T_X-approaches to multiflows and metrics

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Contents:

Part I: T-dual to maximum multiflow problems

H. Hirai, Tight extentions of distance spaces and the dual fractionality of undirected multiflow problems, RIMS Preprint-1606, 2007.
http://www.kurims.kyoto-u.ac.jp/preprint/RIMS1606.pdf

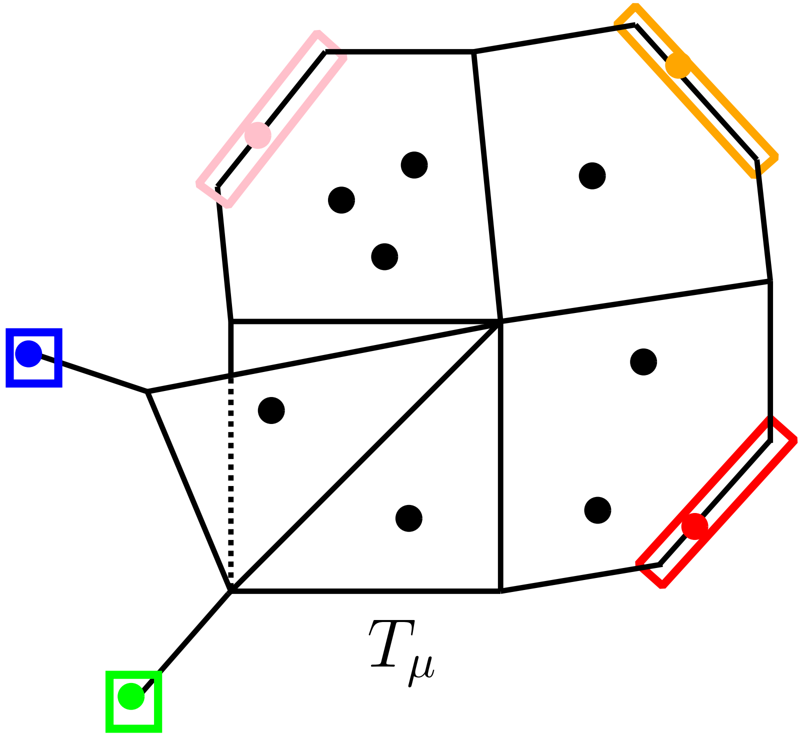
Part II: Metric packing for $K_3 + K_3$ (option)

H. Hirai, Metric packing for $K_3 + K_3$, RIMS-preprent 1608, 2007.
http://www.kurims.kyoto-u.ac.jp/preprint/RIMS1608.pdf

Part I: T-dual to maximum multiflow problems

Main message:

- Multiflow combinatorial duality theorems can be derived from T-dual.



- Geometry of T_μ rules discreteness of multiflow potential.

Notation

$G = (V, E)$: an undirected graph with nonnegative capasity $c : E \to \mathbf{R}_+$
S: the set of terminals $S \subseteq V$
$\mathcal{P}$: the set of paths in G whose ends belong to S.

Definition. $f : \mathcal{P} \to \mathbf{R}_+$ is a *multiflow* (w.r.t $(G, c; S)$) if

$$\sum_{P \in \mathcal{P} : e \in P} f(P) \leq c(e) \quad (e \in E).$$

Maxmimization problem

μ-max problem:
Given $\mu : S \times S \to \mathbf{R}_+$ with $\mu(s,t) = \mu(t,s)$ and $\mu(s,s) = 0$,

$$\begin{array}{ll} \text{Maximize} & \sum_{P \in \mathcal{P}} \mu(s_P, t_P) f(P) \\ \text{Subject to} & f : \text{a multiflow for } (G, c; S), \end{array}$$

where s_P, t_P: endpoints of P.

Philosophy: we shall regard μ as a distance on S

Problem of the bounded fractionality (Karzanov)
When does μ-max problem have integer, half-integer, quarter-integer, or $1/k$-integer (fixed k) optimal flow for $\forall G = (V, E)$ with integer c and $S \subseteq V$?

Some nice examples

- $S = \{s, t\} \Rightarrow$ single commodity flow

Maxflow-Mincut Theorem (Ford-Fulkerson 54)
Max flow value $=$ s-t mincut value,
$\exists$ integer optimal flow if c is integer.

- $S = \{s, s', t, t'\}$, $\mu(s, t) = \mu(s', t') = 1$ and zero otherwise
$\Rightarrow$ two commodity flow

Maxbiflow-Mincut Theorem (Hu 63)
Max flow value $=$ Min (ss'-tt' mincut, st'-ts' mincut),
$\exists$ half-integer optimal flow if c is integer.

- $\mu(s,t) = 1$ $\forall s,t$ with $s \neq t \Rightarrow$ free multiflow problem

Theorem (Lovasz 76, Cherkassky 77)
Max flow value $= \dfrac{1}{2}\sum_{t \in S} t\text{-}S \setminus t$ mincut,
$\exists$ half-integer optimal flow if c is integer.

Notation: If μ is 0-1, the commodity graph $H_\mu = (S, R_\mu)$ is defined by $st \in R_\mu \overset{\text{def}}{\Longleftrightarrow} \mu(s,t) = 1$.

Remark: $H_\mu = K_2$: single commodity, $H_\mu = K_2 + K_2$: two commodity, $H_\mu = K_n$: free multiflow, Assume H_μ has no isolated point and c is integer.

