# ON THE REPRODUCING KERNEL FOR THE SPACE OF SEMI-EXACT ANALYTIC DIFFERENTIALS

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We discuss here some analytic invariants associated with a holomorphic family of Riemann surfaces.

Let $R$ be a bordered Riemann surface of genus $g(\geq 0)$ with a finite number of $C^\omega$-smooth contours $C_j$ $(j = 1, \ldots, \nu)$ in a larger Riemann surface $\widetilde{R}$. Let $S(R)$ be the space of *semi-exact* $L^2$-analytic differentials on $R$. Let $\widetilde{K}(z, \zeta)$ denote the reproducing kernel function for $S(R)$.

**Definition.** Let $R$ be as above. Fix two points $a,\ b \in R$ with local coordinates $U_a : |z - \zeta| < r_a$ and $U_b : |z| < r_b$, where $a$ and $b$ correspond to $\zeta$ and 0, respectively (where $U_a$ and $U_b$ have no relations). Among all harmonic functions $u$ on $R \setminus \{a, b\}$ with two logarithmic poles of $\log|z - \zeta|$ at $a$ and $-\log|z|$ at $b$ normalized so that $\lim_{z\to 0}(u(z) + \log|z|) = 0$, we have uniquely determined functions $h_i$ $(i = 1, 0)$ with the $L_i$-boundary conditions $(i = 1, 0)$: $(L_1)$ for each $C_j$, $h_1$ satisfies $h_1(z) = c_j$ (constant) on $C_j$ and $\int_{C_j} \frac{\partial h_1}{\partial n_z} ds_z = 0$; $(L_0)$ $h_0$ satisfies $\frac{\partial h_0(z)}{\partial n_z} = 0$ on $C_j$. We call $h_i(z)$ the *$L_i$-principal function* and $\mu_i := \lim_{z\to\zeta}(h_i(z) - \log|z - \zeta|)$ the *$L_i$-constant* for $(R, b, a)$ with respect to the local coordinates $U_a$ and $U_b$ (simply, for $(R, 0, \zeta)$).

**Theorem 1** ([3])**.** *Let the notation be as above. We have*

$$\widetilde{K}(z, \zeta) = \frac{2}{\pi} \frac{\partial^2 h_1(z, \zeta)}{\partial z \partial \bar{\zeta}}, \qquad \widetilde{K}(\zeta, \zeta) = \frac{1}{\pi} \frac{\partial^2 \mu_1(\zeta)}{\partial \zeta \partial \bar{\zeta}}.$$

Let $\pi : \widetilde{\mathcal{R}} \to B$ be a holomorphic family such that $\widetilde{\mathcal{R}}$ is a complex 2-dimensional manifold, $\pi$ is a holomorphic projection from $\widetilde{\mathcal{R}}$ onto a disk $B$ in $\mathbb{C}_t$, and each fiber $\widetilde{R}(t) = \pi^{-1}(t)$, $t \in B$ is irreducible and non-singular in $\widetilde{\mathcal{R}}$. We set $\widetilde{\mathcal{R}} = \cup_{t\in B}(t, \widetilde{R}(t))$. Let $\mathcal{R} = \cup_{t\in B}(t, R(t))$ be a subdomain with $C^\omega$ smooth boundary $\partial\mathcal{R} = \cup_{t\in B}(t, \partial R(t))$ in $\widetilde{\mathcal{R}}$ such that $\widetilde{R}(t) \ni R(t) \neq \emptyset$ for $t \in B$, $R(t)$ is a bordered Riemann surface of genus $g(\geq 0)$ in $\widetilde{R}(t)$, and $\partial R(t)$ in $\widetilde{R}(t)$ consists of a finite number of $C^\omega$ smooth contours $C_j(t)$ $(j = 1, \ldots, \nu)$.

**Theorem 2** ([3])**.** *We assume that the total space $\mathcal{R} = \cup_{t\in B}(t, R(t))$ is 2-dimensional pseudoconvex in $\widetilde{\mathcal{R}}$. Then $\log \widetilde{K}(t, \zeta, \zeta)$ is a plurisubharmonic function on $\mathcal{R}$.*

This phenomenon is the same as the Bergman metrics (see [5]).

