Veering triangulations of mapping tori of some pseudo-Anosov maps arising from Penner's construction

Naoki Sakata

Hiroshima University

JSPS Research Fellow (PD)

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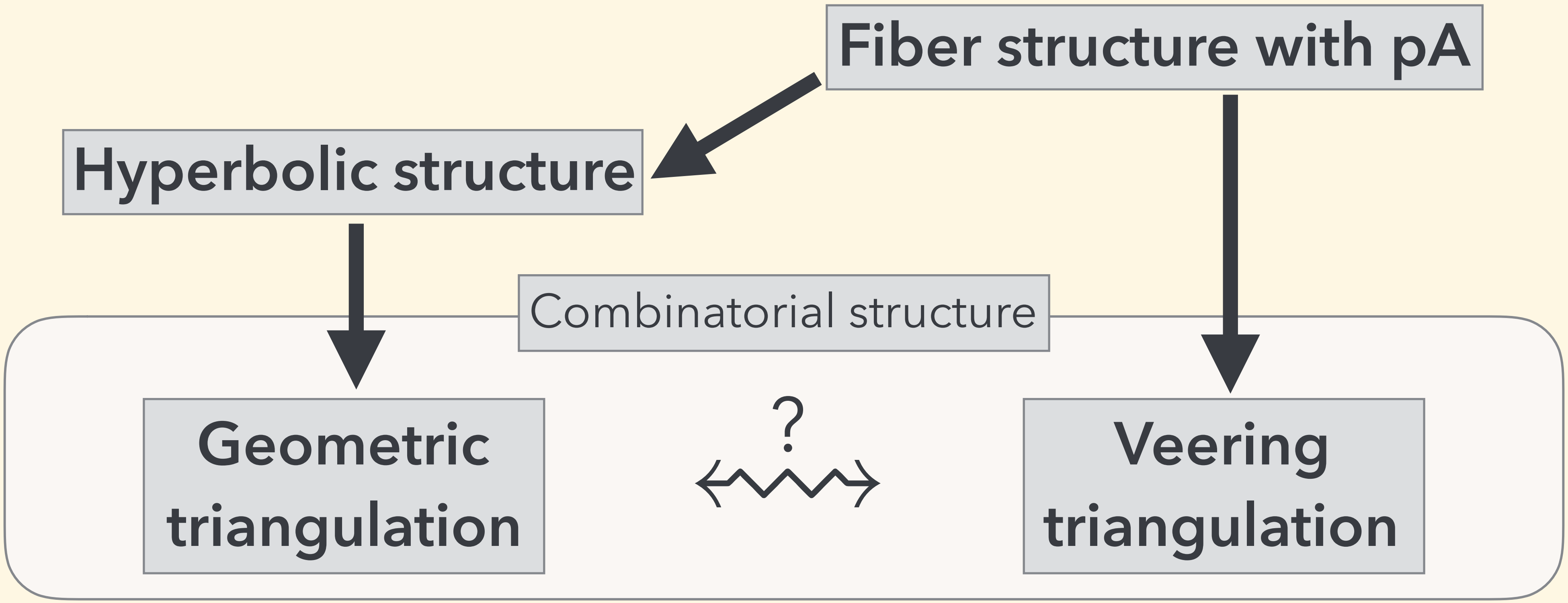
「リーマン面・不連続群論」研究集会

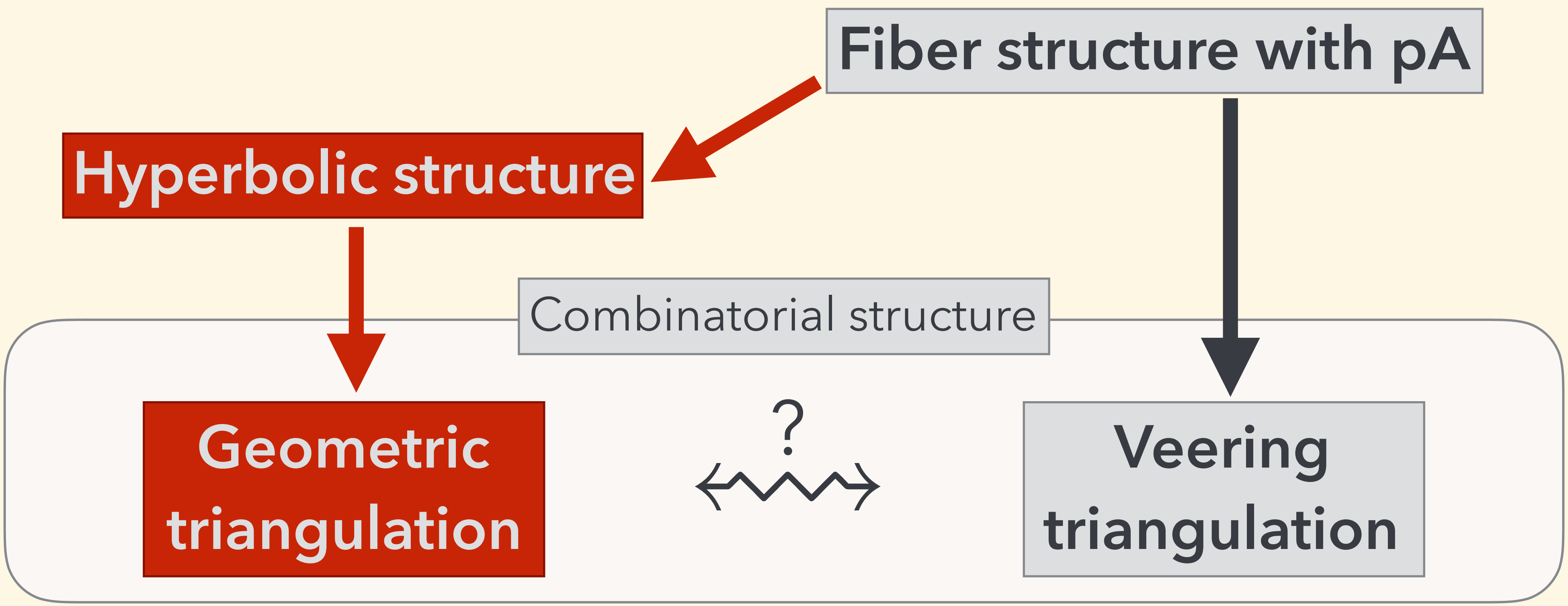
In this talk, we consider **ideal triangulations** of the mapping tori of pseudo-Anosov mapping classes.



In this talk, we consider ideal triangulations of the mapping tori of pseudo-Anosov mapping classes.

Theorem ([Epstein-Penner, 1988])

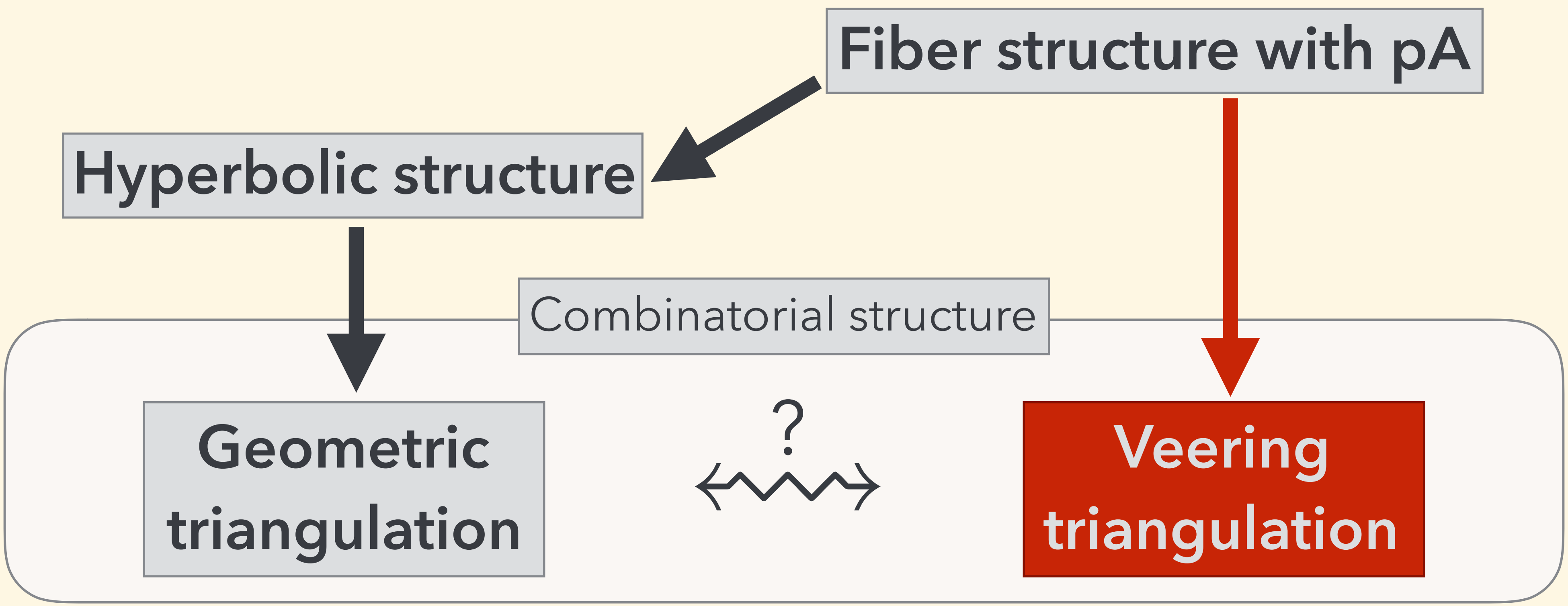
Each cusped hyperbolic manifold of finite volume admits a **canonical** decomposition into ideal polyhedra.