Theorem (Karzanov-Lomonosov 1978)
If the intersection graph Γ of the maximal stable sets in H_μ has no triangle, there exists a quarter-integer optimal flow.

If Γ is bipartite, there exists a half-integer optimal flow.

Rem: $\exists$ combinatorial duality theorem.

Rem: A polymatroidal proof (Frank, Karzanov, and Sebö 1994).

Beyond 0-1 weights

Multiflow Locking Theorem (Karzanov-Lomonosov 1978)

$\mathcal{A}$: 3-cross free family on S

$\mu = \sum_{A \in \mathcal{A}} \delta_A$: sum of cut metrics of $\mathcal{A}$

Max flow value $= \sum_{A \in \mathcal{A}} A$-$S \setminus A$ mincut,

$\exists$ half-integer optimal flow

Theorem (Karzanov & Manoussakis 1996)

(S, μ): the graph metric of $K_{2,n}$

$\exists$ half-integer optimal flow ($+$ combinatrial duality theorem)

Where do these small fractionality phenomena come from ?

LP-dual to μ-max problem

$$\begin{array}{rl} \text{Minimize} & \langle c, d\rangle_E \\ \text{Subject to} & d\text{: metric on } V, \\ & d(s,t) \geq \mu(s,t) \quad (s,t \in S) \end{array}$$

Remark: If μ-max problem has a $1/k$-integer optimal flow for $\forall(G,c)$ with $c \in \mathbf{Z}_+^E$ and μ is integral, the polyhedron

$$\mathcal{P}_{\mu,V} = \{d : \text{metric on } V \mid d(s,t) \geq \mu(s,t)(s,t \in S)\} + \mathbf{R}_+^V$$

is $1/k$-integral (by standard TDI argument).

Remark: This gives a necessary condition for the existence of $1/k$-integral optimal flows

Assume μ is 0-1 distance and H_μ has no isolated point.

Theorem (Karzanov 1989)

(1) If H_μ satisfies:

(P) three pairwise intersecting maximal stable sets A_1, A_2, A_3 in H_μ satisfies $A_1 \cap A_2 = A_2 \cap A_3 = A_3 \cap A_1$,

then $\mathcal{P}_{\mu,V}$ is quarter-integral for $\forall V$ with $S \subseteq V$.

(2) If H_μ violates (P), then there is no integer k such that $\mathcal{P}_{\mu,V}$ is $1/k$-integral for $\forall V$ with $S \subseteq V$.

A. V. Karzanov: Polyhedra related to undirected multicommodity flows, *Linear Algebra and Its Applications* 114/115 (1989) 293–328.

Karzanov Conjecture (ICM, Kyoto, 1990)

(1) If H_μ satisfies (P), then there is $k \in \mathbf{Z}_+$ such that μ-max problem has $1/k$-integer optimal flow for $\forall G = (V, E)$ with $c \in \mathbf{Z}^E$ and $S \subseteq V$.

(2) $k = 4$ will do.

Some special cases beyond Karzanov-Lomonosov Theorem (1978)

- If $H_\mu = K_2 + K_3$, $\exists$ half-integer optimal flow (Karzanov 1998).
- If $H_\mu = K_2 + K_r$, $\exists$ quarter-integer optimal flow (Lomonosov 2004).

μ: an integral metric
$P_\mu := \{p \in \mathbf{R}^S \mid p(s) + p(t) \geq \mu(s,t) \ (s,t \in S)\}$
$T_\mu :=$ the set of minimal elements of P_μ (tight span of μ)

Theorem (Karzanov 1998)

(1) If $\dim T_\mu \leq 2$, then $\mathcal{P}_{\mu,V}$ is quarter-integral for $\forall V$ with $S \subseteq V$.

(2) If $\dim T_\mu \geq 3$, then then there is no k such that $\mathcal{P}_{\mu,V}$ is $1/k$-integral for $\forall V$ with $S \subseteq V$.

A. V. Karzanov:
Minimum 0-extensions of graph metrics, *European J. Combin.* **19** (1998) 71–101.
Metrics with finite sets of primitive extensions, *Ann. Combin.* **2** (1998) 211–241.

μ: an integral distance

Main Theorem (H.07)

(1) If $\dim T_\mu \leq 2$, then $\mathcal{P}_{\mu,V}$ is quarter-integral for every V with $S \subseteq V$.

(2) If $\dim T_\mu \geq 3$, then then there is no k such that $\mathcal{P}_{\mu,V}$ is $1/k$-integral for every V with $S \subseteq V$.

Remark (H.07): Karzanov condition (P) $\Leftrightarrow$ $\dim T_\mu \leq 2$ for 0-1 distance μ.

Generalized Karzanov Conjecture:
If $\dim T_\mu \leq 2$, there is $k \in \mathbf{Z}$ such that μ-max problem has a $1/k$-integral optimal flow for $\forall G = (V, E)$ with $c \in \mathbf{Z}_+^E$ and $S \subseteq V$.

Now I'm trying to solve it !

T_μ: the *tight span*, the *injective hull*, or the T_X-*space*

T_μ is not so common in combinatrial optimization.

Q1. What is T_μ ?

