L-extendable functions and a proximity scaling algorithm for minimum cost multiflow problem

Hiroshi Hirai
The University of Tokyo

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  - Alternating L-convex & L-extendable functions
  - Optimality, proximity, persistency
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L$^\natural$-convex function

Murota 1996, Fujishige-Murota 2000, Favati-Tardella1990

Def: $g : \mathbf{Z}^n \to \mathbf{R}$ is L$^\natural$-convex $\Leftrightarrow$

$$g(p) + g(q) \geq g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right) + g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) \quad (p, q \in \mathbf{Z}^n)$$

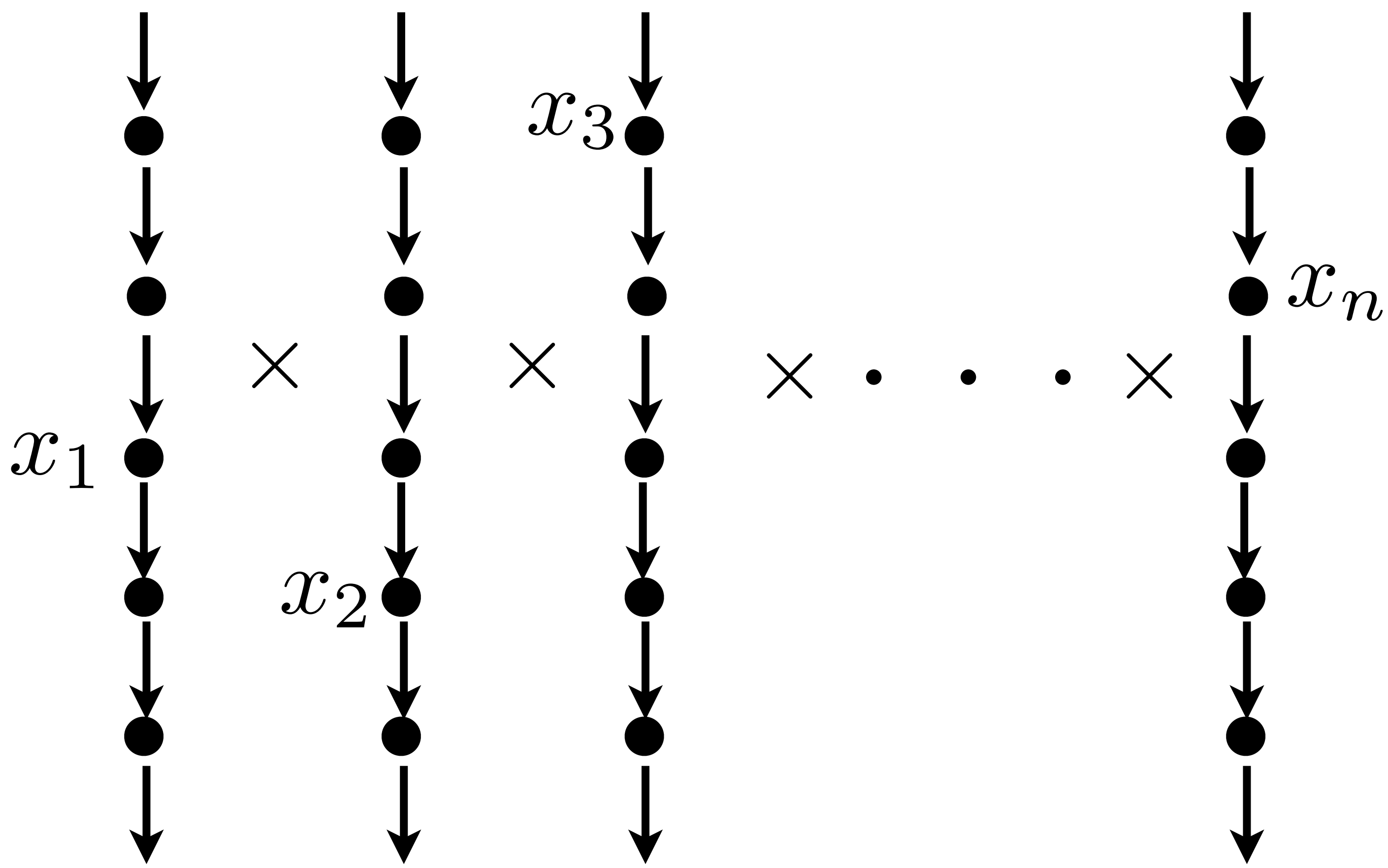
Ex: $p \mapsto \sum_i g_i(p_i) + \sum_{ij} g_{ij}(p_i - p_j)$

g_i, g_{ij} : 1-dim. convex

Ex: dual of mincost flow

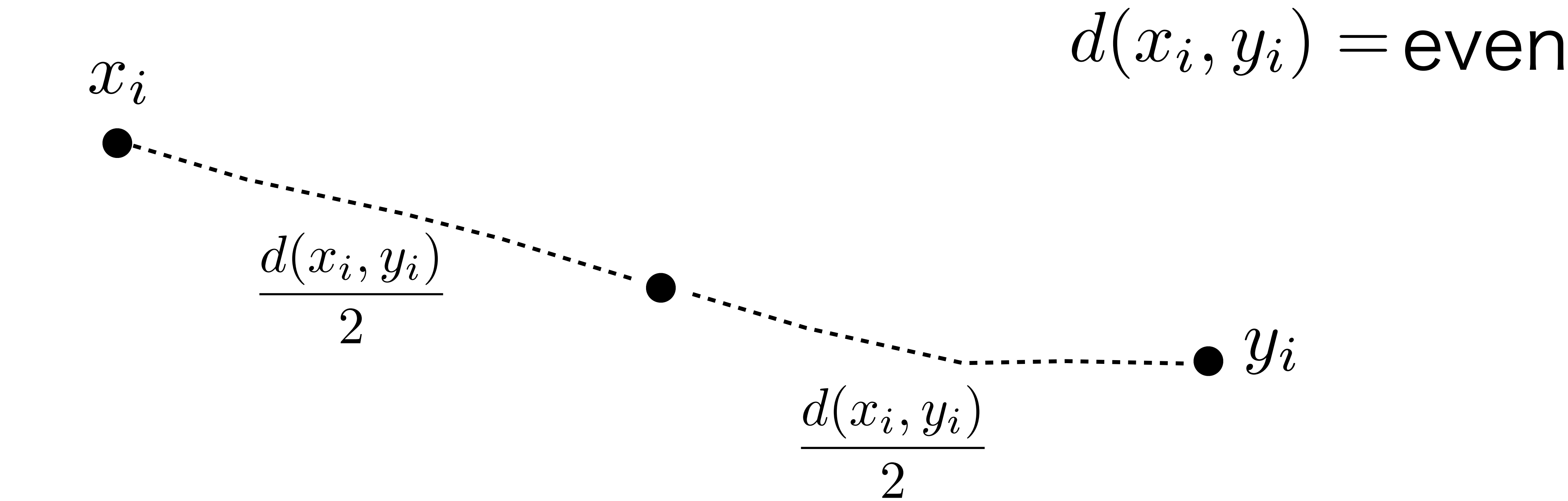
Observation

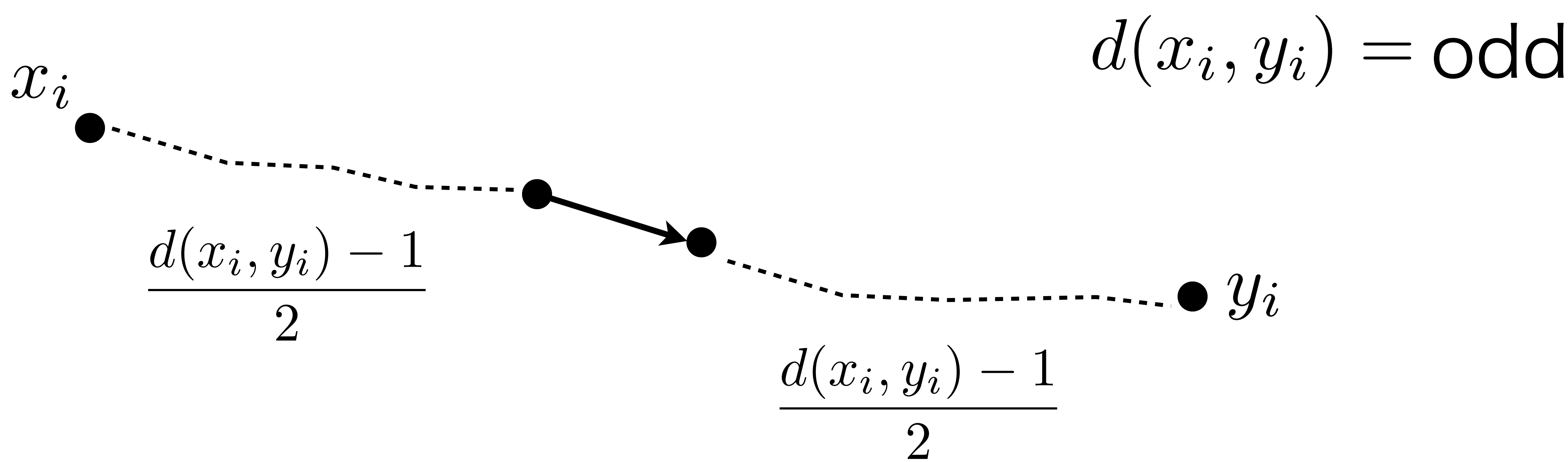
L$^\natural$-convex function is definable on

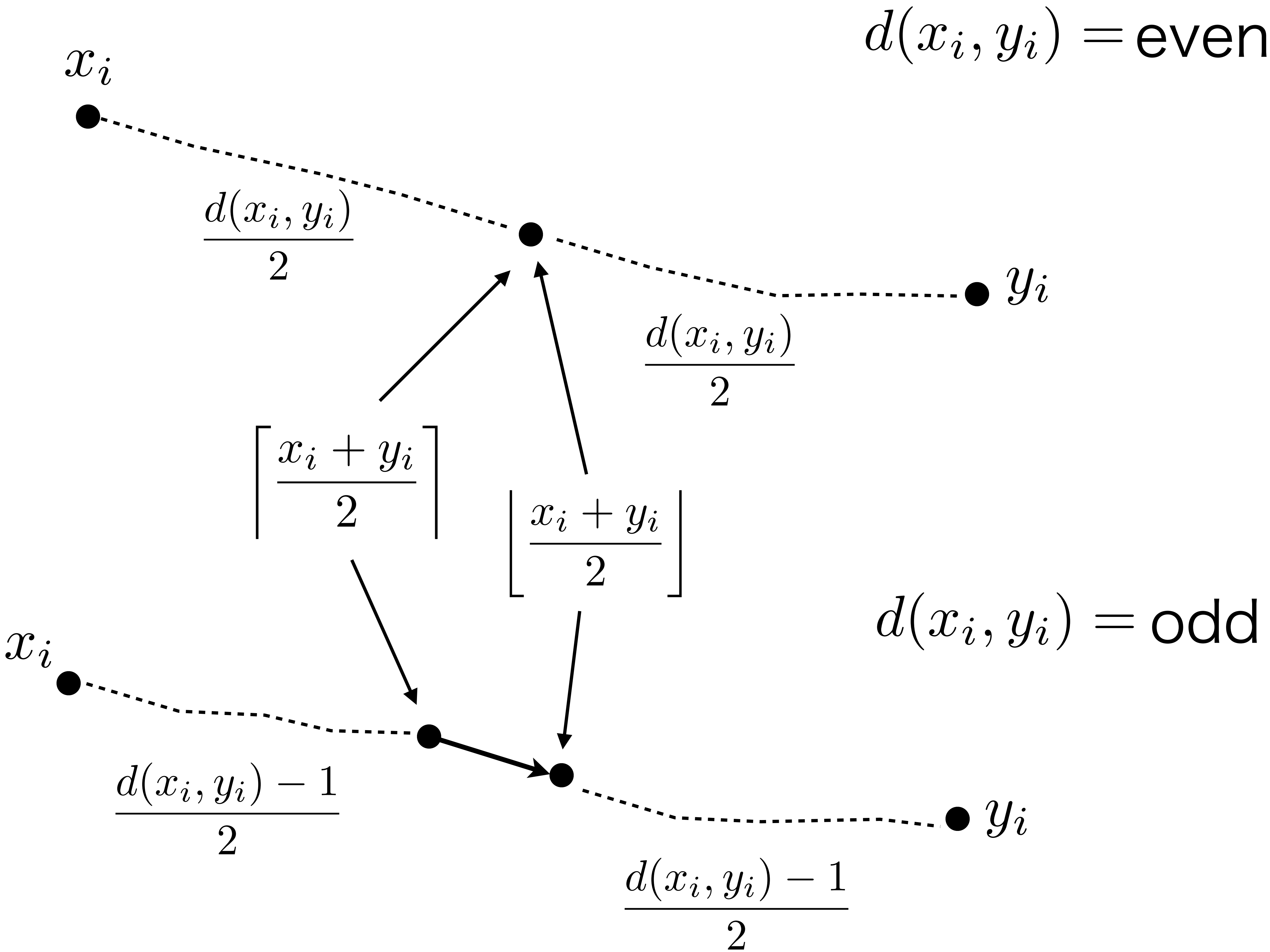


$(\simeq \mathbf{Z}^n)$

graph-theoretically


$d(x_i, y_i) =$ even
x_i
$\frac{d(x_i, y_i)}{2}$
y_i
$\frac{d(x_i, y_i)}{2}$


$d(x_i, y_i) =$ odd
x_i
$\frac{d(x_i, y_i) - 1}{2}$
y_i
$\frac{d(x_i, y_i) - 1}{2}$