Here we recall the definition of Schiffer spans for *planar* Riemann surfaces. Let $R$ be a finite bordered planar Riemann surface. Let $\mathcal{P}(R)$ be the set of

Partly supported by JSPS Grant-in-Aid for Young Scientists (B), 23740098.

all univalent functions $P$ on $R$ with the expression

$$P(z) = \frac{1}{z-\zeta} + 0 + A_1(z-\zeta) + A_2(z-\zeta)^2 + \cdots$$

at a given point $\zeta \in R$. The *Schiffer span* $s$ for $(R, \zeta)$ is defined by

$$s := \frac{2}{\pi} \sup_{P \in \mathcal{P}(R)} \{\text{the Euclidean area } \mathcal{E}_P \text{ of } E_P := \mathbb{C} \setminus P(R)\}.$$

The Schiffer span $s(\zeta)$ induces the metric $s(\zeta)|d\zeta|^2$ on $R$.

**Theorem 3** ([3])**.** *Let $R$ be a finite bordered* planar *Riemann surface. Then we have the following:*

(i) *The metrics $\widetilde{K}(\zeta,\zeta)|d\zeta|^2$, $\frac{1}{\pi}\frac{\partial^2 \mu_1(\zeta)}{\partial\zeta\partial\zeta}|d\zeta|^2$, $-\frac{1}{\pi}\frac{\partial^2 \mu_0(\zeta)}{\partial\zeta\partial\zeta}|d\zeta|^2$, and $s(\zeta)|d\zeta|^2$ are all identical on $R$;*

(ii) *$\widetilde{K}(\zeta,\zeta)|d\zeta|^2$ is of negative curvature at every point $\zeta \in R$;*

(iii) *$\widetilde{K}(\zeta,\zeta)|d\zeta|^2$ is complete on $R$.*

For the proofs of the above theorems, we use the following variational formulas for principal functions and the plurisubharmonic variation of the Schiffer span under pseudoconvexity ([2]).

**Lemma 4** ([1], [4])**.** *Let $R(t)$ is of genus $g(\geq 0)$, and let $\{A_l(t), B_l(t)\}_{l=1}^g$ be a canonical homology basis on $R(t)$ such that each $A_l(t)$ and $B_l(t)$ varies continuously with $t \in B$. Then we have*

$$\frac{\partial^2 \mu_1(t)}{\partial t \partial \bar{t}} = \frac{1}{\pi}\int_{\partial R(t)} k_2(t,z)\left|\frac{\partial h_1(t,z)}{\partial z}\right|^2 ds_z + \frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 h_1(t,z)}{\partial \bar{t}\partial z}\right|^2 dxdy;$$

$$\begin{aligned}\frac{\partial^2 \mu_0(t)}{\partial t \partial \bar{t}} =& -\left(\frac{1}{\pi}\int_{\partial R(t)} k_2(t,z)\left|\frac{\partial h_0(t,z)}{\partial z}\right|^2 ds_z + \frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 h_0(t,z)}{\partial \bar{t}\partial z}\right|^2 dxdy\right) \\ & - \frac{2}{\pi}\mathrm{Im}\left\{\sum_{l=1}^g \frac{\partial}{\partial t}\left(\int_{A_l(t)} *dh_0(t,z)\right)\frac{\partial}{\partial \bar{t}}\left(\int_{B_l(t)} *dh_0(t,z)\right)\right\}.\end{aligned}$$

*Here, for the defining function $\varphi(t,z)$ of $\partial\mathcal{R}$,*

$$k_2(t,z) = \left(\frac{\partial^2\varphi}{\partial t\partial\bar{t}}\left|\frac{\partial\varphi}{\partial z}\right|^2 - 2\mathrm{Re}\left\{\frac{\partial^2\varphi}{\partial\bar{t}\partial z}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial\bar{z}}\right\} + \left|\frac{\partial\varphi}{\partial t}\right|^2\frac{\partial^2\varphi}{\partial z\partial\bar{z}}\right)\left|\frac{\partial\varphi}{\partial z}\right|^{-3}.$$

## References


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