In this talk, we consider ideal triangulations of the mapping tori of pseudo-Anosov mapping classes.

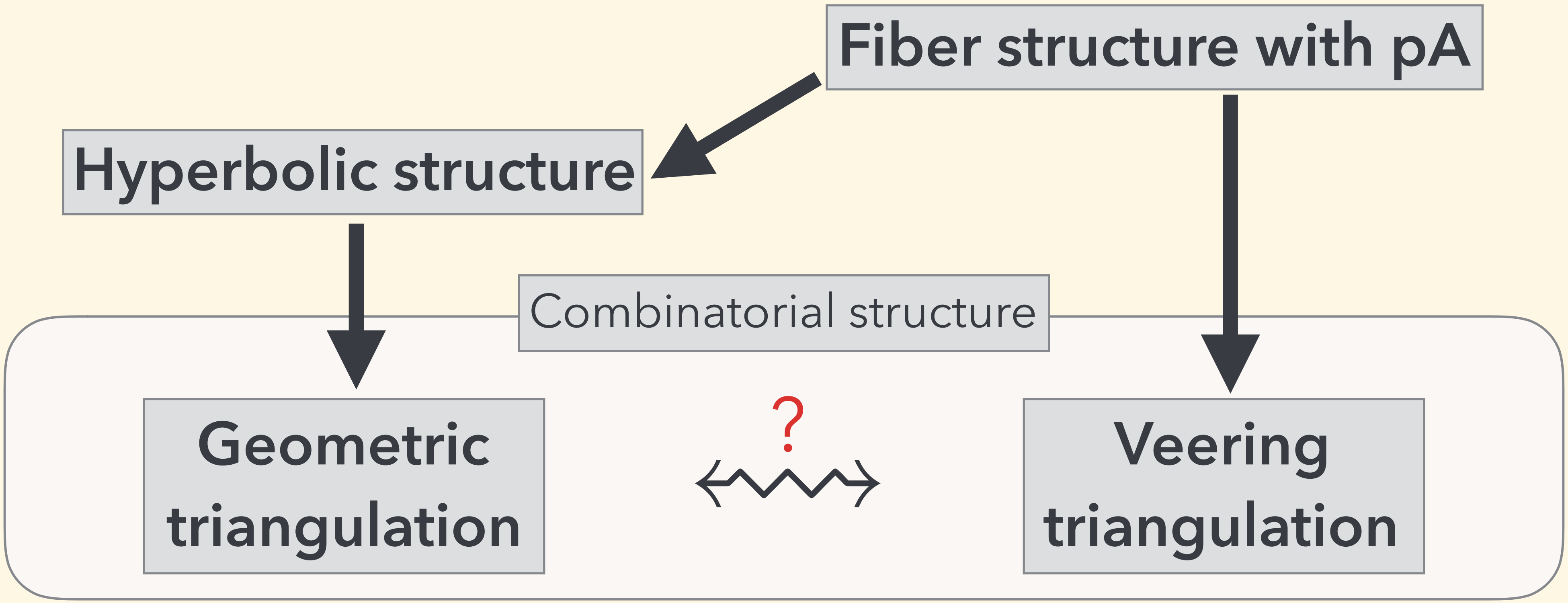
Theorem ([Agol, 2011])

For each pseudo-Anosov surface bundle over S^1, drilled along singular points of the stable/unstable foliations, has a **canonical** "veering" ideal triangulation.



Question

Is there a relationship between them?

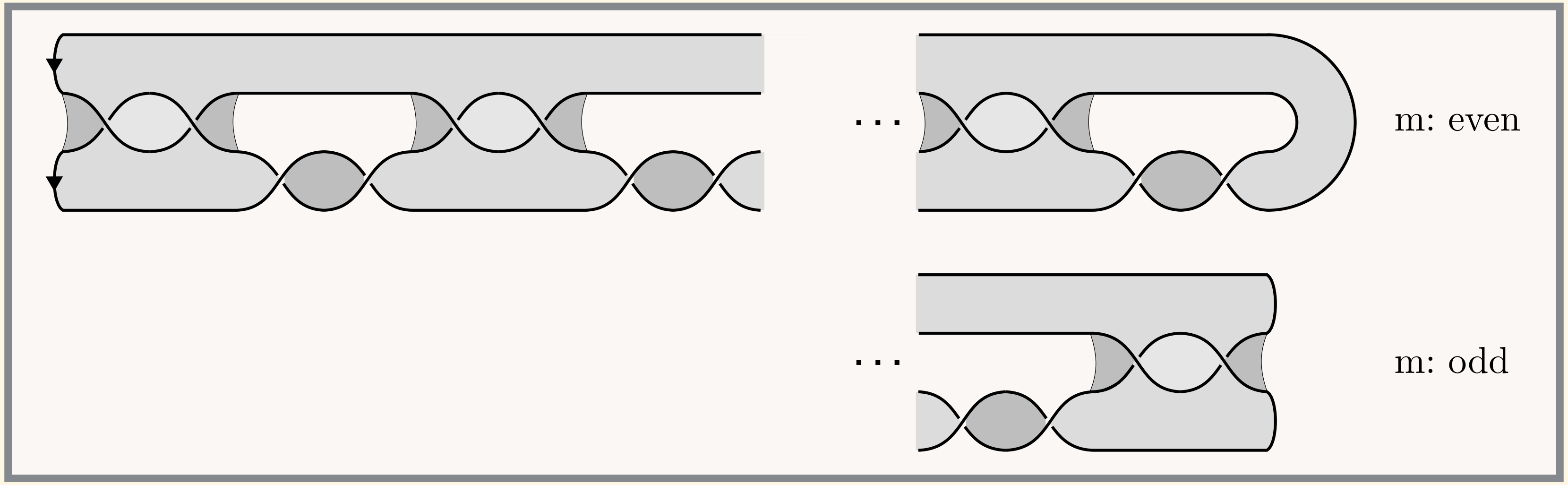


Fact (Jorgensen, etc.)

The Epstein-Penner decomposition of each **once-punctured torus bundle** over S^1 is veering.

Theorem ([S., 2016])

The Epstein-Penner decomposition of a hyperbolic fibered two-bridge link $K(r)$ $(0 < |r| < 1/2)$ is veering $\Longleftrightarrow$ the slope r has the continued fraction expansion $\pm[2, 2, \ldots, 2]$



Question

Are the veering ideal triangulations geometric?

Theorem ([Hodgson-Rubinstein-Segerman-Tillmann, 2011])

Each veering triangulation admits a strict angle structure.

However…

Theorem ([Hodgson-Issa-Segerman, 2016])

$\exists$ a non-geometric veering ideal triangulation.

$\rightsquigarrow$ How can we characterize the geometric veering triangulations?

Definition ([Shin-Strenner, 2015])

A pseudo-Anosov mapping class is **coronal**
$\overset{\text{def}}{\iff}$ the stretch factor has a Galois conjugate on the unit circle

Theorem ([Shin-Strenner, 2015])

A coronal pseudo-Anosov mapping class has no power coming from Penner's construction.

Computer Experiment (S.)

The pseudo-Anosov mapping classes in the list of non-geometric veering triangulations contained in [Hodgson-Issa-Segerman, 2016] are coronal.

φ : pseudo-Anosov map of a surface F

$$F^\circ := F \setminus \{\text{a singular point of the stable/unstable foliation}\}$$

$$\varphi^\circ := \varphi|_{F^\circ}$$

$$M_{\varphi^\circ} := F^\circ \times [0,1]/(x,0) \sim (\varphi^\circ(x),1)$$

: the mapping torus of φ°

(In this talk, we call M_{φ° the mapping torus of φ.)

Question

Is there a relationship between
"geometric veering triangulation of M_{φ°"
and "coronality of φ"?

Main Question

Is the veering triangulation of the mapping torus of each pA mapping class arising from Penner's construction geometric?