Q2. Why does T_μ arise in multiflow problem ?
($\rightarrow$ T-dual)

Q3. Why is $\dim T_\mu \leq 2$ crucial ?
($\rightarrow$ l_∞-plane $\simeq$ l_1-plane)

What is T_μ ? (some history)

1964 Isbell (injective hull)

1984 Dress (phylogenetic tree reconstruction)

1994 Chrobak & Larmore (online algorithm)

2006 Hirai (the tight span of nonmetric distances)

Relation to multiflow theory

1997 Chepoi (T_X-proof to cut packing theorem)

1998 Karzanov (relaxation of 0-extension problem)

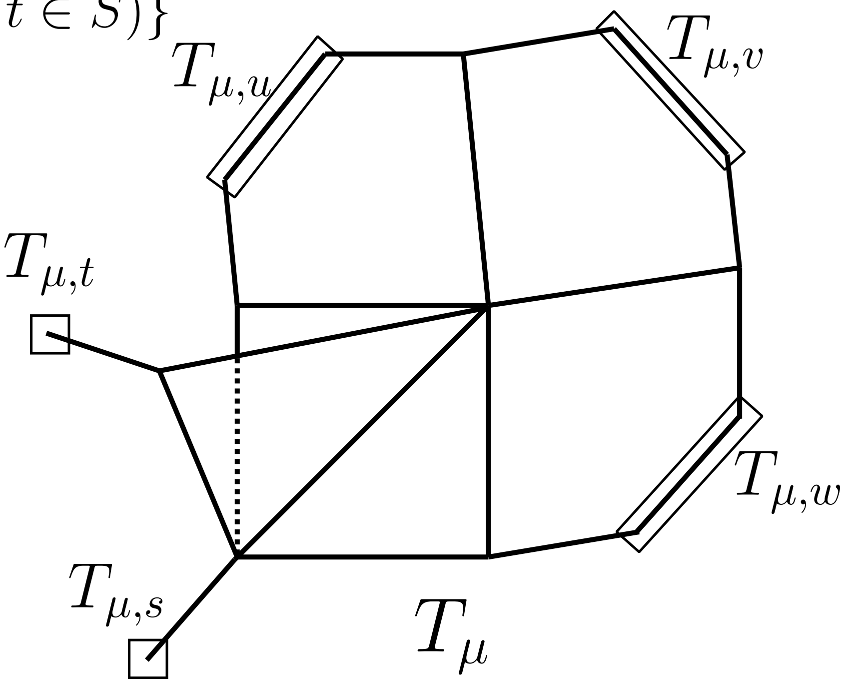
Some interesting properties of T_μ

- μ is isometrically embeded into (T_μ, l_∞) (Dress 84, H. 06)
- metric μ is a tree metric if and only if T_μ is a tree (Dress 84), and more...

$$P_\mu = \{p \in \mathbf{R}^S \mid p(s) + p(t) \geq \mu(s,t) \ (s,t \in S)\}$$
$$T_\mu = \ \text{Minimal } P_\mu$$
$$T_{\mu,s} = \{p \in \mathbf{R}^S \mid p(s) = 0\}$$

$\mu =$

	s	t	u	v	w
s	0	2	3	4	2
t	2	0	3	3	3
u	3	3	0	1	3
v	4	3	1	0	1
w	2	3	3	1	0



Why does T_μ arise in multiflow problem ?

$P_\mu := \{p \in \mathbf{R}^S \mid p(s) + p(t) \geq \mu(s,t) \ (s,t \in S)\}$
$T_\mu :=$ the set of minimal elements of P_μ
$T_{\mu,s} := \{p \in T_\mu \mid p(s) = 0\} \quad (s \in S)$ (the terminal region of s)

T-dual to μ-max problem:

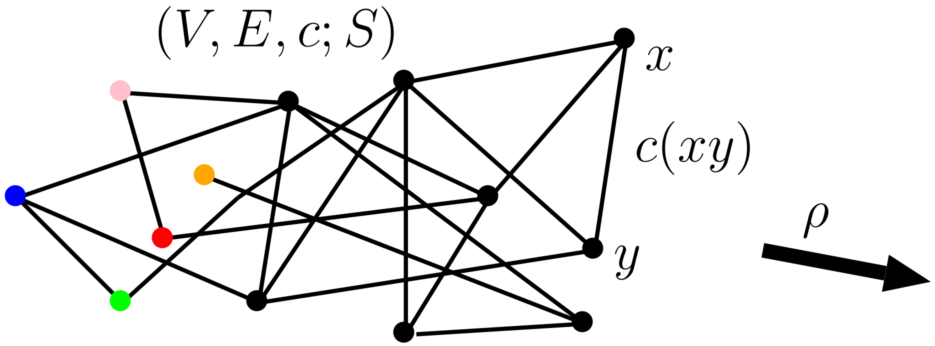
Theorem (H. 07)