$d(x_i, y_i) = \text{even}$
x_i
$\frac{d(x_i, y_i)}{2}$
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$\frac{d(x_i, y_i)}{2}$
$\left\lceil \frac{x_i + y_i}{2} \right\rceil$
$\left\lfloor \frac{x_i + y_i}{2} \right\rfloor$
$d(x_i, y_i) = \text{odd}$
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$\frac{d(x_i, y_i) - 1}{2}$
y_i
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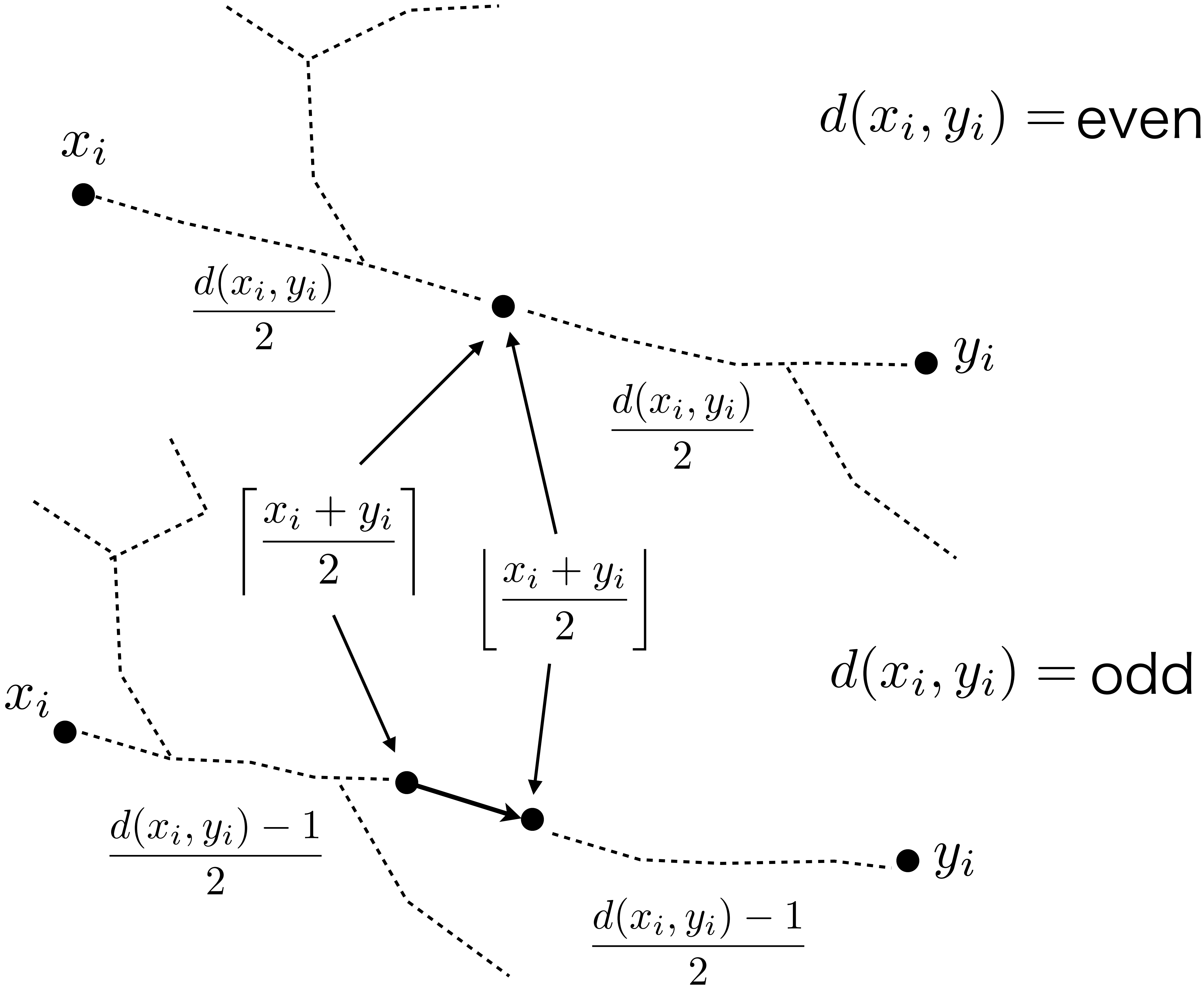

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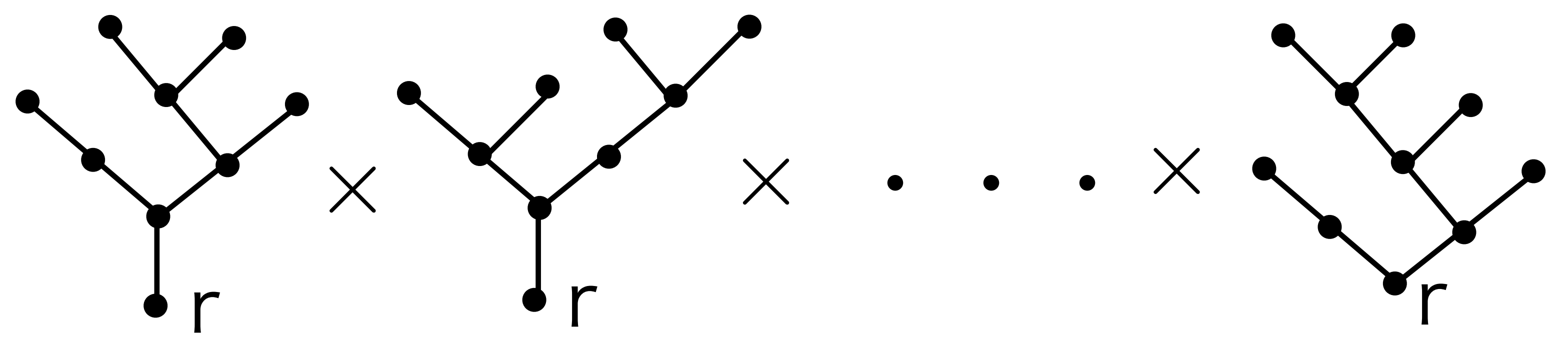
Toward L-convexity on graphs

- Tree-submodular functions (Kolmogorov 11)

$g :$  $\to \mathbf{R}$

s.t. $g(x) + g(y) \geq g\left(\left[\frac{x+y}{2}\right]_{\to r}\right) + g\left(\left[\frac{x+y}{2}\right]_{\leftarrow r}\right)$

Toward L-convexity on graphs

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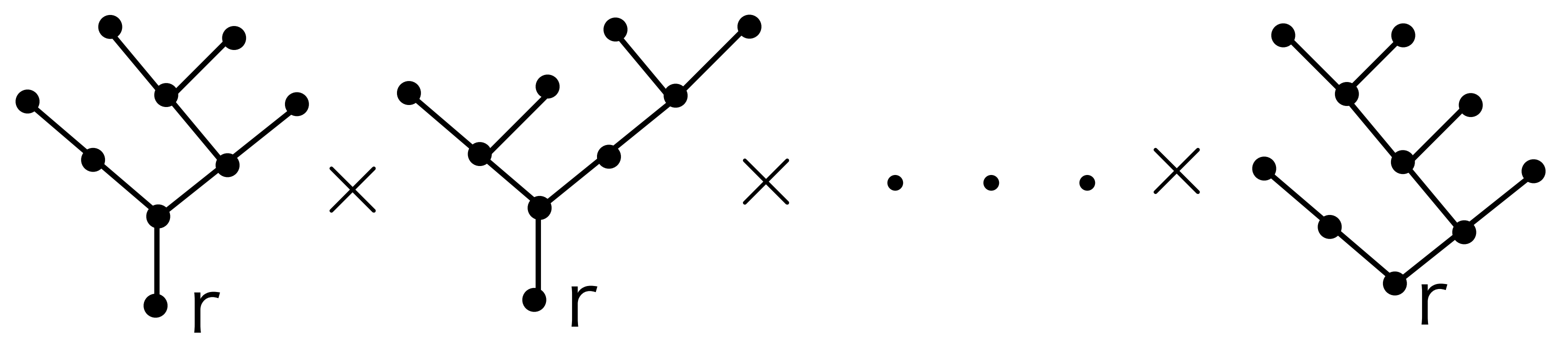
$g :$



$\to \mathbf{R}$

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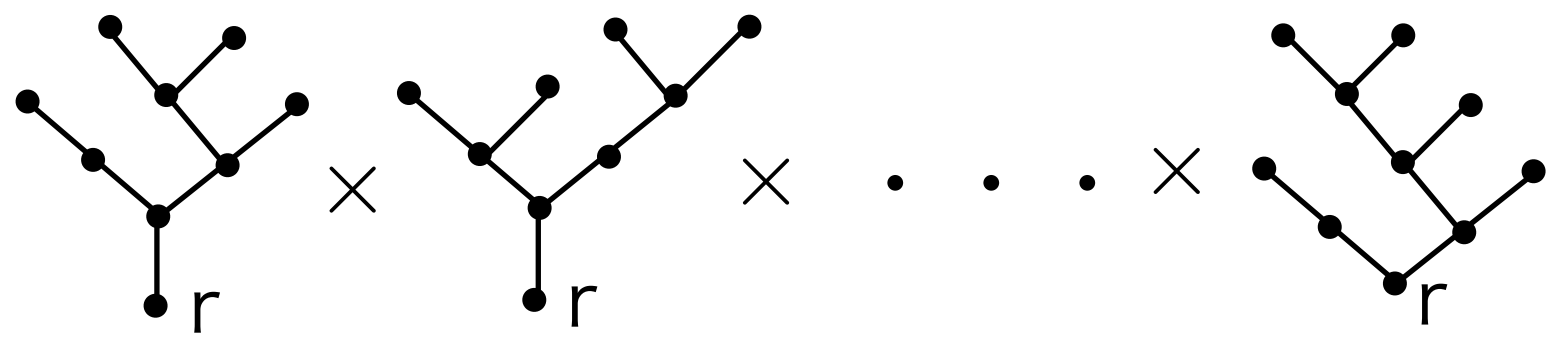
$\cap$ (rotated ⊆: $\subseteq$)

- L-convex functions on oriented modular graphs

(H. MPA to appear)

Toward L-convexity on graphs

- Tree-submodular functions (Kolmogorov 11)

$g :$  $\to \mathbf{R}$

s.t. $g(x) + g(y) \geq g([\frac{x+y}{2}]_{\to r}) + g([\frac{x+y}{2}]_{\leftarrow r})$

$|\cap$

- L-convex functions on oriented modular graphs

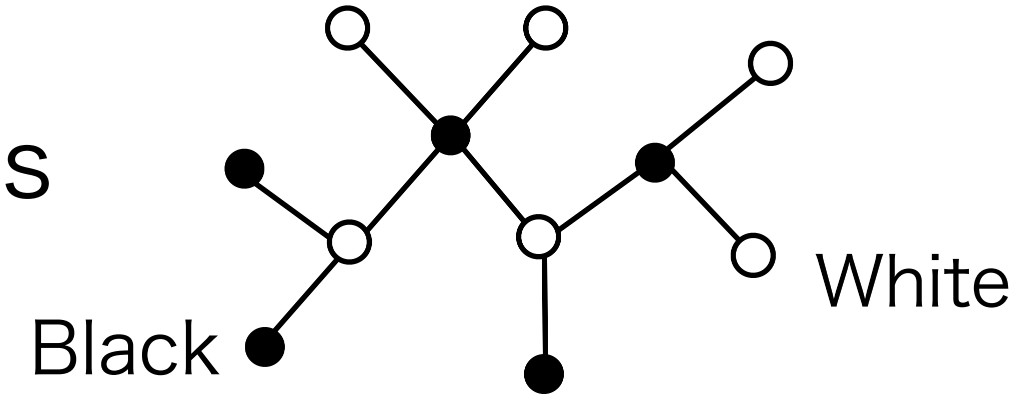
(H. MPA to appear)