Penner's construction

$\mathcal{A} := \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$: mutually disjoint essential simple
$\mathcal{B} := \{\beta_1, \beta_2, \ldots, \beta_\ell\}$: closed curves in F

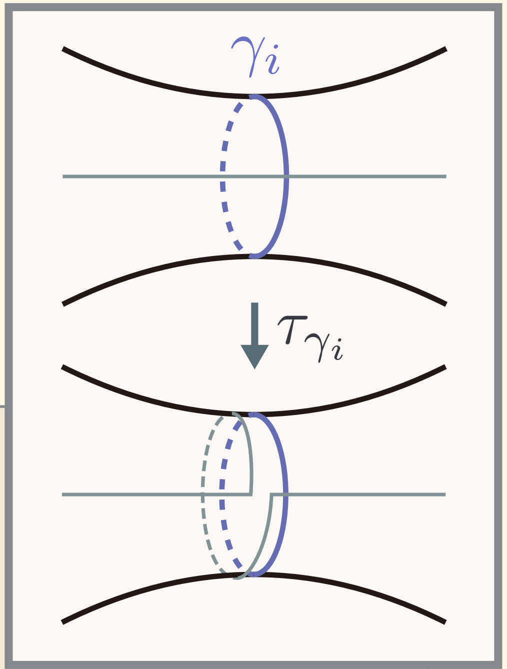
$\mathcal{A} \cup \mathcal{B}$ fills F

$\omega = \gamma_1\gamma_2\cdots\gamma_n$: (positive) word $\quad (\gamma_i \in \mathcal{A} \cup \mathcal{B})$

$\omega \rightsquigarrow \varphi_\omega = \varphi_{\gamma_1} \circ \varphi_{\gamma_2} \circ \cdots \circ \varphi_{\gamma_n} : F \to F$ defined by

$$\varphi_{\gamma_i} = \begin{cases} \tau_{\gamma_i}^{-1} & (\gamma_i \in \mathcal{A}) \\ \tau_{\gamma_i} & (\gamma_i \in \mathcal{B}) \end{cases},$$

where τ_{γ_i} : left-hand Dehn twist along γ_i.

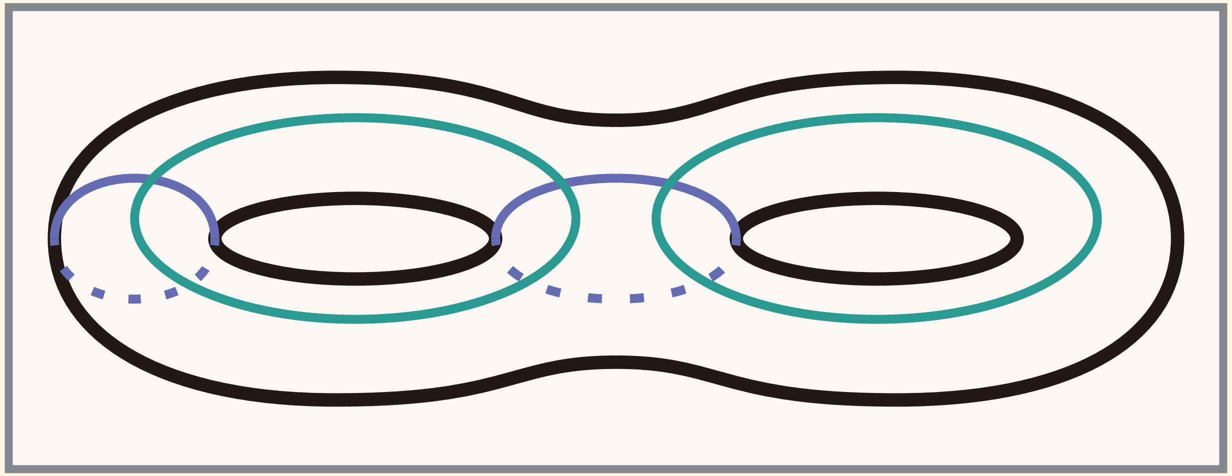


Theorem ([Penner, 1988])

Each α_i and β_i occur at least once in ω
$\implies$ the class of φ_ω is pseudo-Anosov

Main Result

We have a complete combinatorial description of the veering triangulations of the mapping tori of the pA mapping classes arising from Penner's construction such that all complementary regions are **not** quadrilateral.



In this talk, we will describe the veering triangulation of such a mapping torus.

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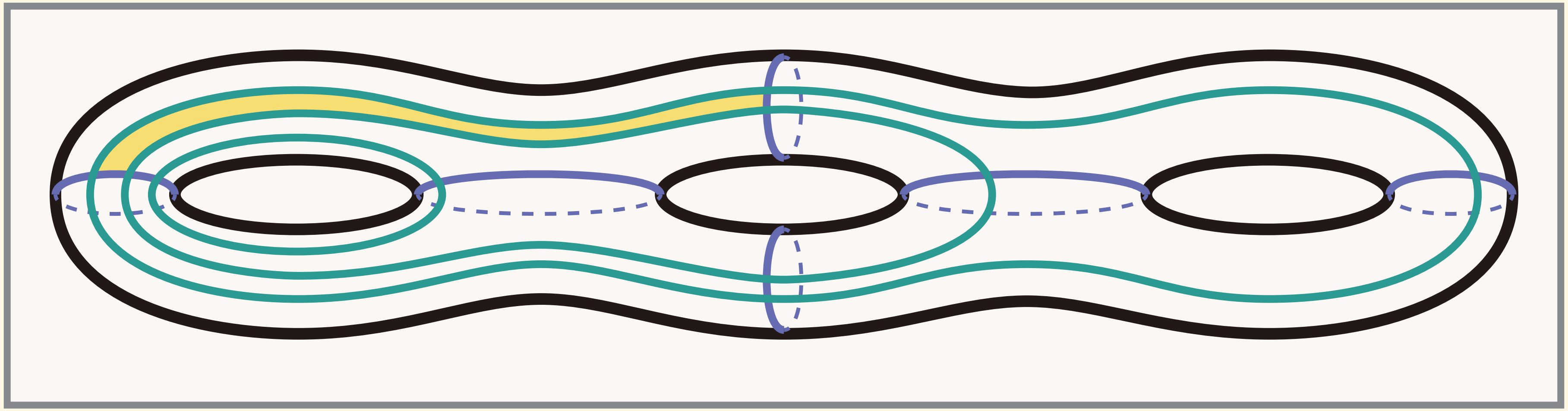


In this talk, we will describe the veering triangulation of such a mapping torus.

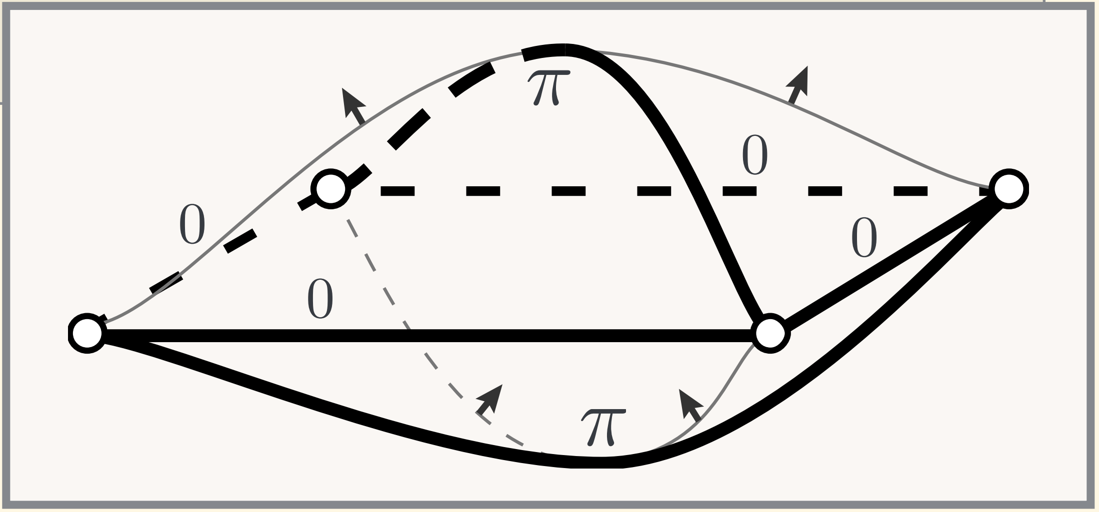
Taut structure (1)

Definition

an ideal tetrahedron: **taut**

$\overset{\text{def}}{\Longleftrightarrow}$ (i) Each face is assigned a co-orientation so that two co-orientations point inwards and the others point outwards.

(ii) Each edge of the tetrahedron is assigned an angle of either π or 0 according to whether the co-orientations on the adjacent faces are the same or different.