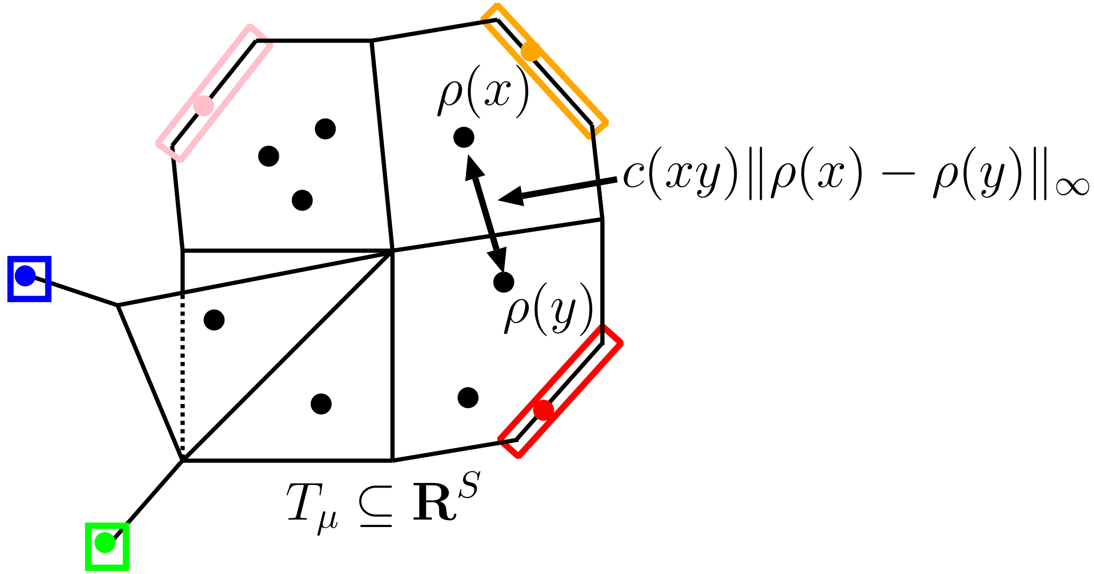
$$\begin{array}{rl} \text{Minimize} & \langle c, d\rangle_E \\ \text{Subject to} & d\text{: metric on } V, \\ & d(s,t) \geq \mu(s,t) \quad (s,t \in S) \end{array}$$

$$\begin{array}{rl} \simeq \ \text{Minimize} & \displaystyle\sum_{xy \in E} c(xy)\|\rho(x) - \rho(y)\|_\infty \\ \text{Subject to} & \rho : V \to T_\mu \\ & \rho(s) \in T_{\mu,s} \quad (s \in S) \end{array}$$

ρ is an analogue of the *potential*

$\|\rho(x) - \rho(y)\|_\infty$ is the *potential difference*





$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{xy \in E} c(xy)\|\rho(x) - \rho(y)\|_\infty \\ \text{Subject to} & \rho : V \to T_\mu \\ & \rho(s) \in T_{\mu,s} \ (s \in S) \end{array}$$

Multifacility location problem (a variation of p-median problems)

Proof of T-dual

$P_\mu = \{p \in \mathbf{R}^S \mid p(s) + p(t) \geq \mu(s,t) \ (s,t \in S)\}$
$P_{\mu,s} := \{s \in P_\mu \mid p(s) = 0\} \ (s \in S)$

Lemma: LP-dual of μ-max problem is equivalent to

$$\begin{array}{rl} \text{Minimize} & \displaystyle\sum_{xy \in E} c(xy)\|\rho(x) - \rho(y)\|_\infty \\ \text{Subject to} & \rho : V \to P_\mu \\ & \rho(s) \in P_{\mu,s} \quad (s \in S) \end{array}$$

Proof: For $\rho : V \to P_\mu$ define metric d by

$$d(x,y) := \|\rho(x) - \rho(y)\|_\infty \quad (x, y \in V).$$

Then

$$d(s,t) = \|\rho(s) - \rho(t)\|_\infty \geq \rho(t)(s) - \rho(s)(s) = \rho(t)(t) + \rho(t)(s) \geq \mu(s,t).$$

Conversely, for metric d with $d|_S \geq \mu$, define $\rho : V \to \mathbf{R}^S$ by

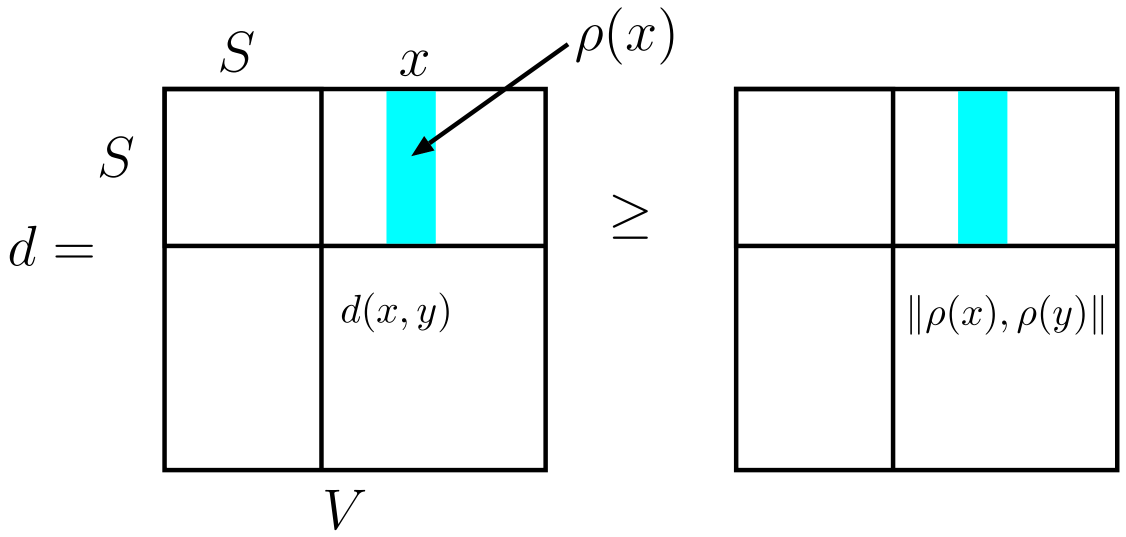
$$(\rho(x))(s) := d(x,s) \quad (s \in S).$$

