# The maximum multiflow problems with bounded fractionality

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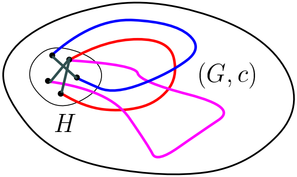
# Multicommodity flows

$(G, c)$: undirected network
$G = (VG, EG)$, $c : EG \to \mathbf{Z}_+$
$H$: commodity graph ($VH \subseteq VG$)




## Definition

*Multiflow* $f = \{\varphi_{st} \mid st \in EH\}$, where $\varphi_{st}$: (fractional) $st$-flow,

$$\sum_{st \in EH} |\varphi_{st}(e)| \leq c(e) \quad (e \in EG).$$

*Communication networks, transpotation, VLSI,*
*LP-relaxations of NP-hard problems*
*(Edge-disjoint paths, Multicut, 0-extension, Sparsest cut, ...)*

# Maximum Multiflow Problem

$(G, c)$: network, $H$: commodity graph
Total flow-value of multiflow $f$
$\|f\| := \sum\{f_{st} \mid st \in EH\}$

## Maximum Multiflow Problem

*Maximize $\|f\|$ over all multiflows $f$ in $(G, c; H)$*

## Fractionality

$\mathrm{frac}(H) :=$ the least positive integer $k$ with property:
$\exists\ 1/k$-integral maximum flow for $(\forall G, \forall c; H)$.

$\mathrm{frac}(\,|\,) = 1$ (Ford-Fulkeson 56)
$\mathrm{frac}(\,|\,|\,) = 2$ (Hu 63)
$\mathrm{frac}(\underbrace{|\,|\,|\cdots|}_{k \geq 3}) =$

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$\mathrm{frac}(\underbrace{|\,|\,|\cdots|}_{k\geq 3}) = +\infty$
$\mathrm{frac}(\triangle) = \mathrm{frac}(\boxtimes) = \mathrm{frac}(K_n) = 2$ (Lovasz 76, Cherkassky 77)
$\mathrm{frac}(|\,\triangle) = 2$ (Karzanov 98)
$\mathrm{frac}(|\,\boxtimes) = \mathrm{frac}(K_2 + K_n) = 4$ (Lomonsov 04)
$\mathrm{frac}(\triangle\,\triangle) = ?$

# Karzanov's conjecture

**Fractionality problem (Kazarnov, ICM Kyoto, 90)**

*Classify commodity graphs having finite fractionality.*

(Property P) For every intersecting maximal stable sets $A, B, C$ of $H$

$$A \cap B = B \cap C = C \cap A.$$

**Theorem (Karzanov 89)**

If $\mathrm{frac}(H) < +\infty \Rightarrow H$ satisfies P.

**Conjecture (Karzanov, ICM Kyoto, 90)**

If $H$ satisfies P $\Rightarrow$

1. $\mathrm{frac}(H) < +\infty$,
2. $\mathrm{frac}(H) \in \{1, 2, 4\}$.

# A weighted generalization

$(G, c)$:network, $S \subseteq VG$: terminal set, $\mu : \binom{S}{2} \to \mathbf{R}_+$: terminal weight
Multiflow $f = \{\varphi_{st} \mid st \in \binom{S}{2}\}$
Flow-value $\|f\|_\mu := \sum\{\mu(st) f_{st} \mid st \in \binom{S}{2}\}$.

**$\mu$-weighted maximum multiflow problem**

*Maximize $\|f\|_\mu$ over all multiflows $f$ in $(G, c; S)$*

- 0-1 weight $\Leftrightarrow$ commodity graph $H$
- $\mathrm{frac}(\mu)$ := the least positive integer $k$ s.t. $\cdots$

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**Theorem (Karzanov 98 for metrics, H. 09 for general)**

If $\mathrm{frac}(\mu) < +\infty \Rightarrow \dim T_\mu \leq 2$,

Tight span (Isbell 64, Dress 84)
$T_\mu :=$ Minimal $\{p \in \mathbf{R}^S_+ \mid p(s) + p(t) \geq \mu(s, t)\ s, t \in S\}$

**Remark (H. 09)**

$H$ satisfies property P $\Leftrightarrow \dim T_{\mu_H} \leq 2$

# Main Theorem

**Main Theorem (H. 09, STOC 2010)**

If dim $T_\mu \leq 2$,

1. $\exists$ 1/12-integral maximum multiflow
   in $\mu$-max multiflow problem for every Eulerian network,
   ($\rightarrow$ $\exists$ 1/24-integral max multiflow for every network)
2. $\exists$ strongly polytime algorithm to find it *provided $\mu$ is 0-1*.