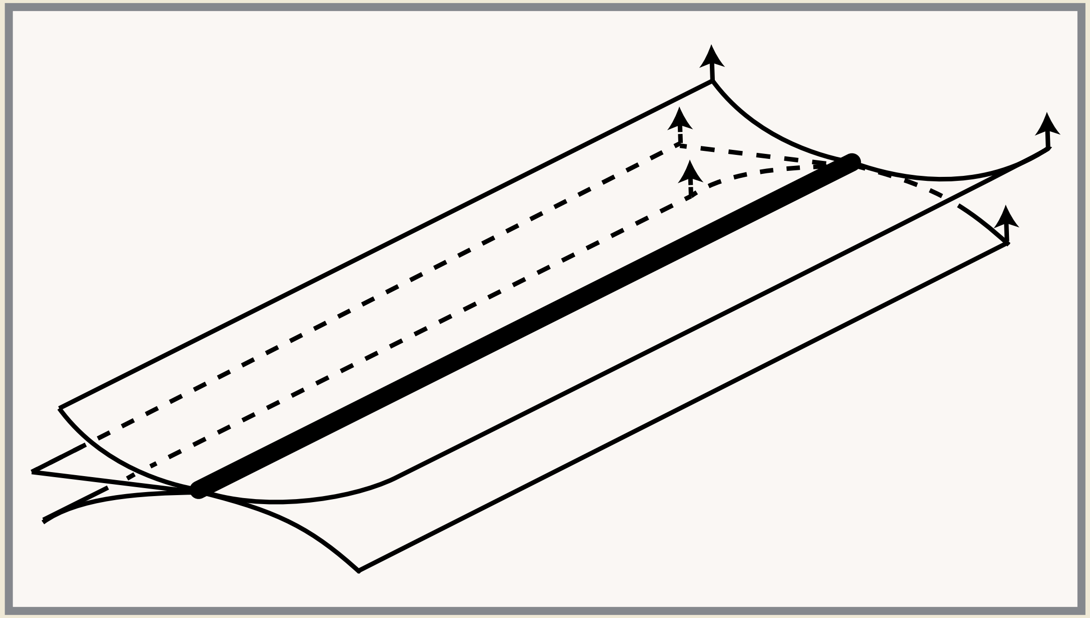
Taut structure (2)

Definition

An ideal triangulation of M_{φ° : **taut**

$\overset{\text{def}}{\Longleftrightarrow}$ (i) $\exists$a co-orientation assigned to each faces $\mathrm{s.t.}$ each ideal tetrahedron is taut.

(ii) The sum of the angles around each edge is 2π.

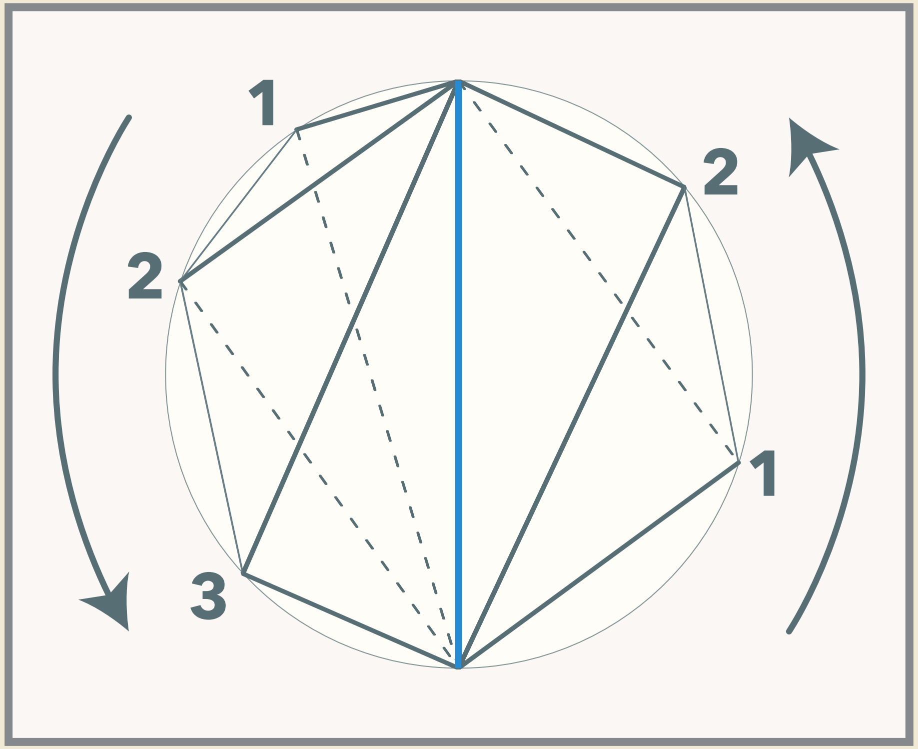


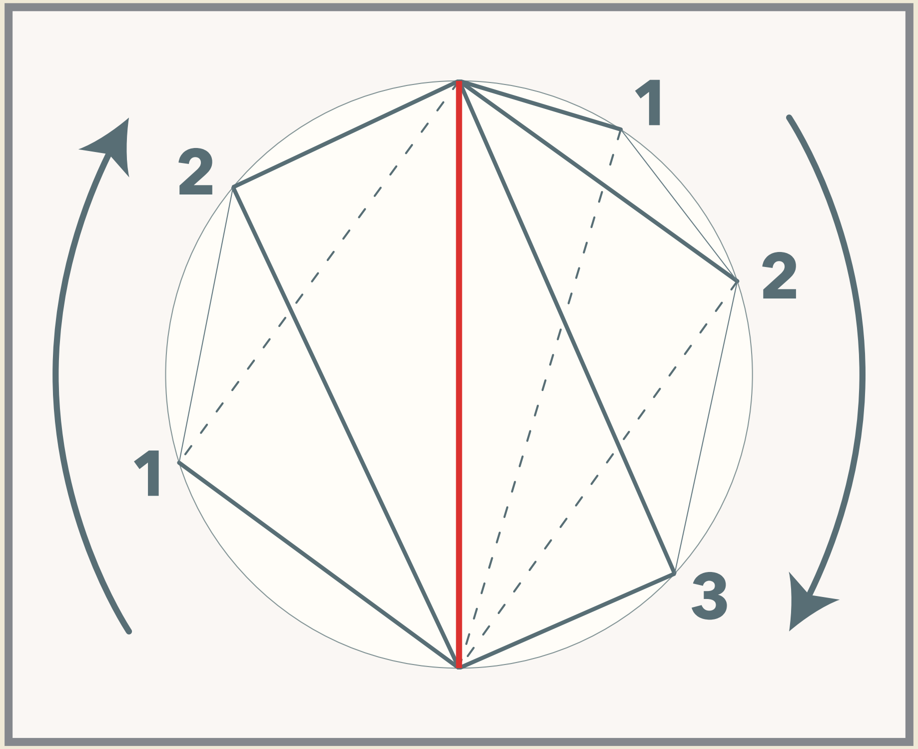
Veering triangulation (1)

Definition ([Agol, 2011])

taut triangulation $\mathcal{D}$: **veering**

$\overset{\text{def}}{\Longleftrightarrow}$ Each ideal edge of $\mathcal{D}$ is one of the following:



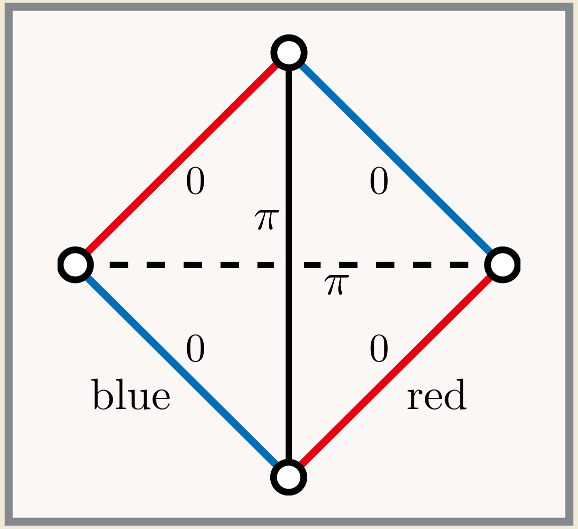


Veering triangulation (2)

Proposition ([Hodgson-Rubinstein-Segerman-Tillmann, 2011])

taut triangulation $\mathcal{D}$: **veering**

$\Longleftrightarrow$ $\exists$ assignment of two colors, red and blue, to all ideal edges of $\mathcal{D}$ so that every ideal tetrahedron can be sent by an orientation preserving homeomorphism to the following tetrahedron.