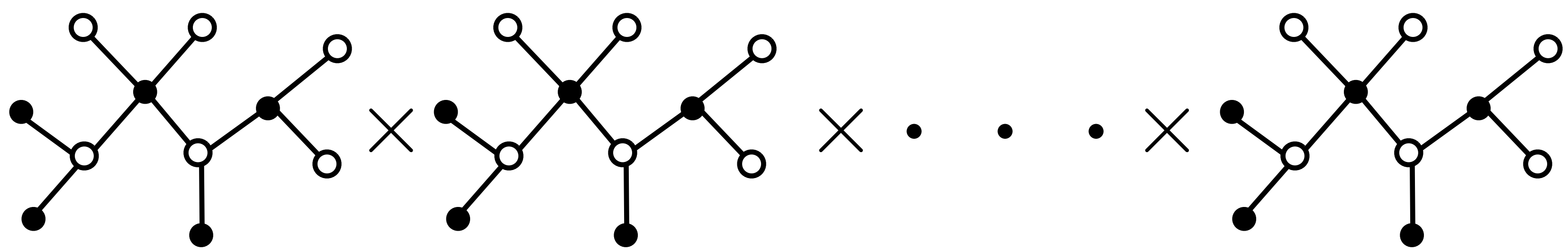
$\cup|$

We introduce a new useful class

Alternating L-convex functions

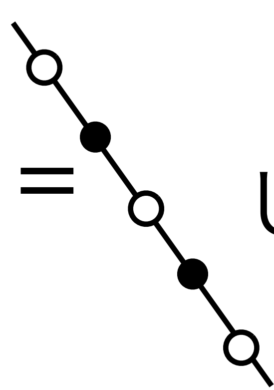
T :tree, B,W: color classes



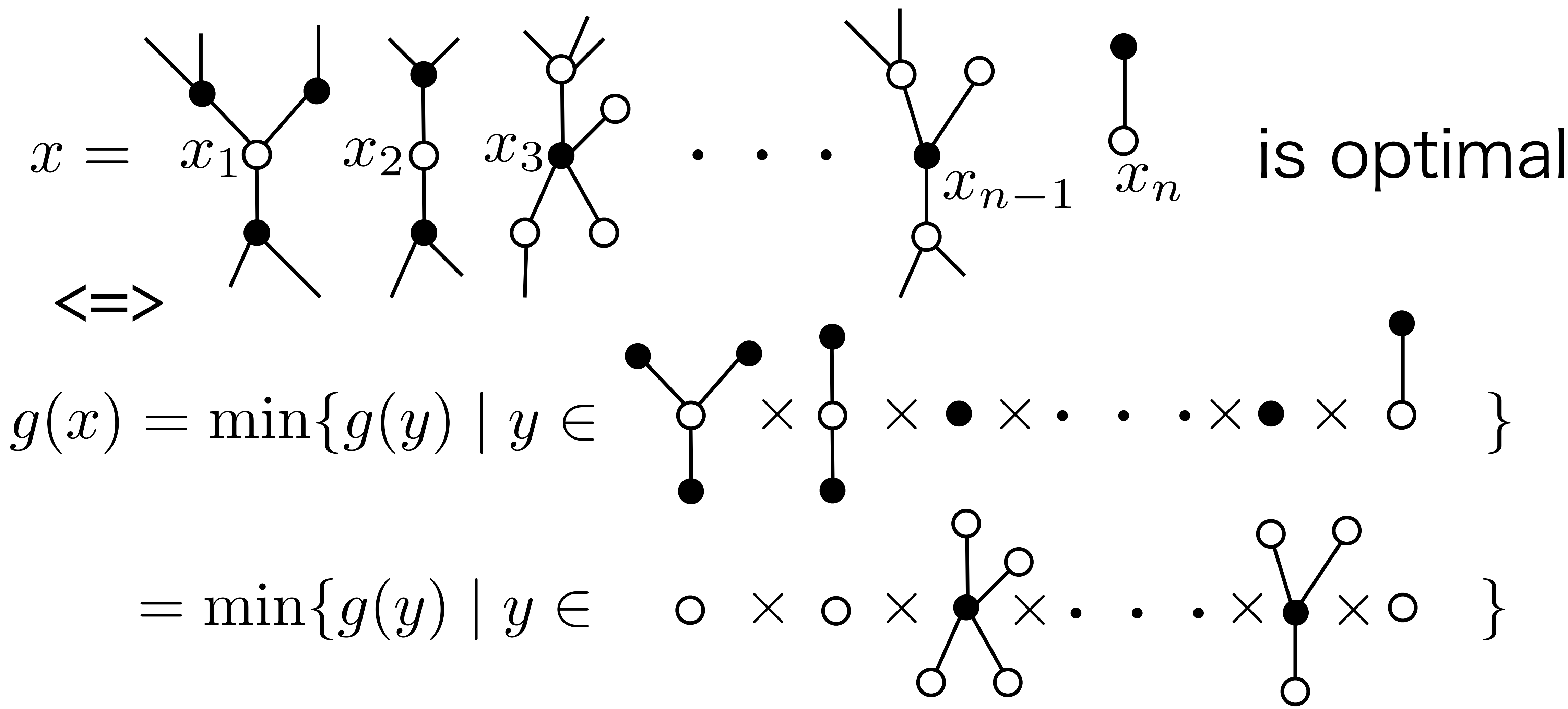
Def: $g :$  $\to \mathbf{R}$

is alternating L-convex

<=> $g(x) + g(y) \geq g(\left[\frac{x+y}{2}\right]_{\bullet}) + g(\left[\frac{x+y}{2}\right]_{\circ})$

c.f. T =  UJ-convex fn. (Fujishige 14)

Thm [L-optimality] g : L-convex



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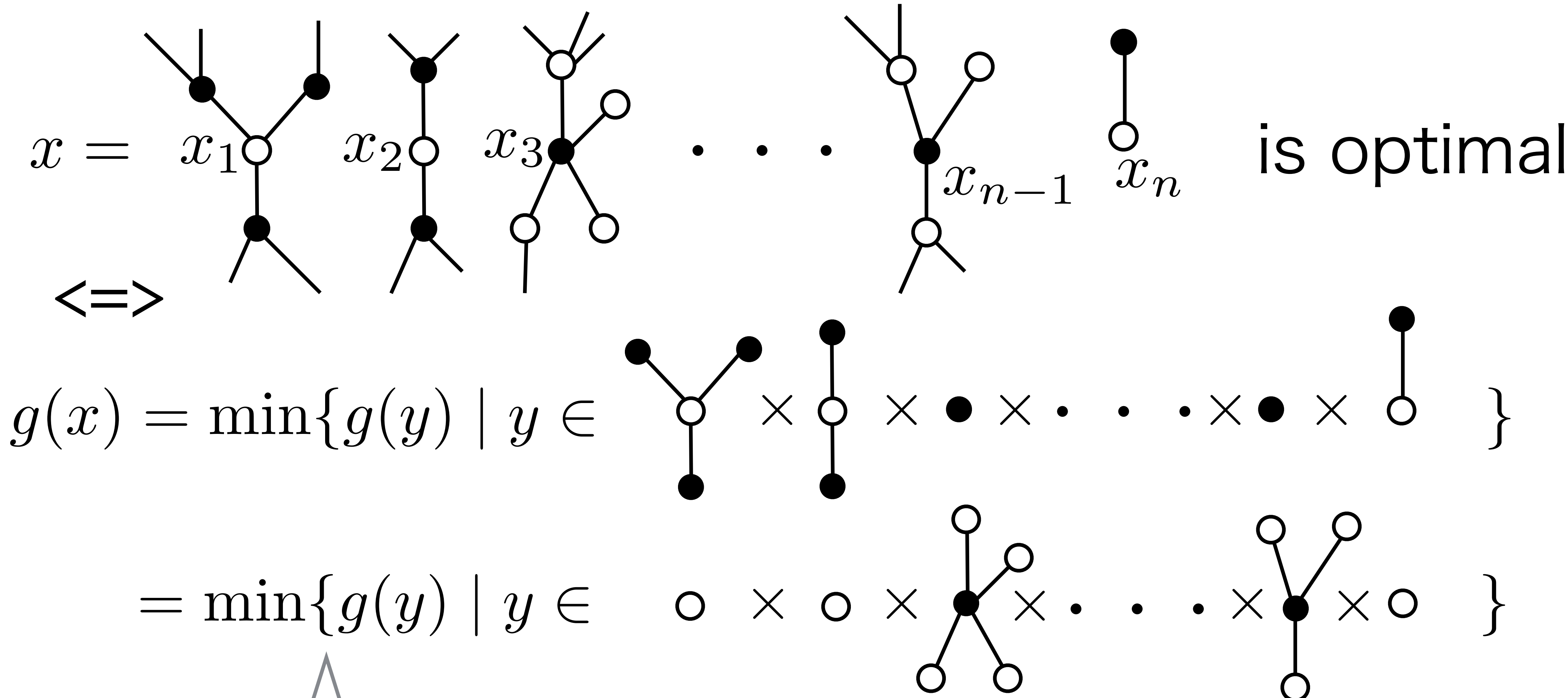

k-submodular fn. minimization (Huber-Kolmogorov 12)

k-SFMin $\overset{?}{\in}$ P $\quad x \overset{\text{oracle}}{\mapsto} f(x)$

k-SFVcsp $\in$ P $\quad x \mapsto \sum_{\boldsymbol{i}} f_{\boldsymbol{i}}(x_{i_1}, x_{i_2}, \ldots, x_{i_K})$

(Thapper-Zivny 12) $\quad K$: fixed

Steepest descent algorithm

k-SFMin

0: initial point x

1: Take z in Argmin { g(y) | y: B or W-neighbor of x}

2: If g(x) = g(z), then x is opt; stop.

3: Otherwise (g(z) < g(x)), let x := z; go to 1

Steepest descent algorithm

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3: Otherwise (g(z) < g(x)), let x := z; go to 1

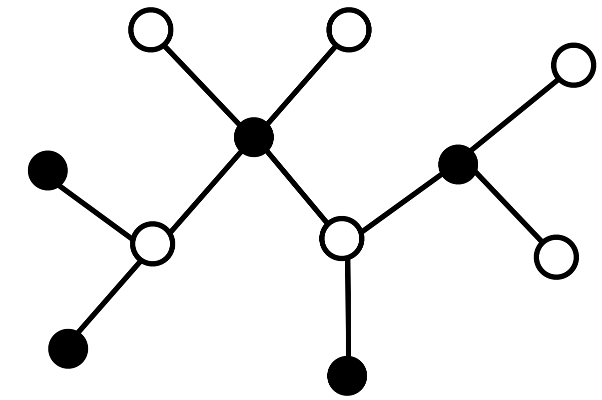
Thm :

#iteration $\leq d_\infty(x, \mathrm{opt}(g)) + 2$

$= d_\infty(x, \mathrm{opt}(g))$ if x in B^n or W^n

c.f. Murota-Shioura 14 for $L^\natural$-convex fn.