Then we have

$$\rho(x)(s) + \rho(x)(t) = d(x,s) + d(x,t) \geq d(s,t) \geq \mu(s,t) \Rightarrow \rho(x) \in P_\mu,$$
$$\rho(s)(s) = d(s,s) = 0 \Rightarrow \rho(s) \in P_{\mu,s}.$$

Moreover,

$$\|\rho(x) - \rho(y)\| = |d(x,s) - d(y,s)| \leq d(x,y).$$



Lemma (Dress 84)

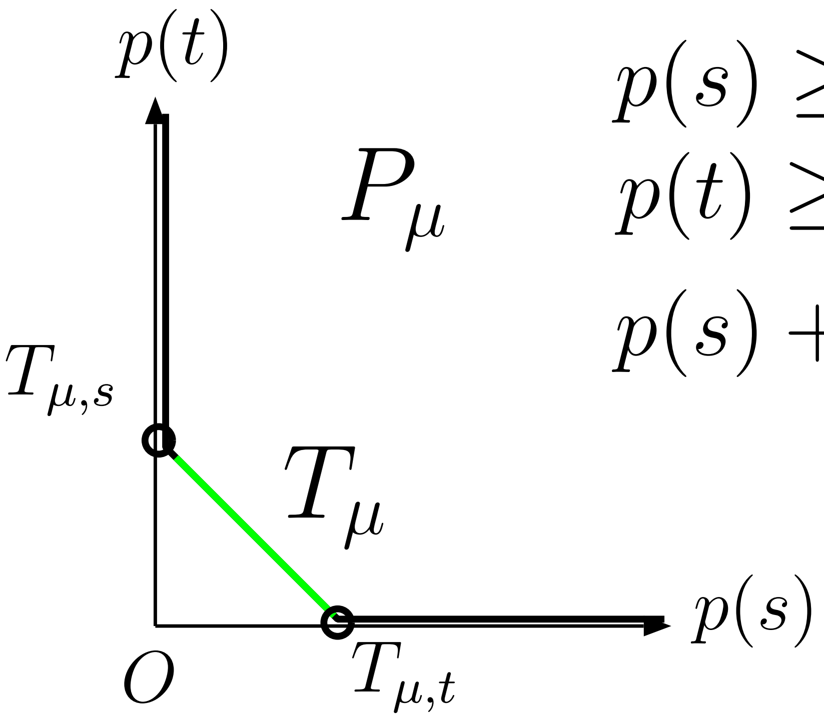
There is $\phi : P_\mu \to T_\mu$ such that

- $\phi(p) \leq p$ for $p \in P_\mu$ (, and thus $\phi(p) = p$ for $p \in T_\mu$),
- $\|\phi(p) - \phi(q)\|_\infty \leq \|p - q\|_\infty$ for $p, q \in P_\mu$.

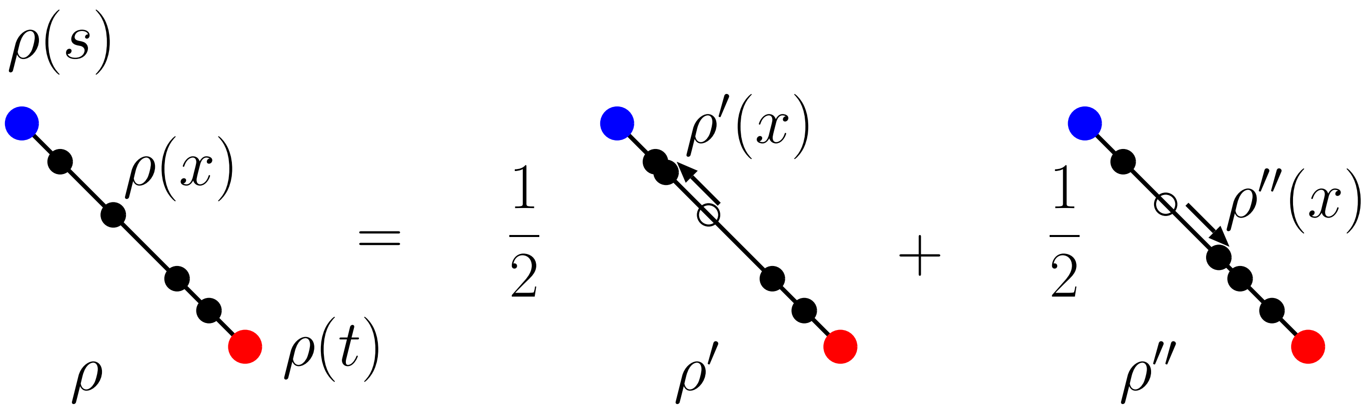
A. W. M. Dress: Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces. *Advances in Mathematics* **53** (1984), 321–402.