A construction of the veering triangulation (1)

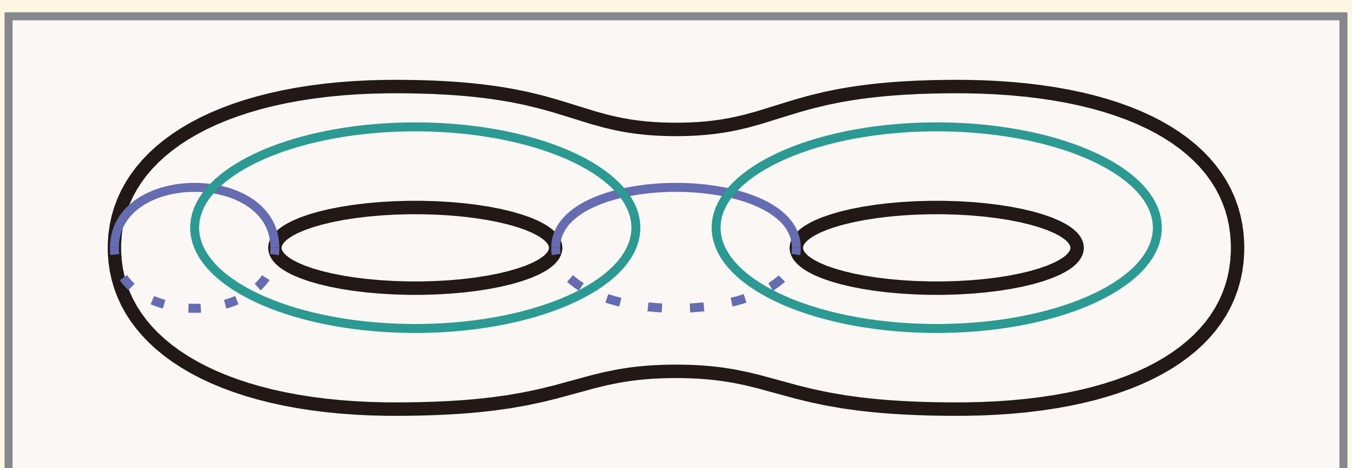
$\mathcal{A} := \{\alpha_1, \ldots, \alpha_k\}$: mutually disjoint essential simple closed curves in F
$\mathcal{B} := \{\beta_1, \ldots, \beta_\ell\}$

- $\mathcal{A} \cup \mathcal{B}$ fills F
- all complementary regions are *not* quadrilateral

$\omega = \gamma_1 \gamma_2 \cdots \gamma_n$: (positive) word $\quad (\gamma_i \in \mathcal{A} \cup \mathcal{B})$

$\omega \rightsquigarrow \varphi_\omega = \varphi_{\gamma_1} \circ \varphi_{\gamma_2} \circ \cdots \circ \varphi_{\gamma_n} : F \to F$

We'll construct the veering triangulation of $M_{\varphi^\circ_\omega}$.



A construction of the veering triangulation (2)

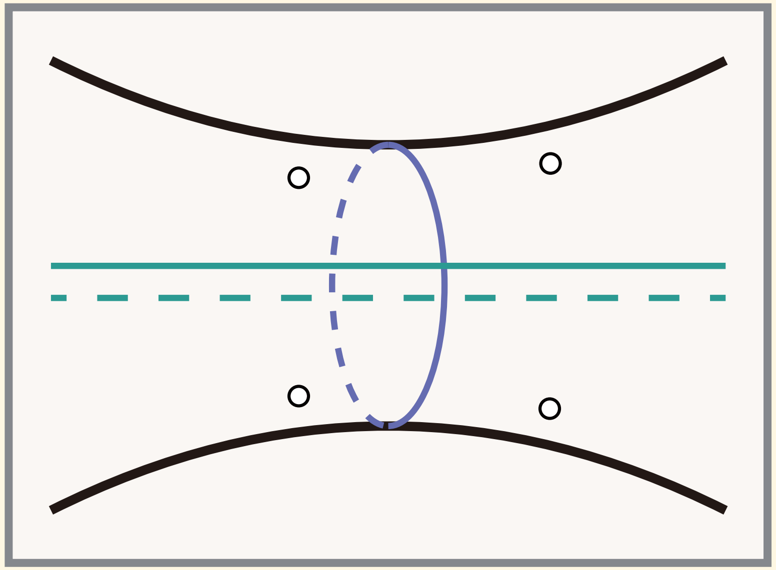
Step 1:

We construct an ideal triangulation of F.

To construct the ideal triangulation,

(i) We consider the cell complex dual to the 1-skeleton of the curves $\mathcal{A} \cup \mathcal{B}$.

(ii) We add a diagonal of each 2-cell.



Note

Each complementary region has a puncture.

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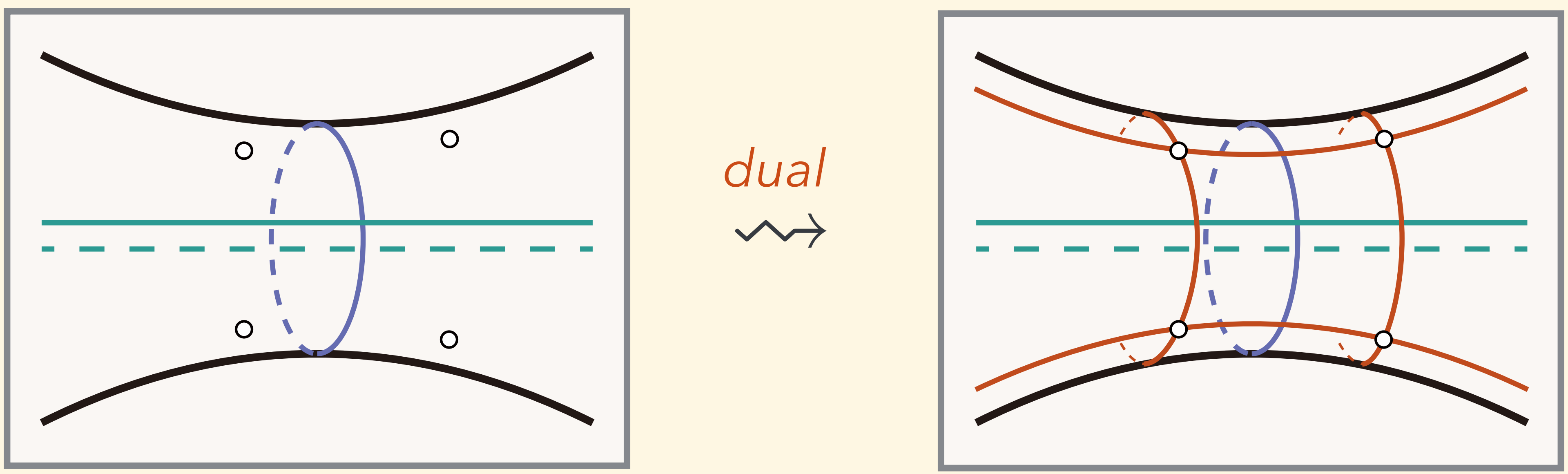

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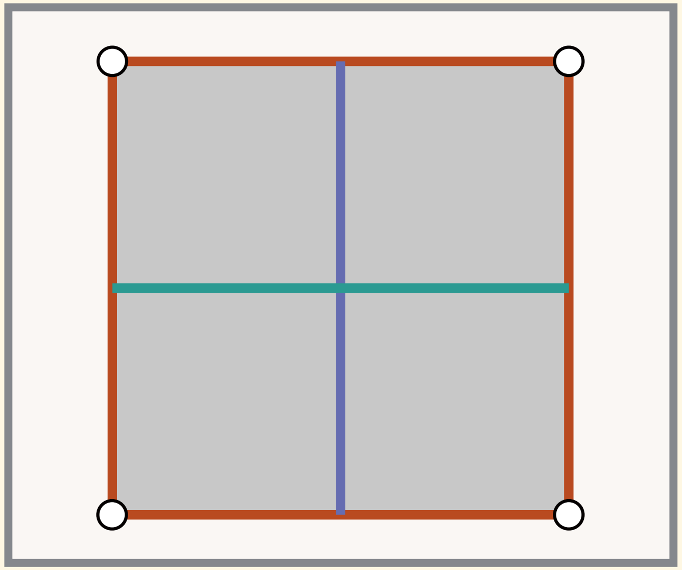

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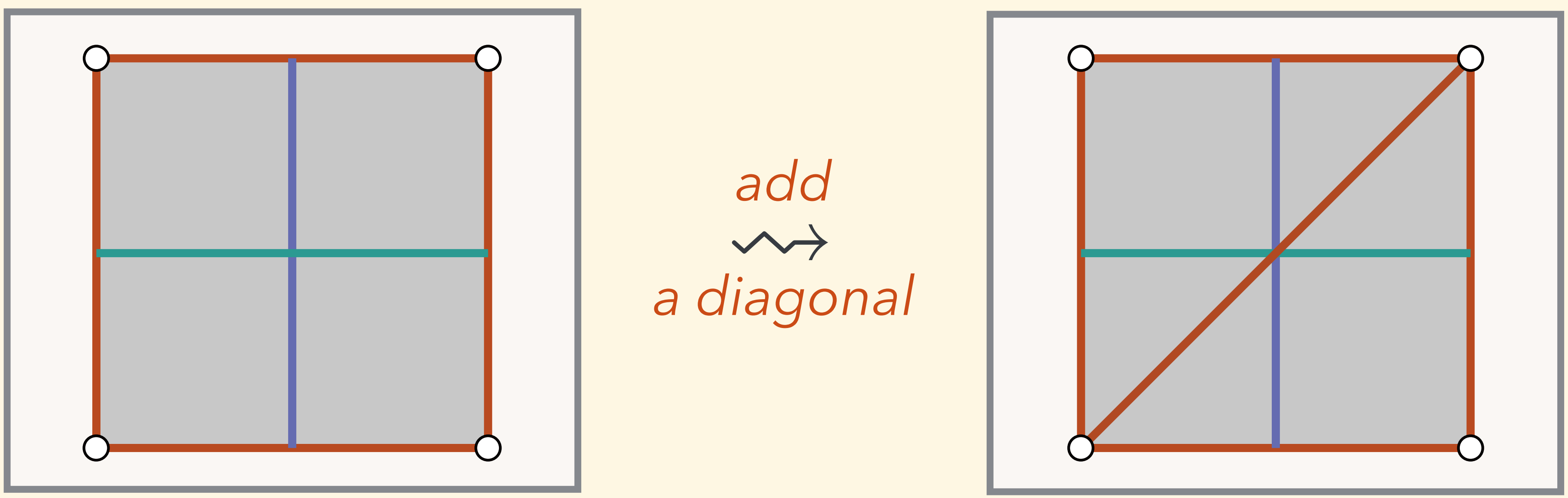