**Corollary**

- $\mathrm{frac}(\mu) \in \{1, 2, 3, 4, 6, 8, 12, 24, +\infty\}$
- $\mathrm{frac}(\mu) < +\infty \Leftrightarrow$ dim $T_\mu \leq 2$
- $\mathrm{frac}(H) < +\infty \Leftrightarrow$ Property P

*We do not know whether* $1/24$ *is tight !!*

# Proof Idea

Synchronizing Primal & Dual

**Dual**: LP-dual $\rightarrow$ *minimum 0-extension on a folder complex*

**Primal**: Fractional splitting-off & potential update

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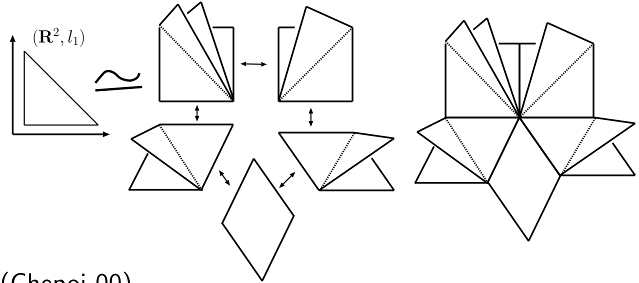
Synchronizing Primal & Dual
**Dual**: LP-dual $\rightarrow$ *minimum 0-extension on a folder complex*
**Primal**: Fractional splitting-off & potential update

A key concept: folder complex ($\simeq$ 2-dimensional tight span)
$\mathcal{K}$: *metrized* polygonal complex obtained by gluing *folders*:




(Chepoi 00)

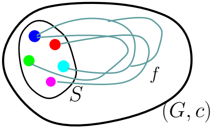
$\mathcal{K}$: folder complex $\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} \mathcal{K} \text{ is simply-connected, and} \\ \text{link graph of each vertex has girth} \geq 8. \end{cases}$

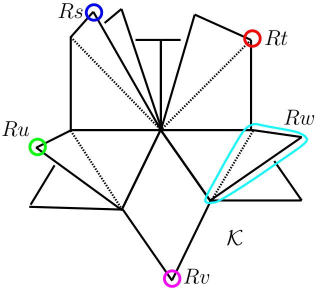
$\Longleftrightarrow$ *CAT(0)* under $l_2$-metrization.

# Folder complex and multiflow combinatorial duality

$\mu =$

| | $t$ | $u$ | $v$ | $w$ |
|---|---|---|---|---|
| $s$ | 3 | 3 | 3 | 2 |
| $t$ | | 4 | 4 | 1 |
| $u$ | | | 2 | 3 |
| $v$ | | | | 1 |




## Theorem (H. 09)

Suppose $\exists$ folder complex $\mathcal{K}$ with *normal regions* $\{R_s\}_{s \in S}$ s.t.

$$\mu(s,t) = d_{\mathcal{K},l_1}(R_s, R_t) \quad (s,t \in S).$$

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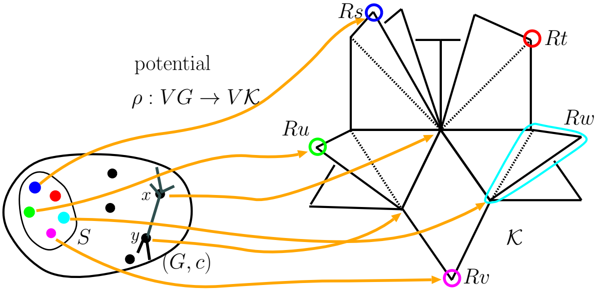


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$$\mu(s,t) = d_{\mathcal{K},l_1}(R_s, R_t) \quad (s,t \in S).$$

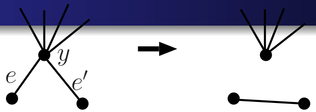
$$\max_f \|f\|_\mu \quad = \quad \min \sum_{xy \in EG} c(xy) d_{\mathcal{K},l_1}(\rho(x), \rho(y))$$

$$\text{s.t. } \rho : VG \to V\mathcal{K}, \ \rho(s) \in R_s \quad \leftarrow \textit{potential}$$

- $\dim T_\mu \leq 2$ if and only if $\mu$ is realized by a folder complex.
- optimal potential $\rho$ is obtained by solving LP-dual.