Ford-Fulkerson reconsidered ($S = \{s, t\}$)

The tight span is a segment

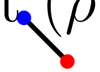
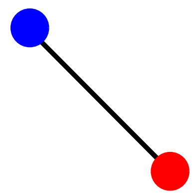


$$p(s) \geq 0$$
$$p(t) \geq 0$$
$$p(s) + p(t) \geq 1$$



$$
\begin{aligned}
d^{\rho} &= 1/2(d^{\rho'} + d^{\rho''}) \\
d^{\rho}(x,y) &:= \|\rho(x) - \rho(y)\|_{\infty}
\end{aligned}
$$

$\Rightarrow$ T-dual is equivalent to

Minimize $\sum_{xy \in E} c(xy)\mathrm{dist}$$(\rho(x), \rho(y))$ Subject to $\rho : V \to$ 

$\rho(s) =$  $\rho(t) =$ 

$\Rightarrow$ finding s-t mincut.

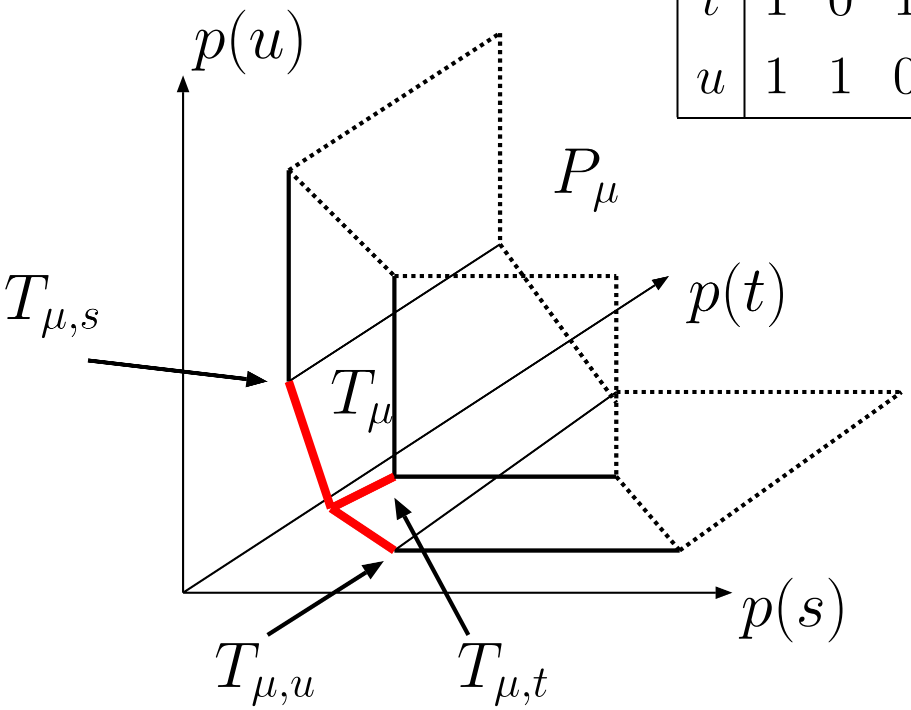
Lovasz-Cherkassky reconsidered ($H_\mu = K_n$)

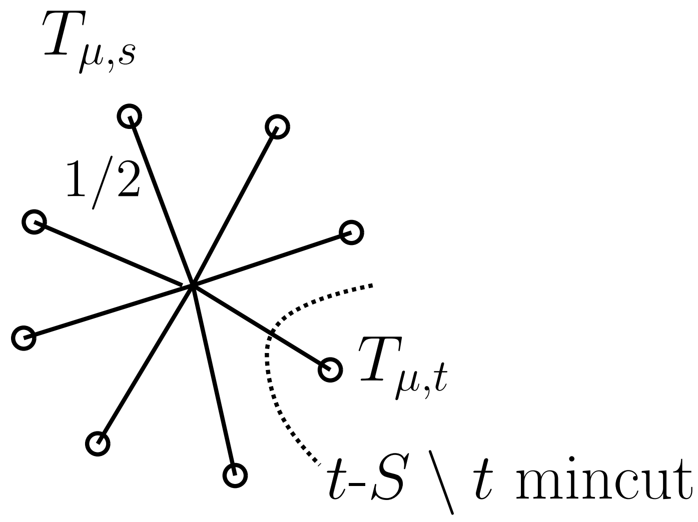
The tight span is a star

$$S = \{s, t, u\}$$

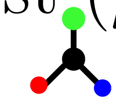
$\mu =$

	s	t	u
s	0	1	1
t	1	0	1
u	1	1	0

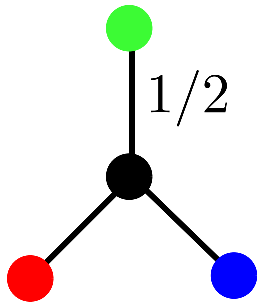




$\Rightarrow$ T-dual is equivalent to

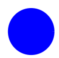
Minimize $\displaystyle\sum_{xy \in E} c(xy)\mathrm{dist}$  $(\rho(x), \rho(y))$