L-extendable functions



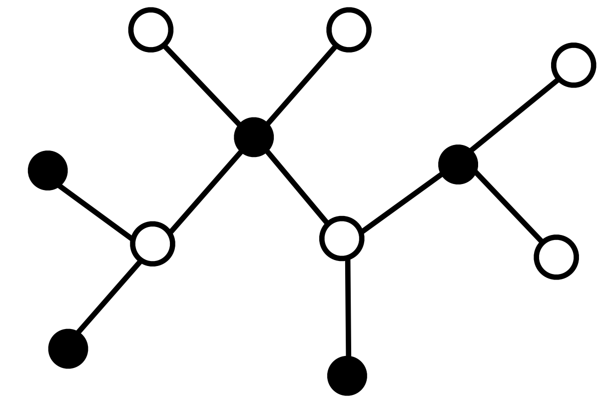
T: tree, B,W: color classes

Def: $h : B^n \to \mathbf{R}$ is L-extendable if

∃ alt. L-convex fn. $g : T^n \to \mathbf{R}$ with $g|_{B^n} = h$

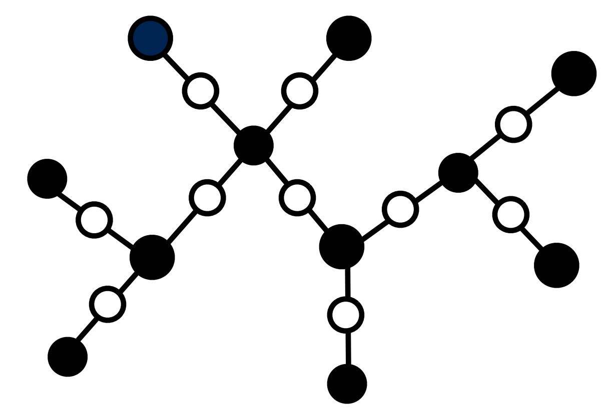
L-extendable functions

T: tree, B,W: color classes



Def: $h : B^n \to \mathbf{R}$ is L-extendable if
$\exists$ alt. L-convex fn. $g : T^n \to \mathbf{R}$ with $g|_{B^n} = h$

T* : subdivision of T

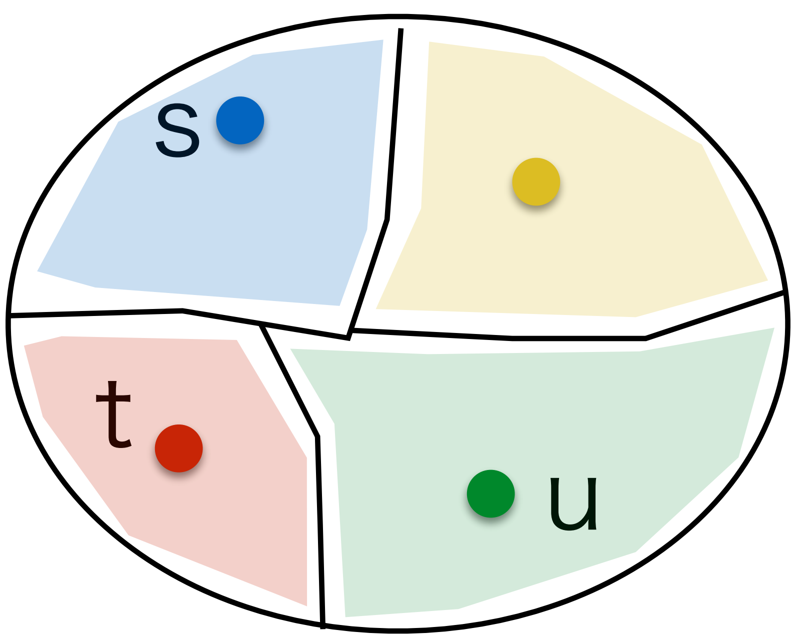


Def: $h : T^n \to \mathbf{R}$ is midpoint L-extendable if
$\exists$ alt. L-convex fn. $g : (T^*)^n \to \mathbf{R}$ with $g|_{T^n} = h$

- Inspired by k-submodular relaxation

(Kolmogorov 12, Gridchyn-Kolmogorov 13, Iwata-Wahlstrom-Yoshida 14)

- Include vertex cover, multiway cut, ⋯



Min. $\sum_{ij \in E} c(ij) 1_{\neq}(x_i, x_j)$

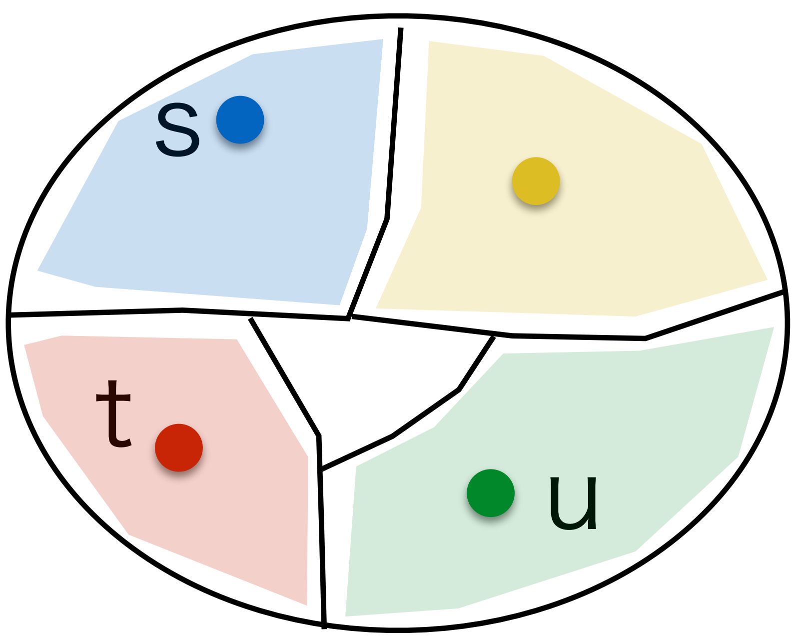
s.t. $(x_1, x_2, \ldots, x_n) \in \{\ \}^n$

- Minimizing L-ext. fn. is NP-hard

- Inspired by k-submodular relaxation

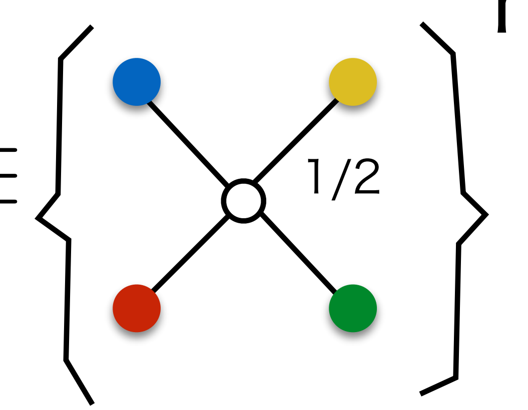
(Kolmogorov 12, Gridchyn-Kolmogorov 13, Iwata-Wahlstrom-Yoshida 14)

- Include vertex cover, multiway cut, ⋯



Min. $\sum_{ij \in E} c(ij) d(x_i, x_j)$

s.t. $(x_1, x_2, \ldots, x_n) \in$



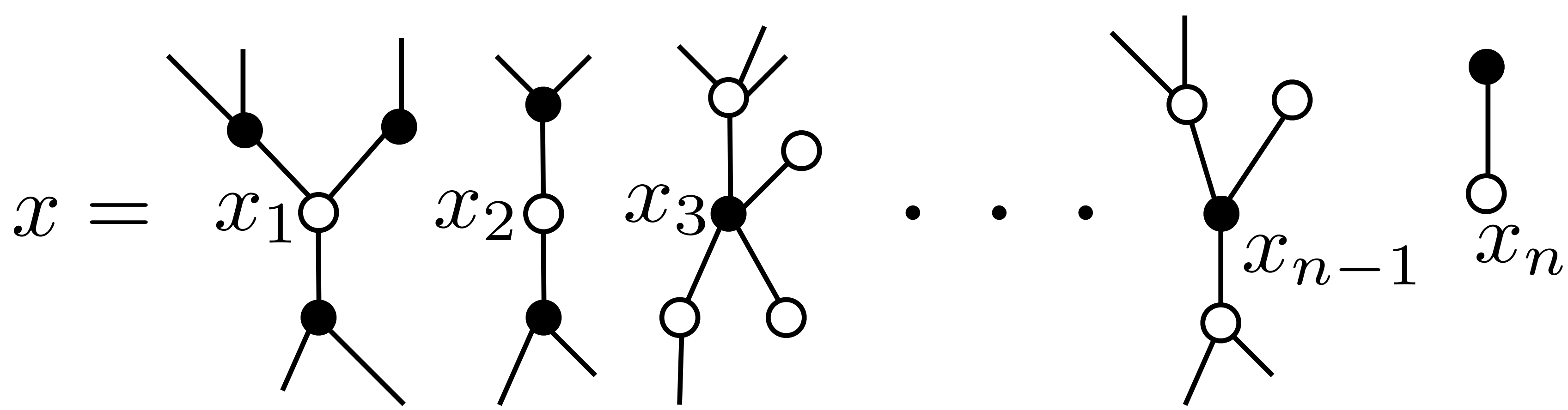
- Minimizing L-ext. fn. is NP-hard

Thm [persistency]

h: L-extendable, $B^n \to R$

g: L-convex relaxation of h, $(B \cup W)^n \to R$

x: minimizer of g



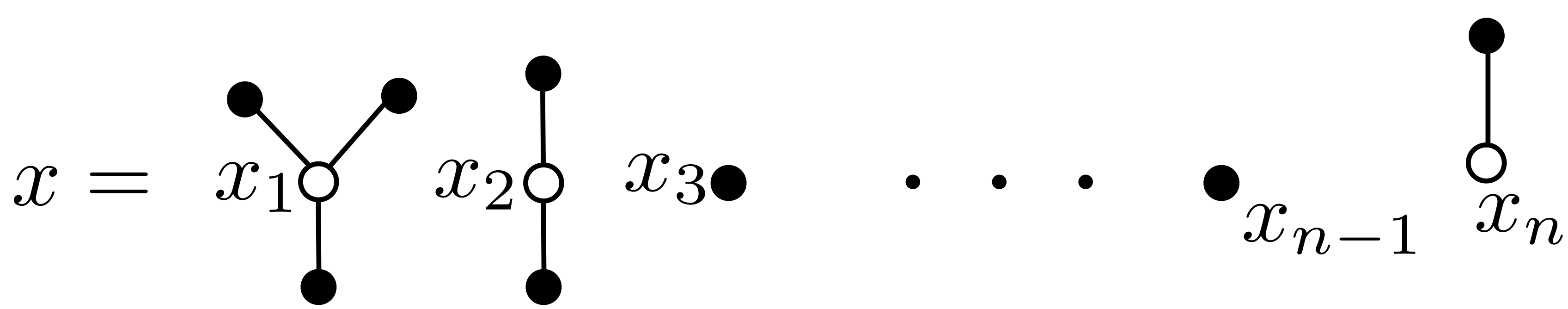
Thm [persistency]

h: L-extendable, $B^n \to R$

g: L-convex relaxation of h, $(B \text{ U } W)^n \to R$

x: minimizer of g

$\Rightarrow$ $\exists$ minimizer of h in B-neighborhood of x



Thm [proximity]

g: L-extendable/convex, $(B \cup W)^n \to R$

x: minimizer over B^n

$\Rightarrow \exists$ y: minimizer over $(B \cup W)^n$ s.t.

$$d_\infty(x, y) \leq 2n$$

Important example

$$T^n \ni x \mapsto \sum_i g_i(x_i) + \sum_{ij} g_{ij}(d(x_i, x_j))$$

g_i : convex

g_{ij} : nondecreasing convex

is L-extendable

Important example

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is L-extendable

$$(T^*)^n \ni x \mapsto \sum_i \bar{g}_i(x_i) + \sum_{ij} \bar{g}_{ij}(d_{\frac{1}{2}}(x_i, x_j))$$

$\bar{g}_i, \bar{g}_{ij}$: linear interpolation

is L-convex relaxation

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is L-convex relaxation

is locally “basic k-submodular”

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is L-convex relaxation

is locally “basic k-submodular”

minimizable in MF(kn,km) time
(Iwata-Wahlstrom-Yoshida 14)

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is L-convex relaxation

is locally "basic k-submodular"

minimizable in MF(kn,km) time
(Iwata-Wahlstrom-Yoshida 14)

is minimizable in $O(d_\infty(x, \mathrm{opt})\mathrm{MF}(kn, km))$ time

k: = maxdeg of T

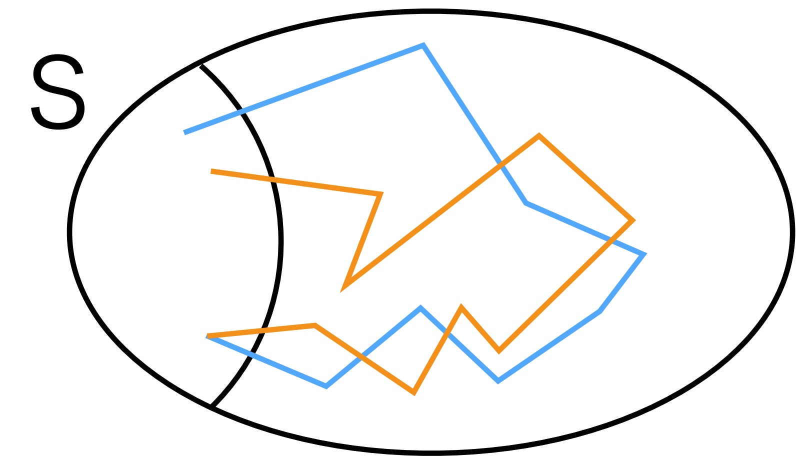
Application: Minimum cost multiflow

N = (V,E,c,a,S): undirected network

c: E → Z+: edge-capacity

a: E → Z+: edge-cost

S: terminal set (⊂ V)



Def: Multiflow <=> f: { S-paths } → R+ satisfying
capacity-constraint

Find a multiflow f of max flow-value Σ f(P)
with min-cost Σ a(e) f(e)

Karzanov 79: half-integrality & pseudo-polytime algorithm

Karzanov 94: strongly-polytime algorithm

(LP-solver + Tardos)

Goldberg-Karzanov 97:

"combinatorial" weakly-polytime algorithms

…but O(?)