# Splitting-off

- fork: $\tau = (e, y, e')$



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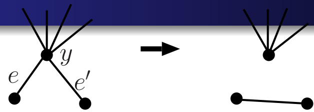


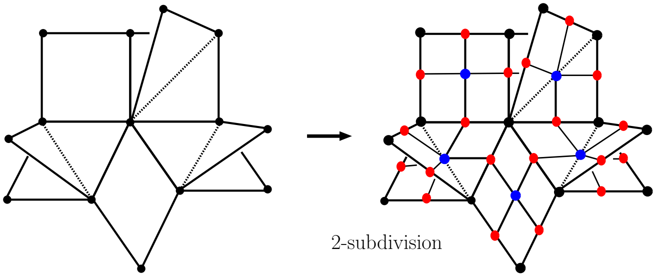

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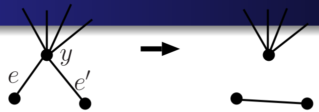
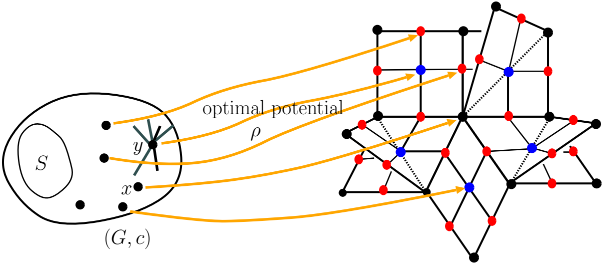



## Proposition (H. 09)

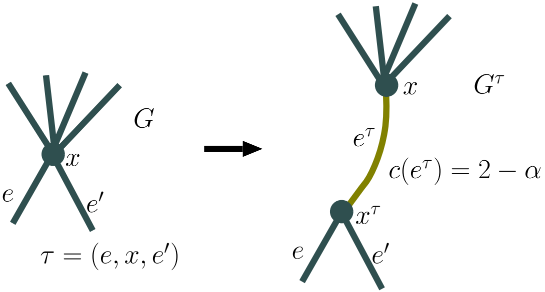
$(G, c)$: Eulerian, $\rho : VG \to V\mathcal{K}$: optimal potential, $y$: inner node
If $\rho(y) = \bullet$, then $y$ has a splittable fork.

## Approach (after some preprocessing on terminals)

$(G, c)$: Eulerian, $\rho$: optimal potential
$(G, c; \rho) \to \cdots \to (G', c'; \rho')$ until $\rho(V^{in}G') = \bullet$ with $(G', kc')$ Eulerian

# Fractional splitting-off & potential update

- Fractional splitting-off




- Maximum possible value: $\quad (c \cdot d^\rho := \sum_{xy \in EG} c(xy) d_{\mathcal{K}}(\rho(x), \rho(y)))$

$$\alpha(\tau) = \min \left\{ \frac{c \cdot d^{\rho'} - c \cdot d^\rho}{d^{\rho'}(e^\tau)} \;\middle|\; \begin{array}{c} \rho' : \text{potential}, \\ d^{\rho'}(e^\tau) > 0 \end{array} \right\}$$

Update $(G, c; \rho) \to (G^\tau, c; \rho')$, $c(e^\tau) := 2 - \alpha(\tau)$

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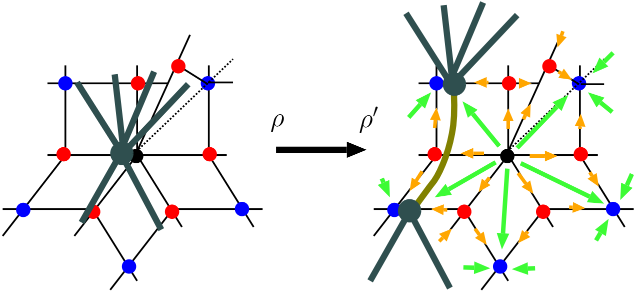


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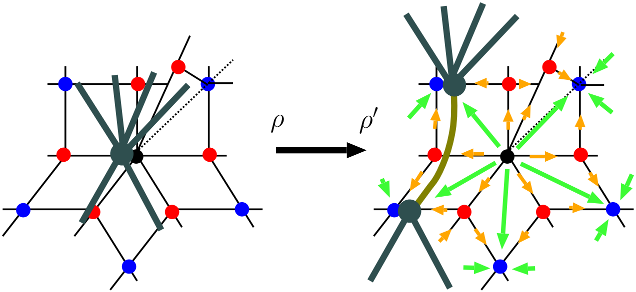


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*We can take $\rho'$ so that potentials move toward $\bullet$*

*$\to$ Capacities of edges between $\rho^{-1}(\bullet)$ cancel out.*

*$\to$ We can make $(G, c; \rho)$ so that $\rho(V^{in}G) = \bullet$ and $(G, 12c)$ is Eulerian.*

# Concluding remarks

- we do not know whether 24 is tight
  ... conjectured tight upper bound is 4.
- proof is lengthy and complicated
- polytime algorithm provided the size of $\mathcal{K}$ is fixed
  ($\mu$ is 0-1 $\Rightarrow \exists\ O(|S|^2)$-size $\mathcal{K}$ realizing $\mu$).
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*Thank you for your attention !*