Subject to

$\rho : V \rightarrow$ 

$\rho(s) =$ 

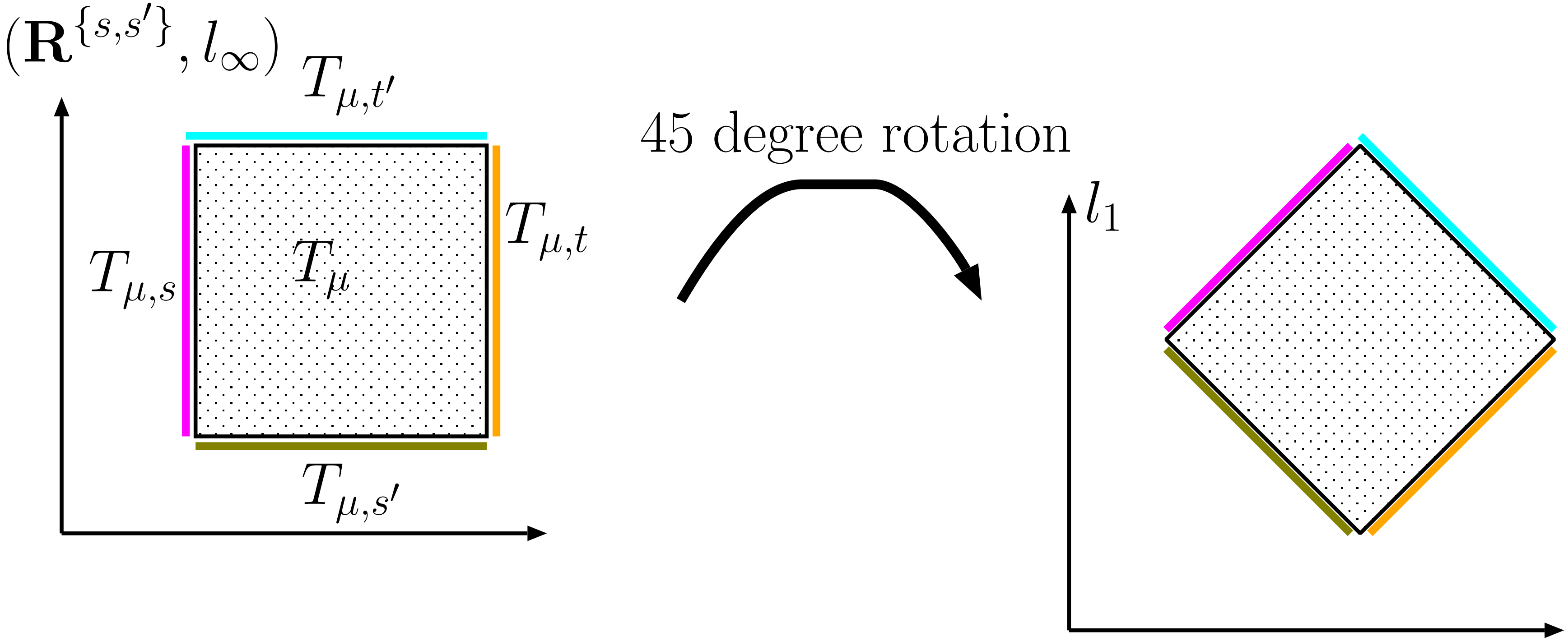
$\rho(t) =$ 

$\rho(u) =$ 

$\Rightarrow \dfrac{1}{2}\displaystyle\sum_{t \in S} t$-$S \setminus t$ mincut.

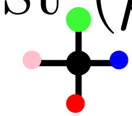
Two-commodity reconsidered ($S = \{s, t, s', t'\}$)

The tight span is a square in l_∞-plane.

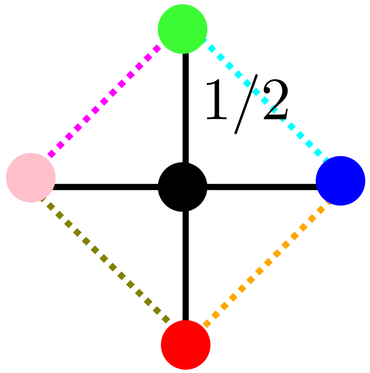


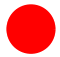
Rem: $(x_1, x_2) \mapsto \left(\frac{x_1 + x_2}{2}, \frac{x_1 - x_2}{2}\right)$.