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"combinatorial" weakly-polytime algorithms

…but O(?)

Our result:

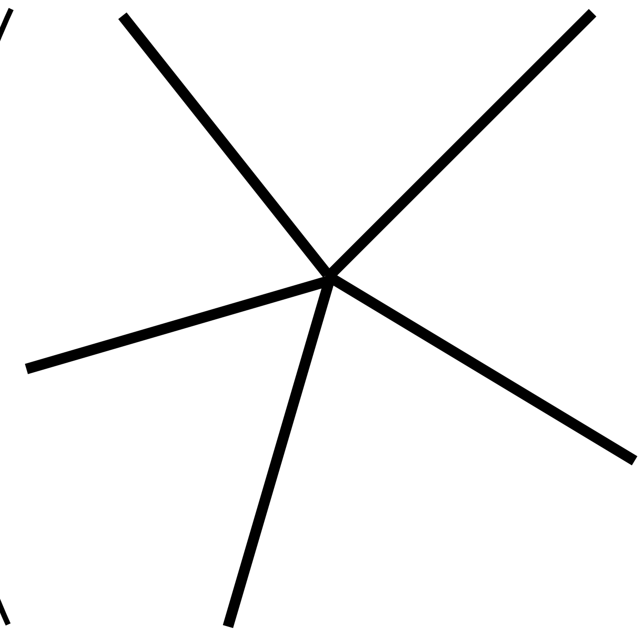
combinatorial $O(n \log(nAC)\, MF(kn, km))$-time algorithm

$k := \#S$

Dual problem: V={1,2,···,n}

Min. $$\sum_{s\in S} g_s(x_s) + \sum_{ij\in E} c(ij)\max\{D(x_i,x_j)-a(ij),0\}$$

s.t. $$(x_1, x_2, \ldots, x_n) \in \Big(\quad \Big)^n$$



Outline

Dual problem: V={1,2,⋯,n}

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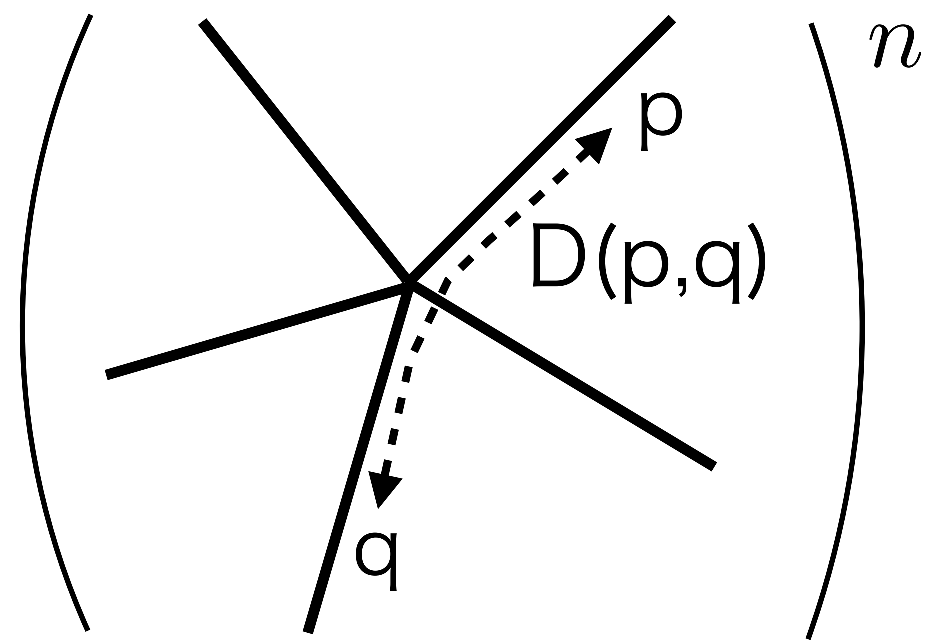

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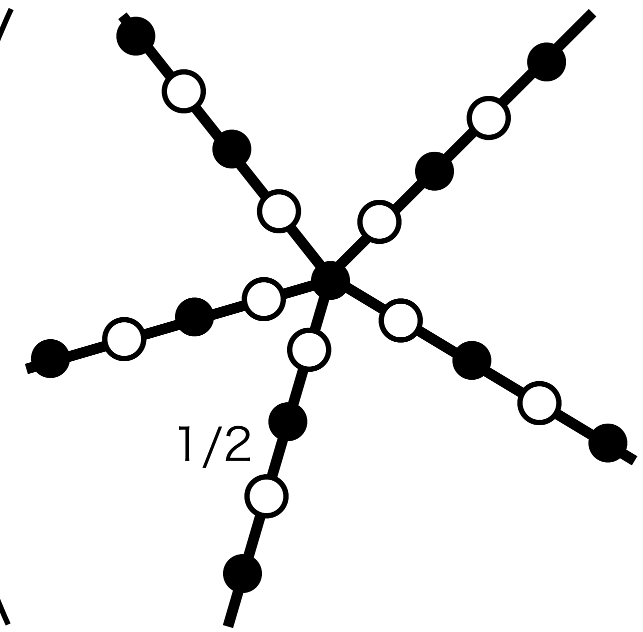
s.t. $(x_1, x_2, \ldots, x_n) \in$ ( $)^n$

- Half-integrality

Outline

Dual problem: V={1,2,⋯,n}

alt. L-convex

Min. $$\sum_{s \in S} g_s(x_s) + \sum_{ij \in E} c(ij) \max\{D(x_i, x_j) - a(ij), 0\}$$

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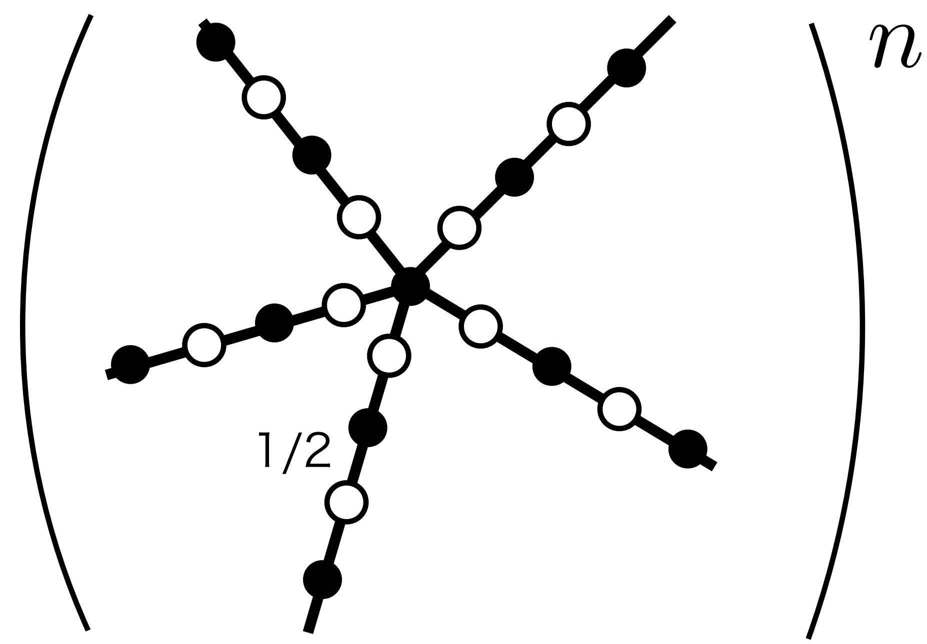

- Half-integrality

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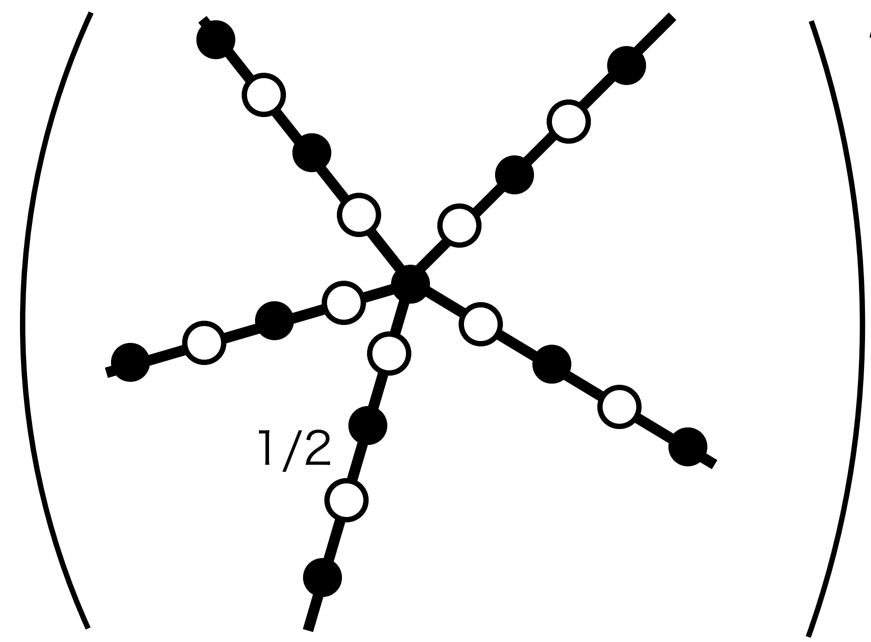
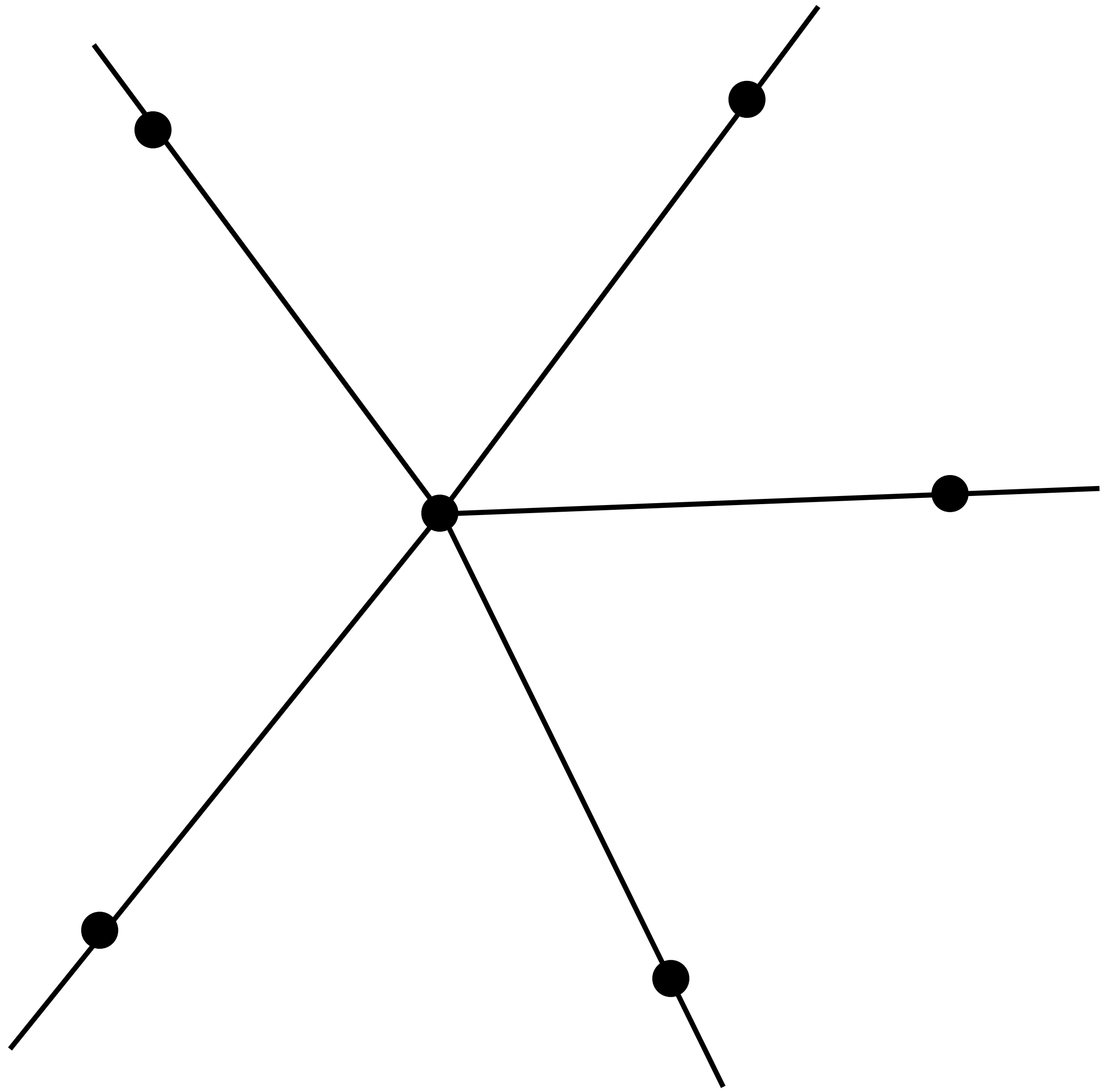
s.t. $(x_1, x_2, \ldots, x_n) \in$  ^{n}