A construction of the veering triangulation (3)

Step 2:

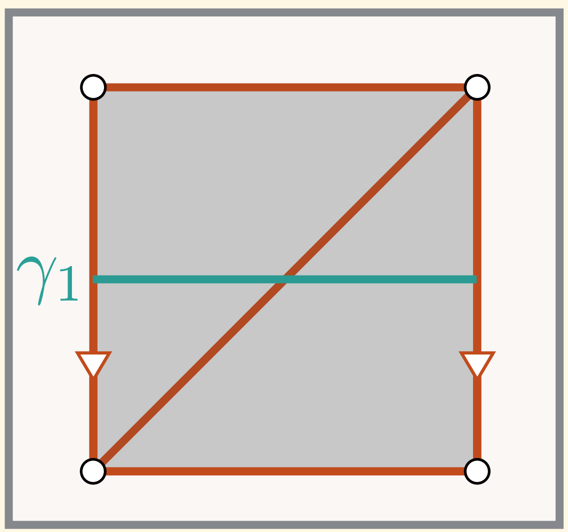
We attach veering tetrahedra.

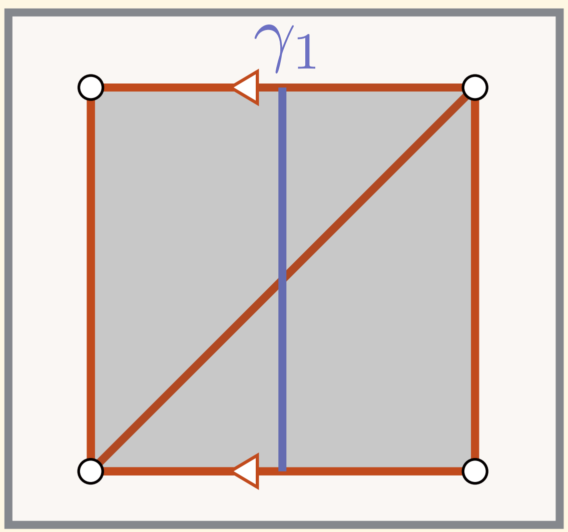
$\omega = \gamma_1\gamma_2\cdots\gamma_n$: (positive) word $\quad (\gamma_i \in \mathcal{A} \cup \mathcal{B})$

$n(\gamma_i) := \#$ of $\gamma_i \cap (\mathcal{A} \cup \mathcal{B})$

Case $i = 1$:

We attach $n(\gamma_1)^2$ tetrahedra to the ideal triangulation of F along γ_1.





$(n(\gamma_1) = 1)$

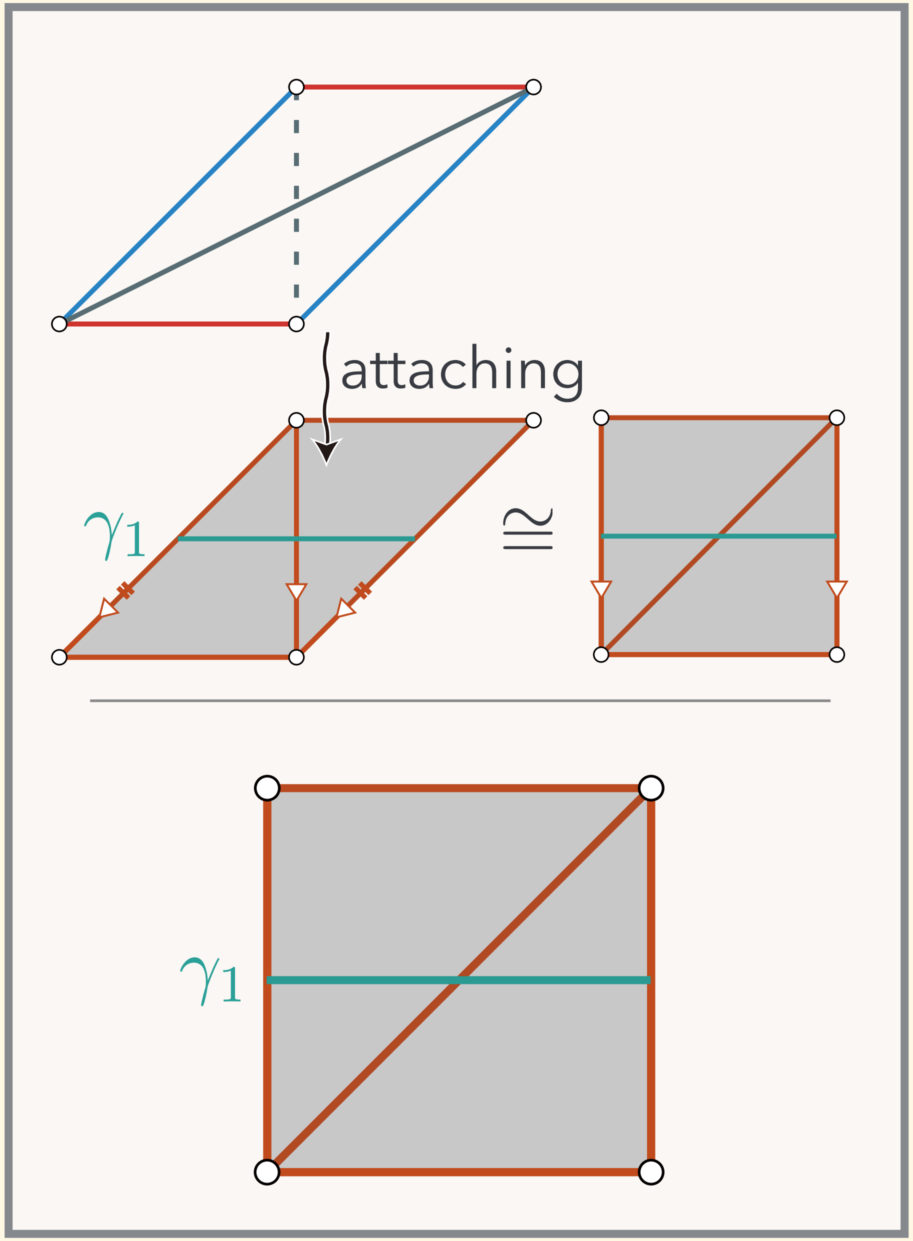
A construction of the veering triangulation (4)

$\gamma_1 \in \mathcal{A}$

We attach a veering tetrahedron to the triangulation along γ_1.

Then there is a natural simplicial homeomorphism ψ_1 from the bottom annulus to the top annulus.

In fact, ψ_1 is the *right-hand Dehn twist* along γ_1.



