}$ be a fibre space over $T_{g,m}$ with fibre $\{(\alpha_1, \ldots, \alpha_m) \in (0, +\infty)^m : \alpha_1 + \cdots + \alpha_m = 2g - 2 + m\}$. (We can think that $\tilde{T}_{g,m} = T_{g,m} \times (0, +\infty)^m$.) We denote by $\Pi_2 : \tilde{T}_{g,m} \to (0, +\infty)^m$ the projection onto the second factor. Then $\Pi_2^{-1}(\alpha)$ can be identified with $\mathrm{Teich}(X, \alpha)$. We now observe that

$$\dim_{\mathbb{R}} \tilde{T}_{g,m} = (m - 1) + (6g - 6 + 2m) = 6g + 3m - 7.$$

We have a natural mapping $\tilde{F} : V_0 \to \tilde{T}_{g,m}$. We expect that this is locally injective.

Thanks

Thank you very much for your attention!