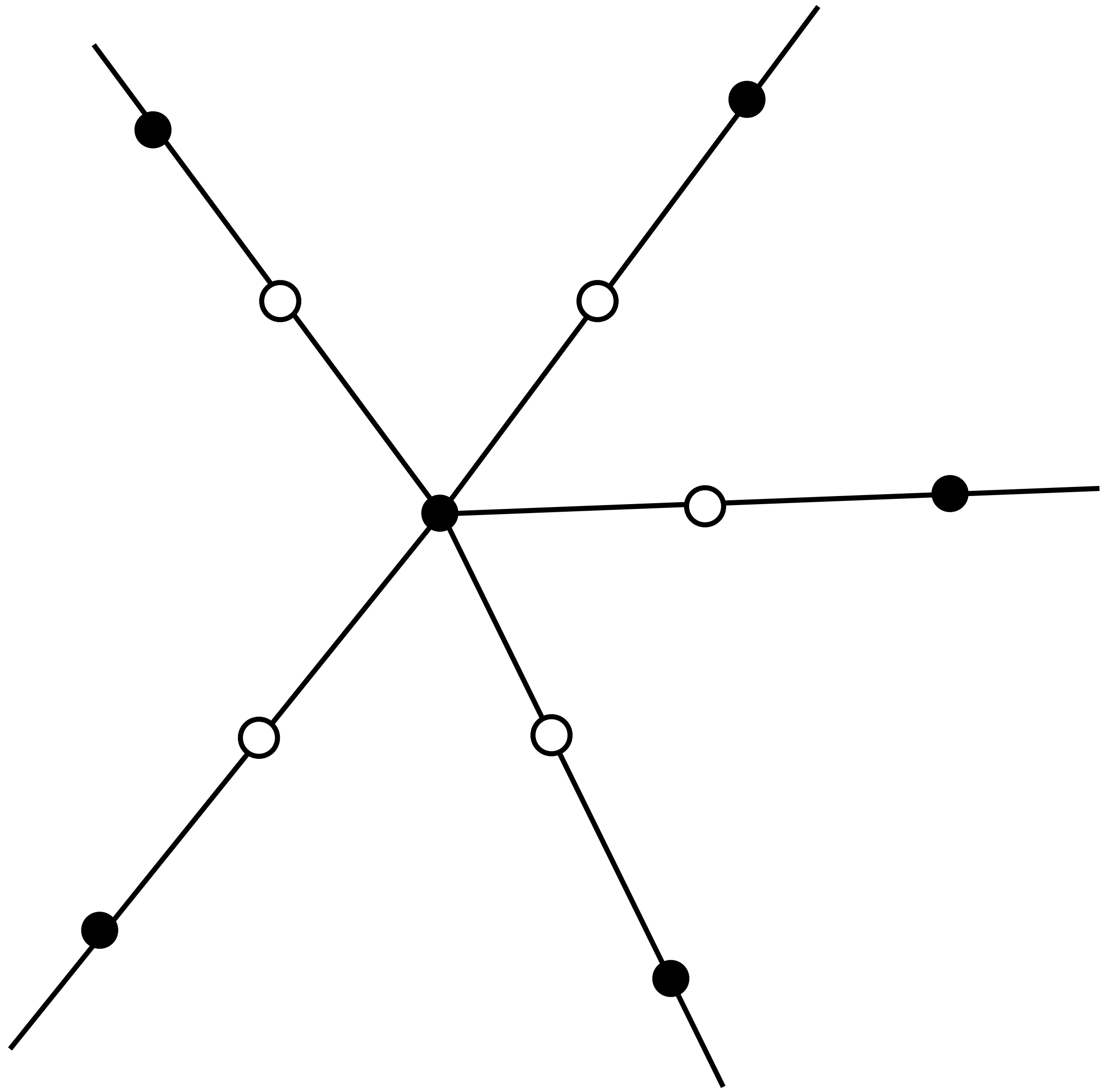
- Half-integrality
- From dual opt, we can recover an optimal multiflow by solving one feasible circulation problem