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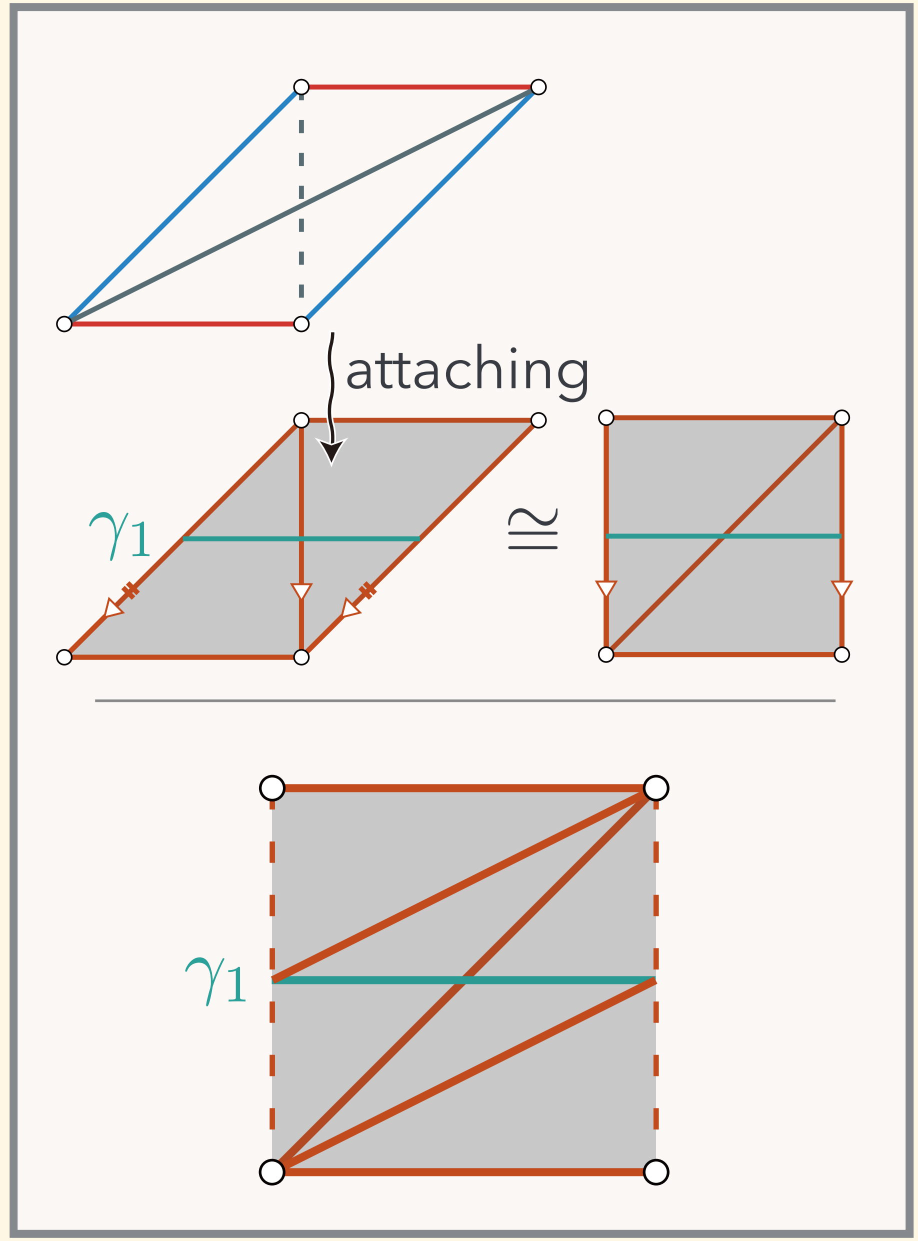

A construction of the veering triangulation (4)

$\gamma_1 \in \mathcal{B}$

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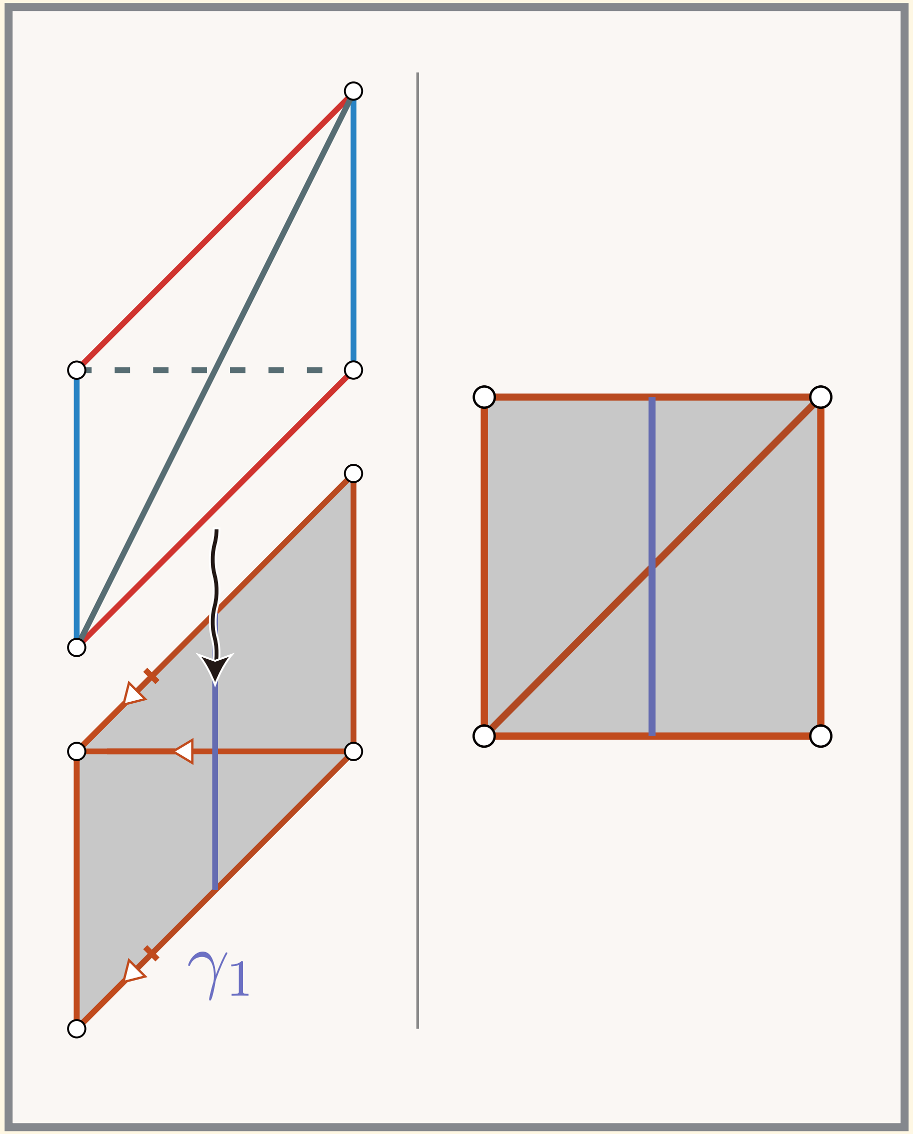

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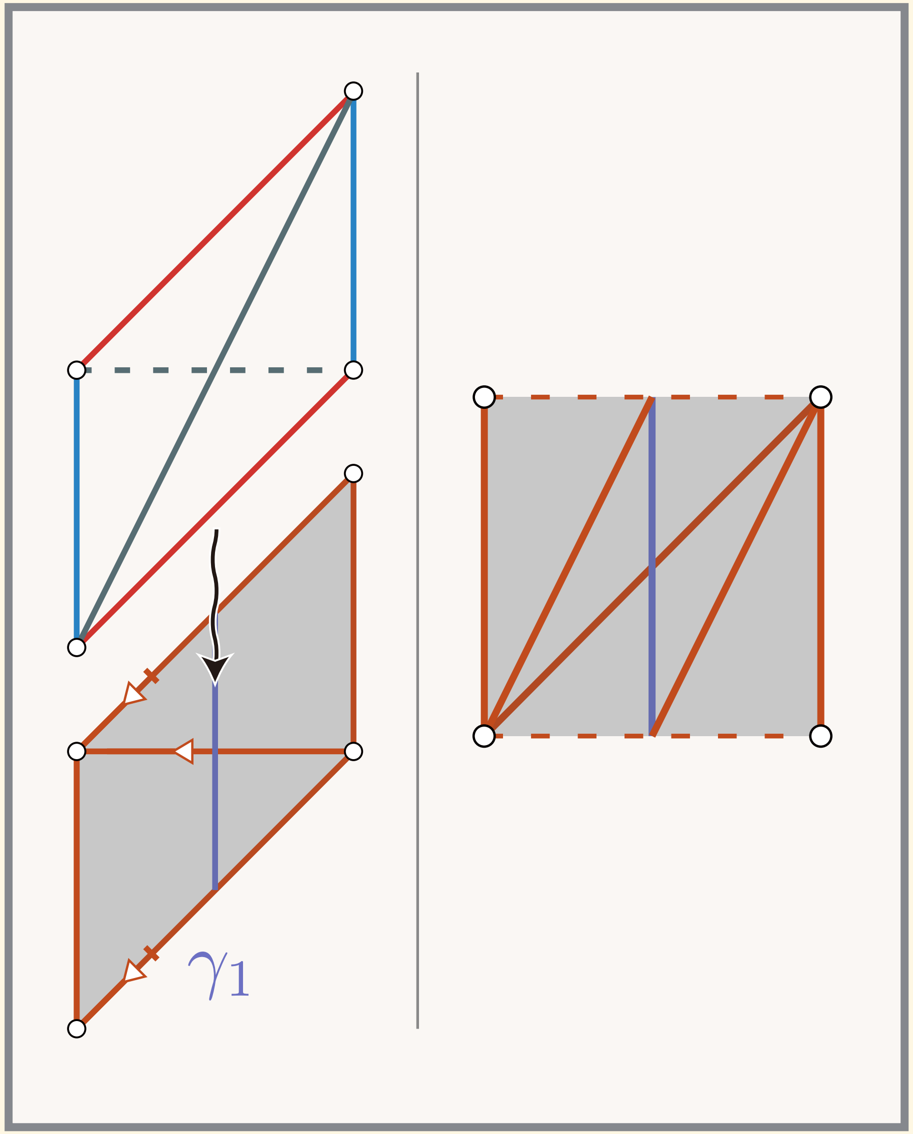


A construction of the veering triangulation (5)

The top of the attached tetrahedra gives a *new* ideal triangulation $\mathcal{T}_1$ of the surface F.