Proximity scaling algorithm



g^j : restriction of obj

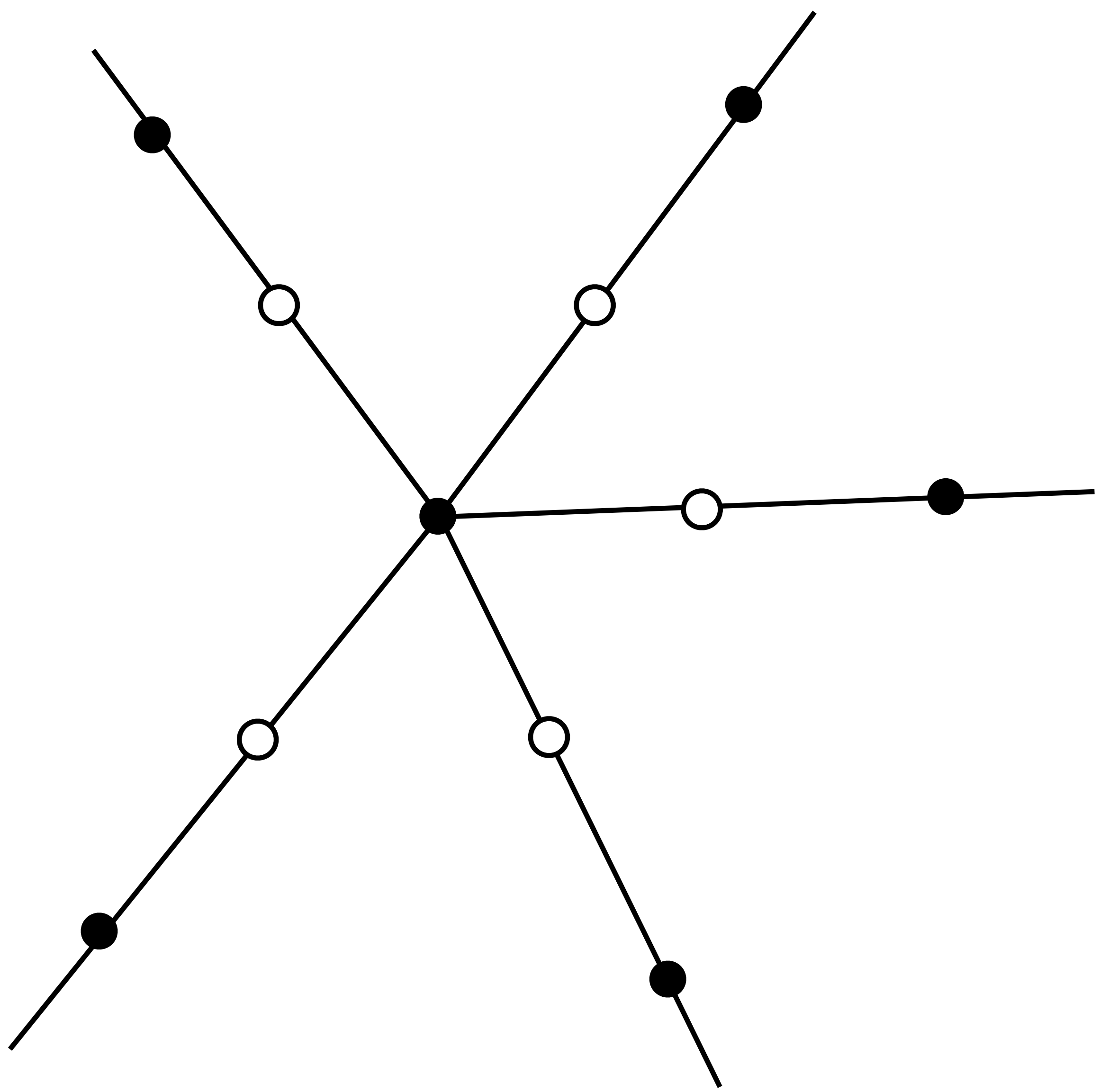
Proximity scaling algorithm



g^j : restriction of obj

$\bar{g}^j$: L-convex relax

Proximity scaling algorithm

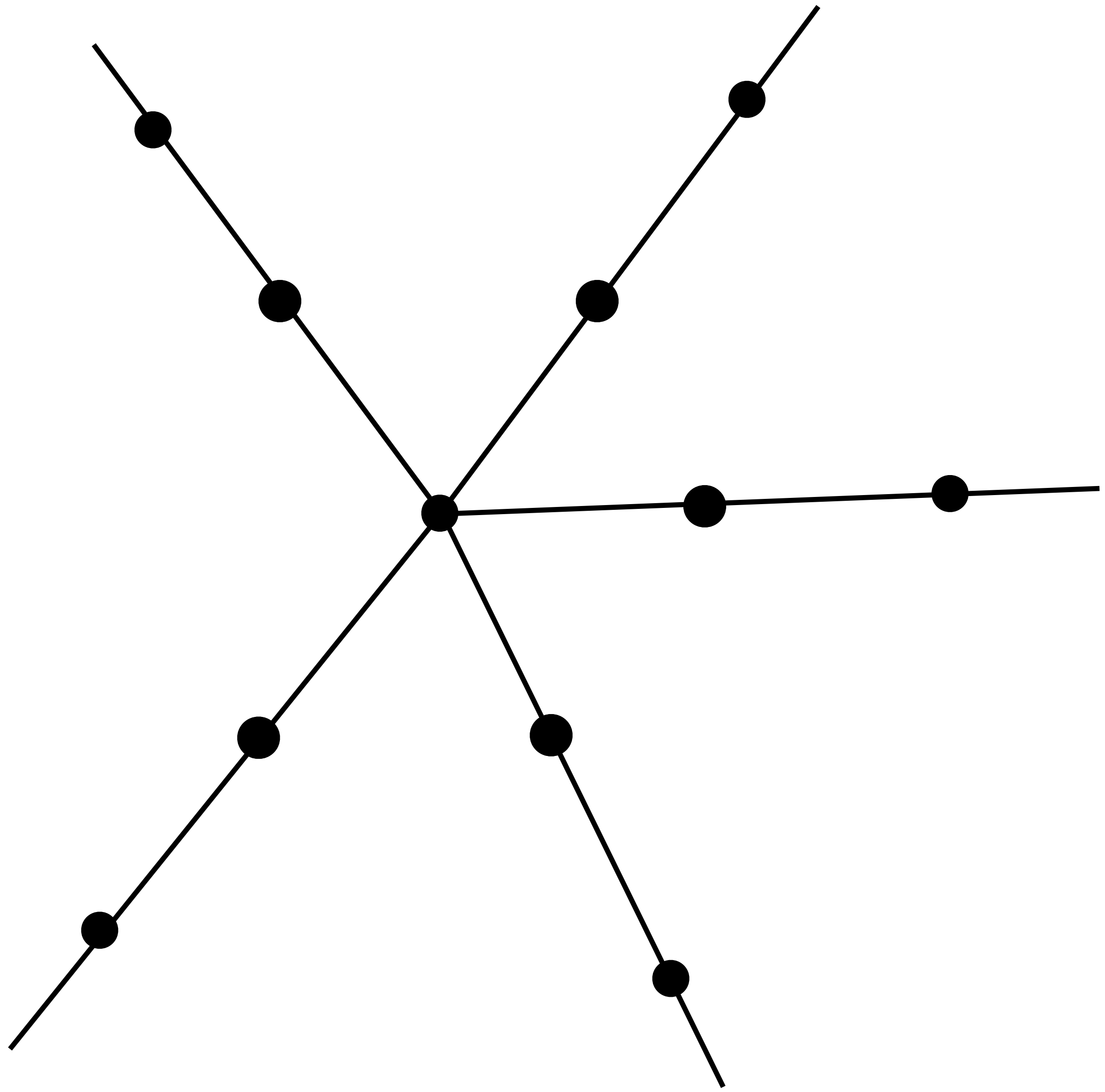


g^j : restriction of obj

$\bar{g}^j$: L-convex relax

x^j :opt of $\bar{g}^j$

Proximity scaling algorithm



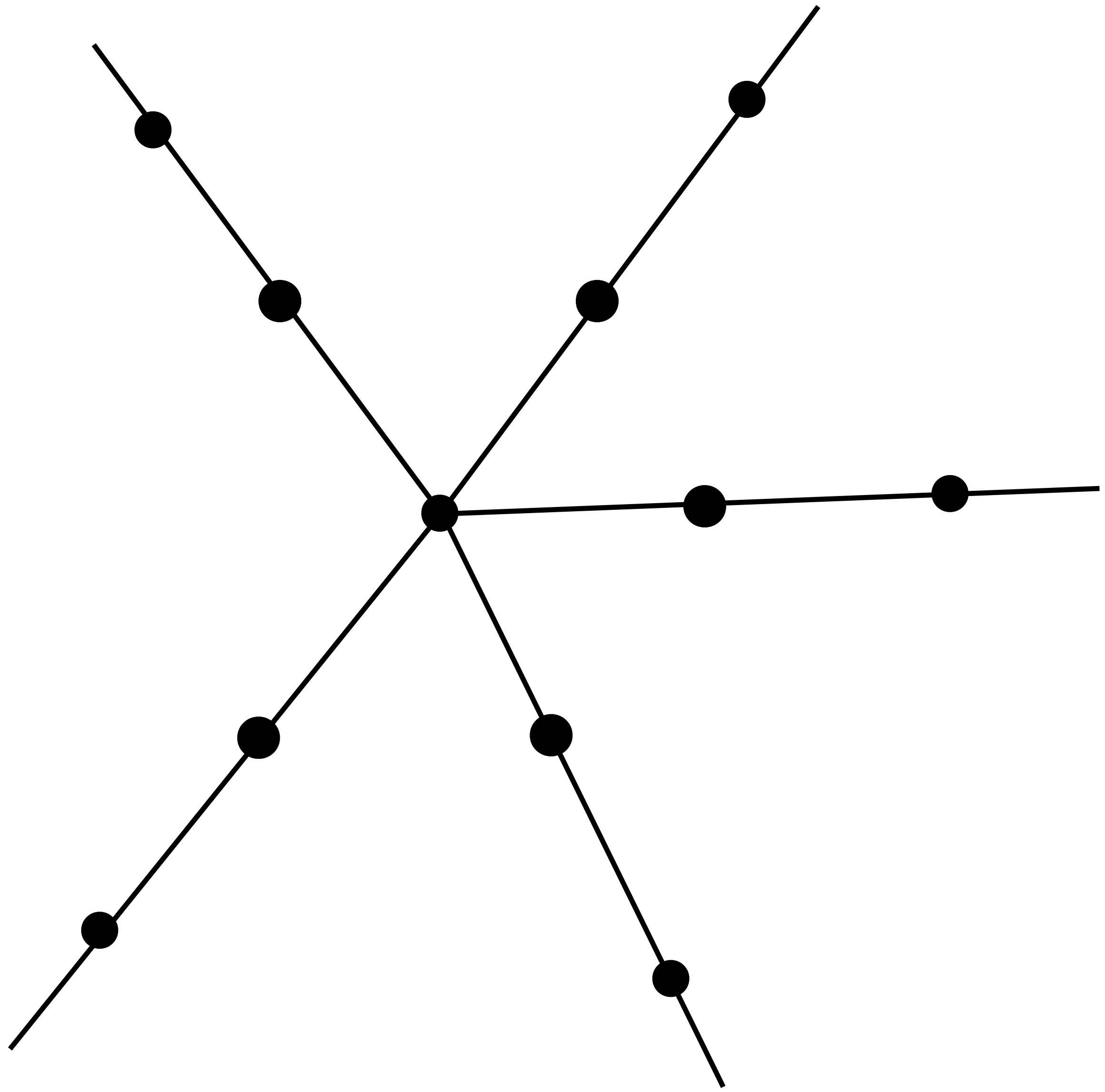
g^j : restriction of obj

$\bar{g}^j$: L-convex relax

x^j :opt of $\bar{g}^j$

g^{j+1}: restriction of obj

Proximity scaling algorithm



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$\nparallel$ x^j :opt of $\bar{g}^j$

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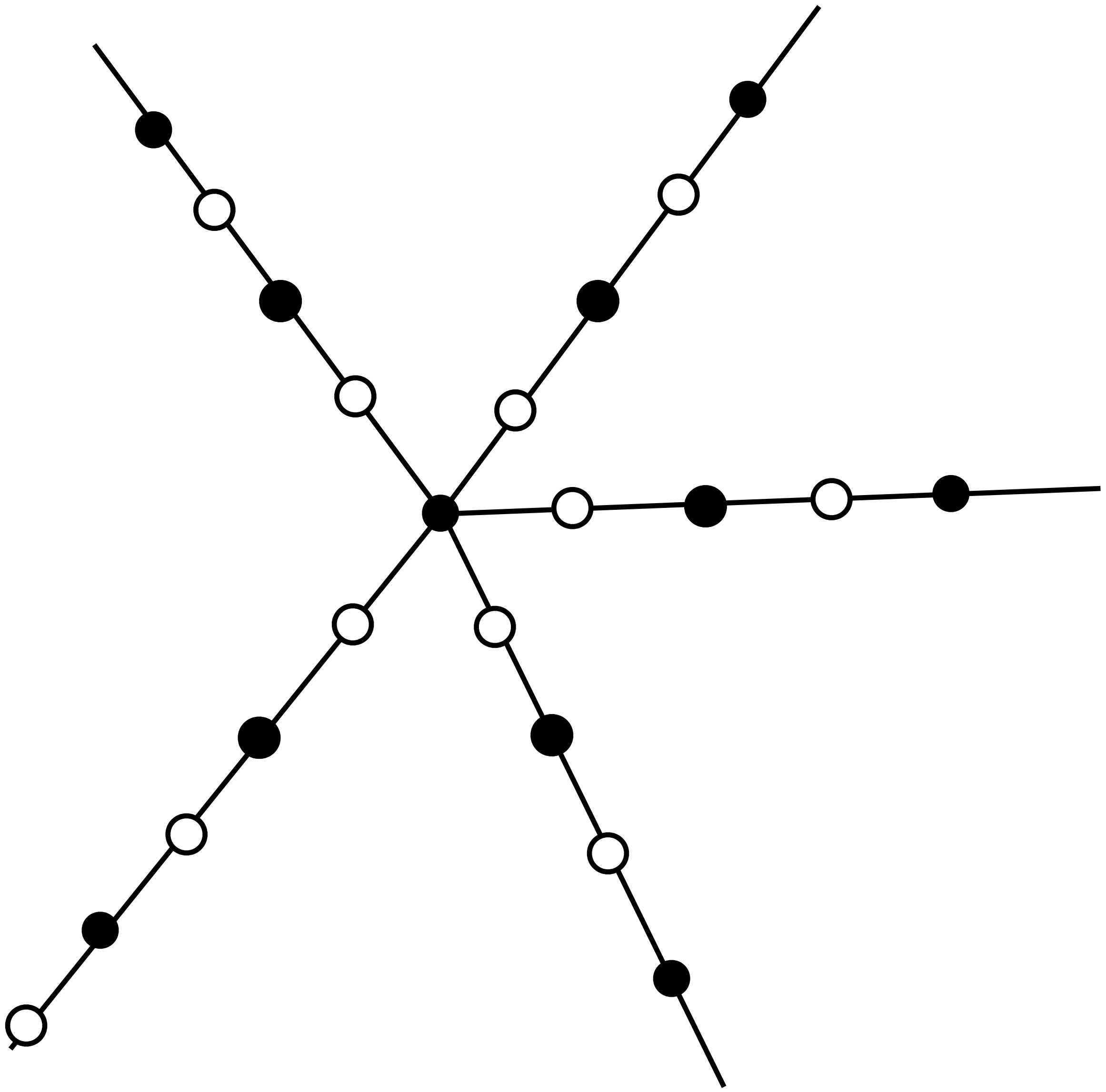
Proximity scaling algorithm



g^j : restriction of obj

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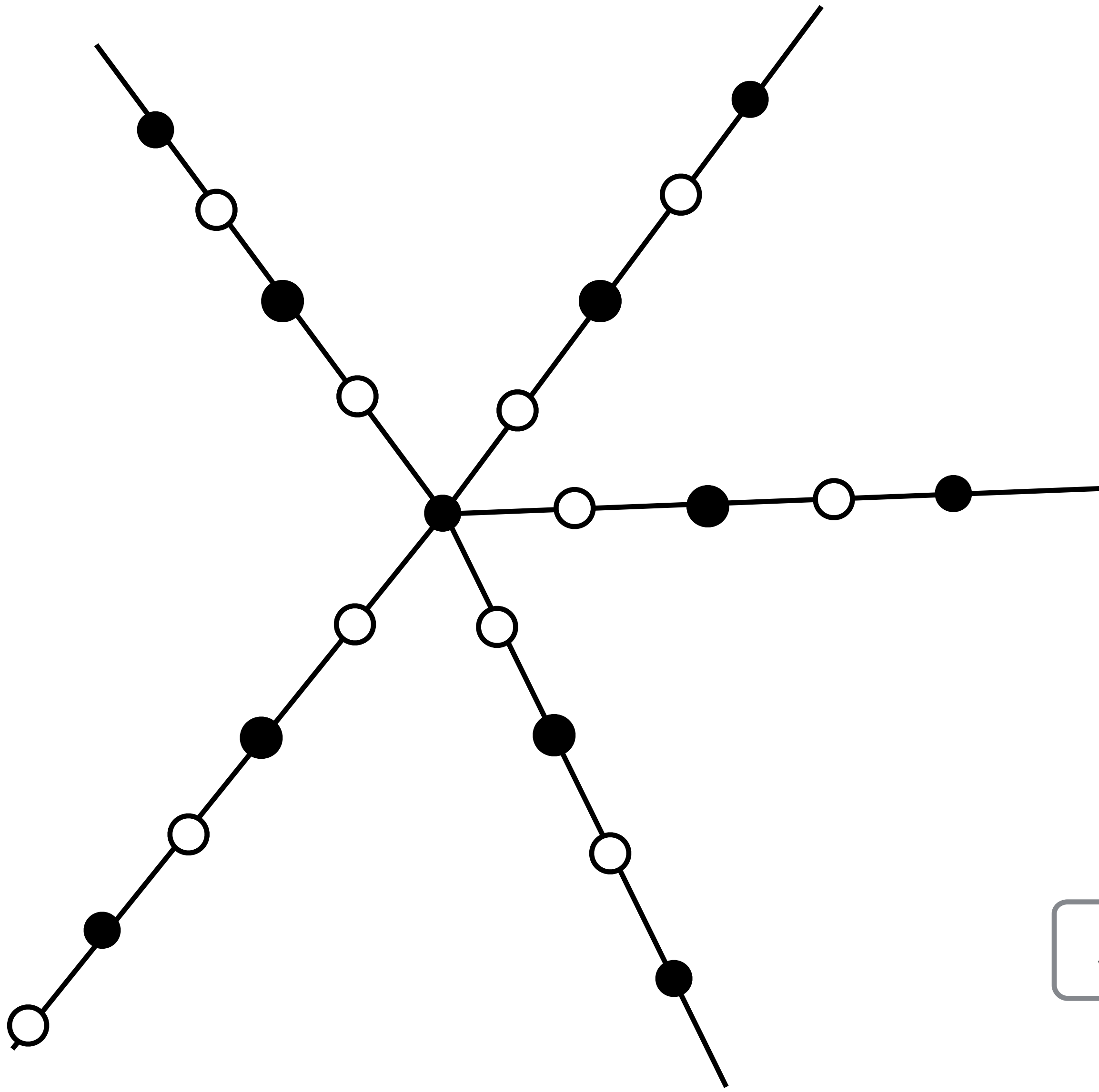
g^{j+1}: restriction of obj

$\bar{g}^{j+1}$: L-convex relax

Proximity scaling algorithm



g^j : restriction of obj

$\bar{g}^j$: L-convex relax

$\nparallel$ x^j :opt of $\bar{g}^j$

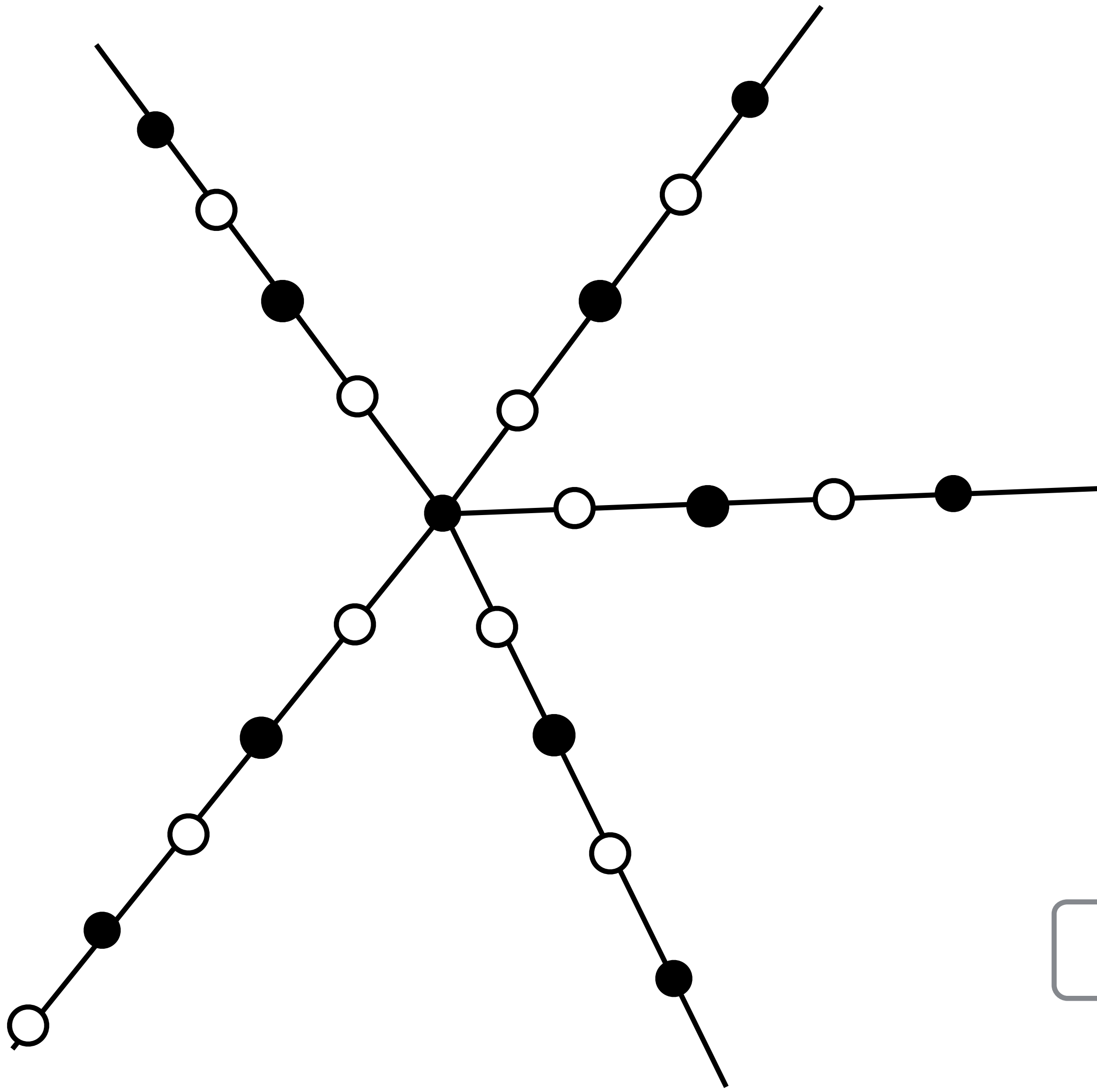
g^{j+1}: restriction of obj

$\bar{g}^{j+1}$: L-convex relax

$x^j \longrightarrow x^{j+1}$:opt of $\bar{g}^{j+1}$

steepest descent

Proximity scaling algorithm



g^j : restriction of obj

$\bar{g}^j$: L-convex relax

$\nparallel$ x^j :opt of $\bar{g}^j$

g^{j+1}: restriction of obj

$\bar{g}^{j+1}$: L-convex relax

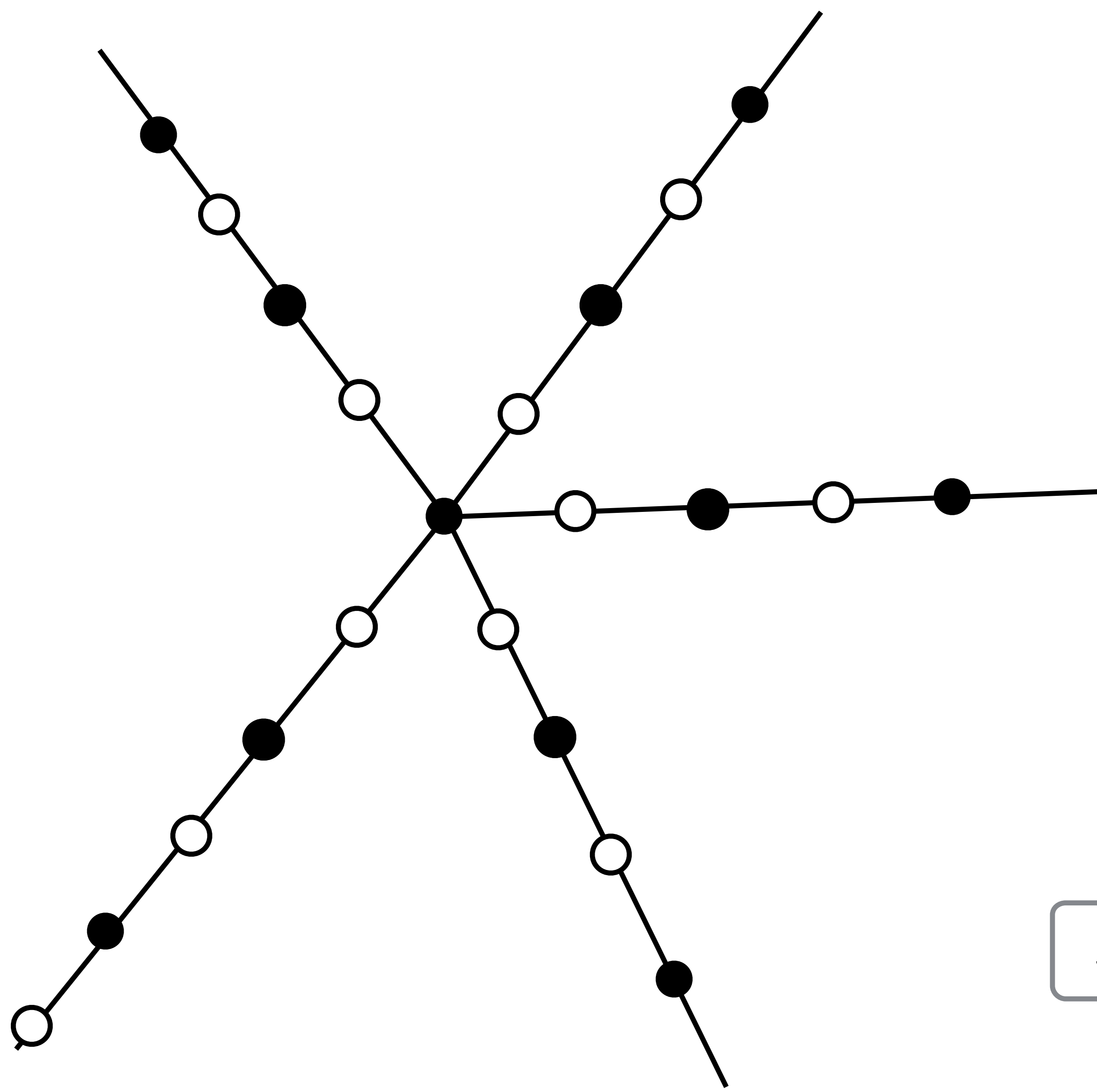
$x^j \longrightarrow x^{j+1}$:opt of $\bar{g}^{j+1}$

steepest descent

$$d_\infty(x^j, x^{j+1}) = O(n)$$

proximity & persistency

Proximity scaling algorithm



g^j : restriction of obj

$\bar{g}^j$: L-convex relax

$\nparallel$ x^j :opt of $\bar{g}^j$

g^{j+1}: restriction of obj

$\bar{g}^{j+1}$: L-convex relax

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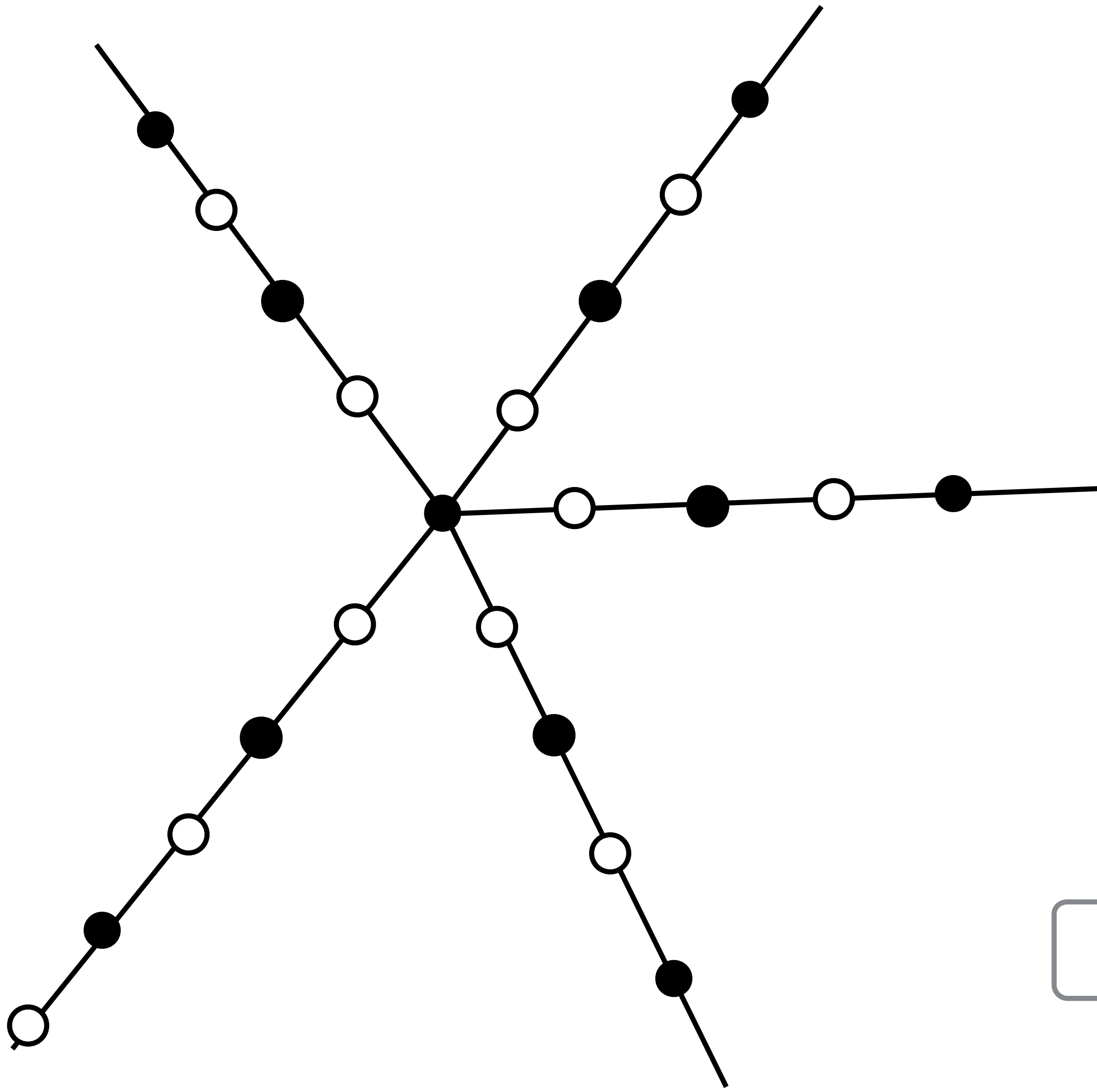
scaling phases = O(log nAC)

Proximity scaling algorithm



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$\bar{g}^{j+1}$: L-convex relax

$x^j \longrightarrow x^{j+1}$:opt of $\bar{g}^{j+1}$

steepest descent

$d_\infty(x^j, x^{j+1}) = O(n)$

proximity & persistency

scaling phases = O(log nAC)

Total time = O(n log (nAC)MF(kn,km))

Concluding remarks

- Our algorithm solves mincost node-demand multiflow problem, considered by Fukunaga 14, as LP-relaxation of capacitated terminal backup problem (Anshelevich-Karagiozova 11, Bernath-Kobayashi-Matsuoka 13)
- Efficient implementation of his 4/3-approx. algorithm
- Algorithm for node-capacitate multiflow & node-multiway cut (in preparation)