Case $i = 2, \ldots, n$:

We attach $n(\gamma_i)^2$ tetrahedra to the *new* ideal triangulation $\mathcal{T}_{i-1}$ of F along $\psi_{i-1} \circ \cdots \circ \psi_1(\gamma_i)$.

The top of the attached tetrahedra gives a *new* ideal triangulation $\mathcal{T}_i$ of the surface F.

Then we have a simplicial homeomorphism $\psi_i : \mathcal{T}_{i-1} \to \mathcal{T}_i$.

ψ_i is a Dehn twist along $\psi_{i-1} \circ \cdots \circ \psi_1(\gamma_i)$.

Hence $\psi_i = (\psi_{i-1} \circ \cdots \circ \psi_1) \circ \varphi_{\gamma_i} \circ (\psi_{i-1} \circ \cdots \circ \psi_1)^{-1}$.

A construction of the veering triangulation (6)

Step 3:

Glue the bottom and the top of $F \times [0,1]$.

We have an ideal triangulation of $F \times [0,1]$.

$$\underbrace{\psi_n \circ \psi_{n-1} \circ \cdots \circ \psi_1}_{\psi} = (\psi_{n-1} \circ \cdots \circ \psi_1) \circ \varphi_{\gamma_n} \circ (\psi_{n-1} \circ \cdots \circ \psi_1)^{-1} \circ \psi_{n-1} \circ \cdots \circ \psi_1$$

$$= (\psi_{n-1} \circ \cdots \circ \psi_1) \circ \varphi_{\gamma_n}$$

$$= \cdots$$

$$= \varphi_{\gamma_1} \circ \cdots \circ \varphi_{\gamma_{n-1}} \circ \varphi_{\gamma_n}$$

$$= \varphi_\omega$$

We glue $F \times \{0\}$ and $F \times \{1\}$ by ψ.

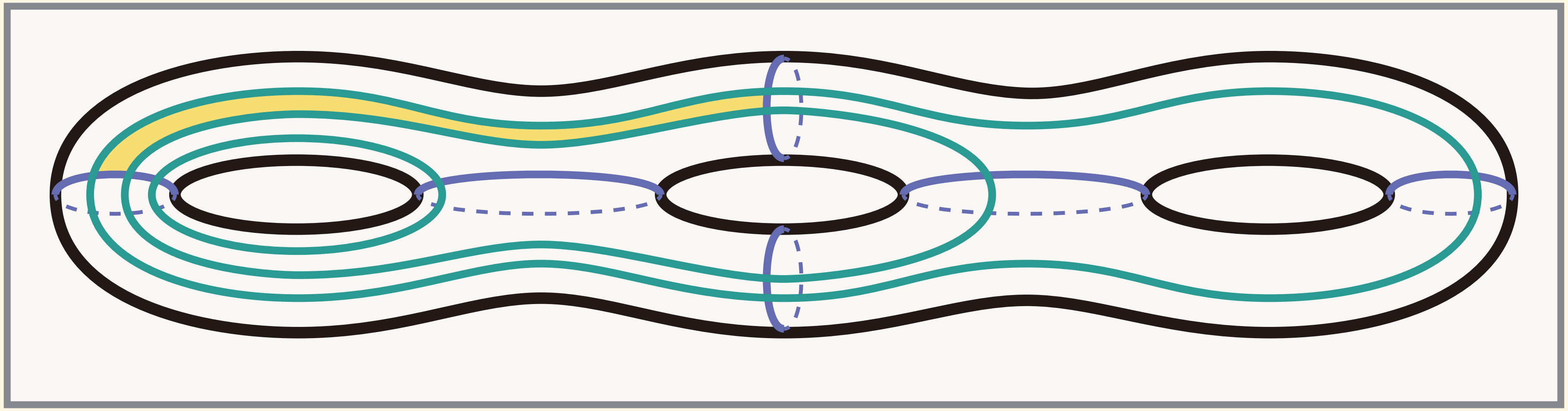
Therefore, we obtain the veering triangulation of $M_{\varphi_\omega^\circ}$.

Future work (1)

The above construction assumes that each complementary region has a singular point of the stable/unstable foliation. However, a quadrilateral region does not have a singular point.

Question

How can we describe the veering triangulation when some complementary regions are quadrilateral?



Future work (2)

Recall (Main Question)

Is the veering triangulation of the mapping torus of each pA mapping class arising from Penner's construction geometric?

↓

Can we apply Casson-Rivin volume maximization theory?

Guéritaud has proved that the veering triangulation of each once-punctured torus bundle over S^1 is geometric by using the theory.

Future work (3)

Note

The combinatorial structure of the veering triangulation of such a mapping torus bundle is similar to that of a once-punctured torus bundle over S^1.

Example:

φ : pA map of a once-punctured torus

Then we can regard φ as an element of $\mathrm{SL}(2, \mathbb{Z})$.
Hence, we have

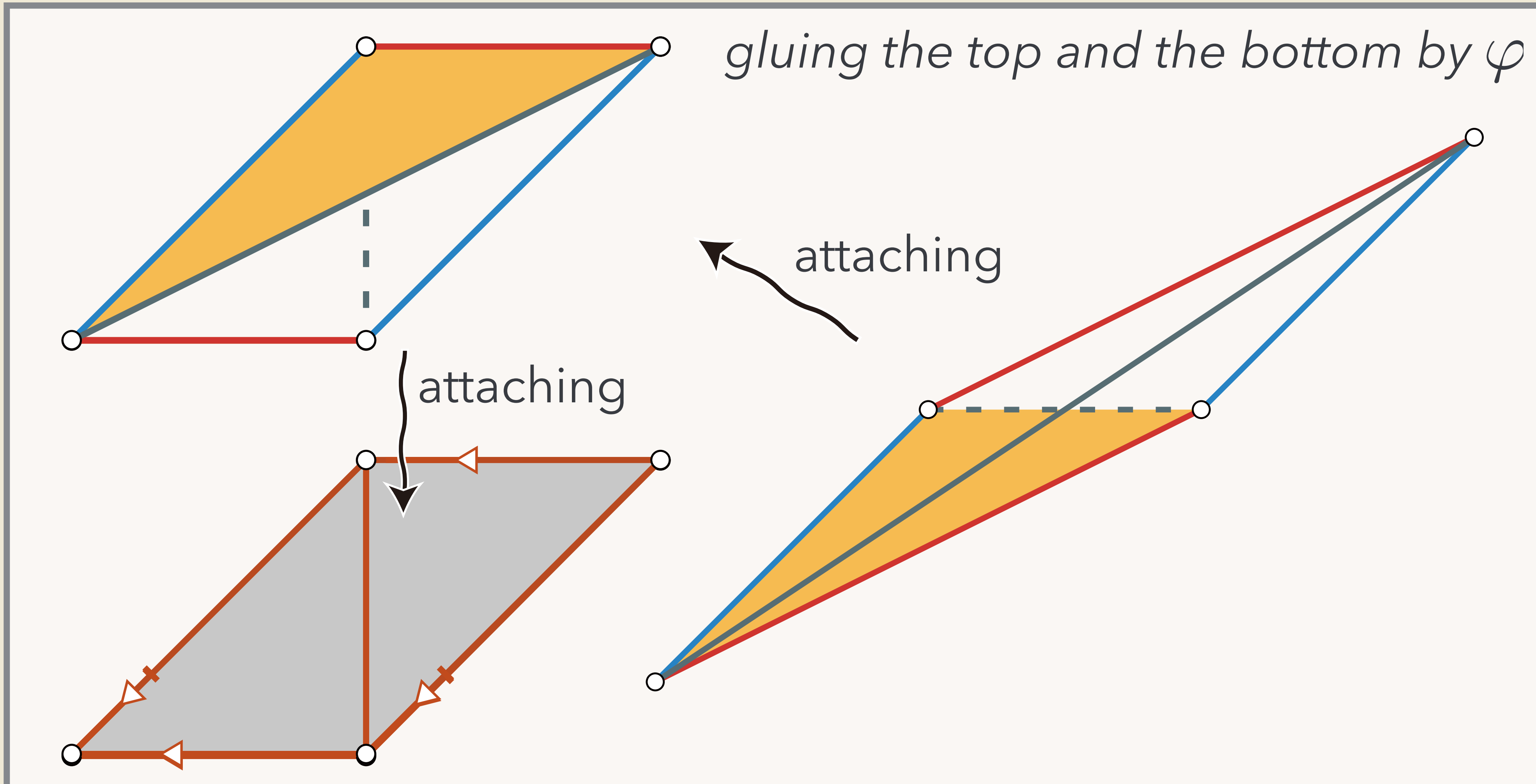
$$\varphi \overset{\text{conj}}{\sim} R^{a_1} L^{b_1} \cdots R^{a_n} L^{b_n},$$

where $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $a_i,\ b_i \in \mathbb{Z}_+$.

Future work (4)

Example (conti.): $\varphi \overset{\text{cong}}{\sim} RL$

The veering triangulation of M_{φ° is obtained as follows.



Future work (4)

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