T-dual is equivalent to

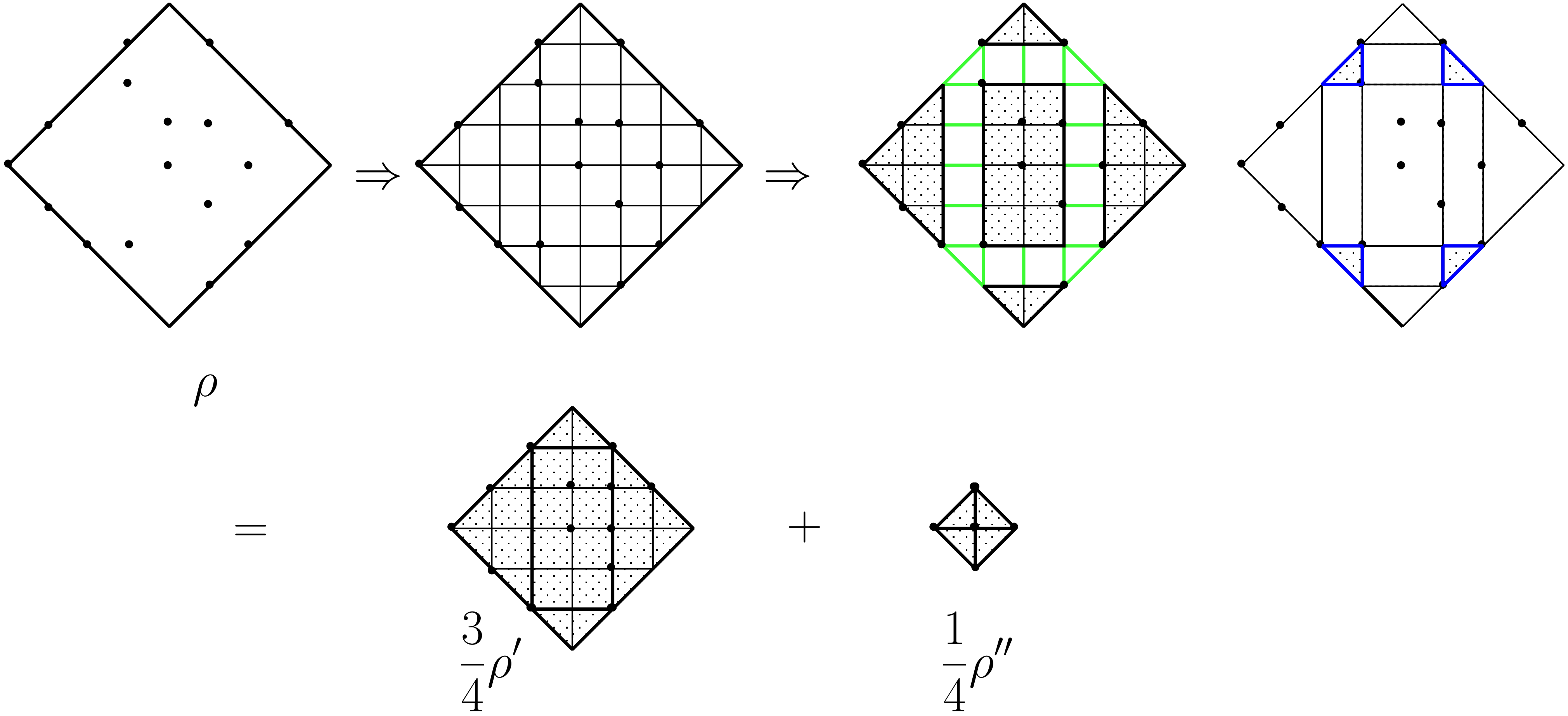
$$\text{Minimize} \quad \sum_{xy \in E} c(xy)\,\mathrm{dist}$$  $$(\rho(x), \rho(y))$$

Subject to

$\rho : V \to$ 

$\rho(s) =$  or $\rho(s') =$ or 

$\rho(t) =$  or  $\rho(t') =$  or 



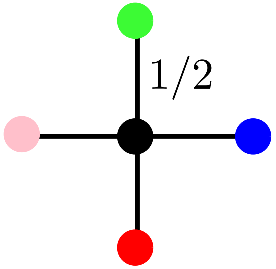
$$d^{\rho} = \frac{3}{4} d^{\rho'} + \frac{1}{4} d^{\rho''}$$

One more step to maxbiflow-mincut (left to audience)

Minimize $\langle c, d^\rho \rangle$

Subject to

$\rho : V \to$



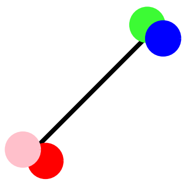
$\rho(s) =$ or $\quad \rho(s') =$ or

$\rho(t) =$ or $\quad \rho(t') =$ or

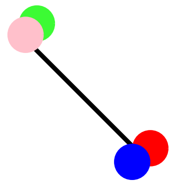
$=$

Minimize $\langle c, d^\rho \rangle$

Subject to $\quad \rho : V \to$



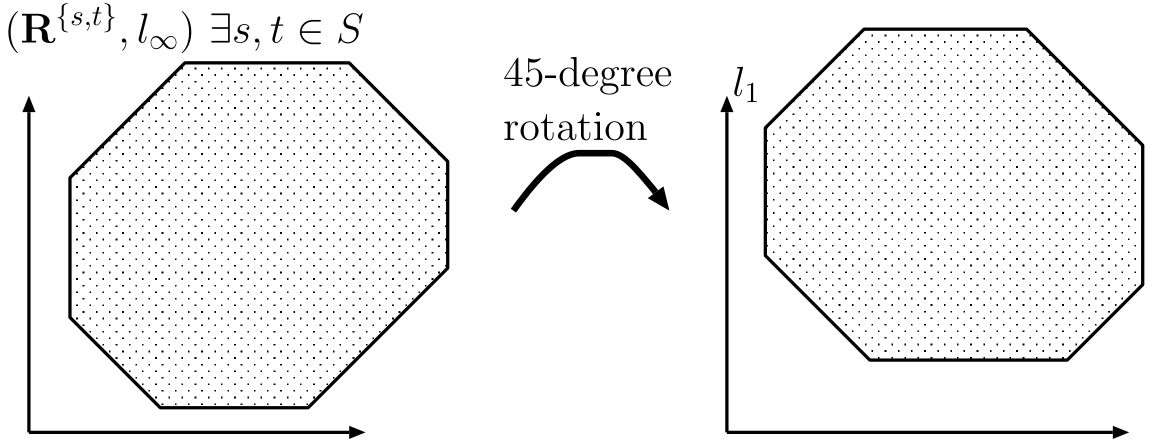
or



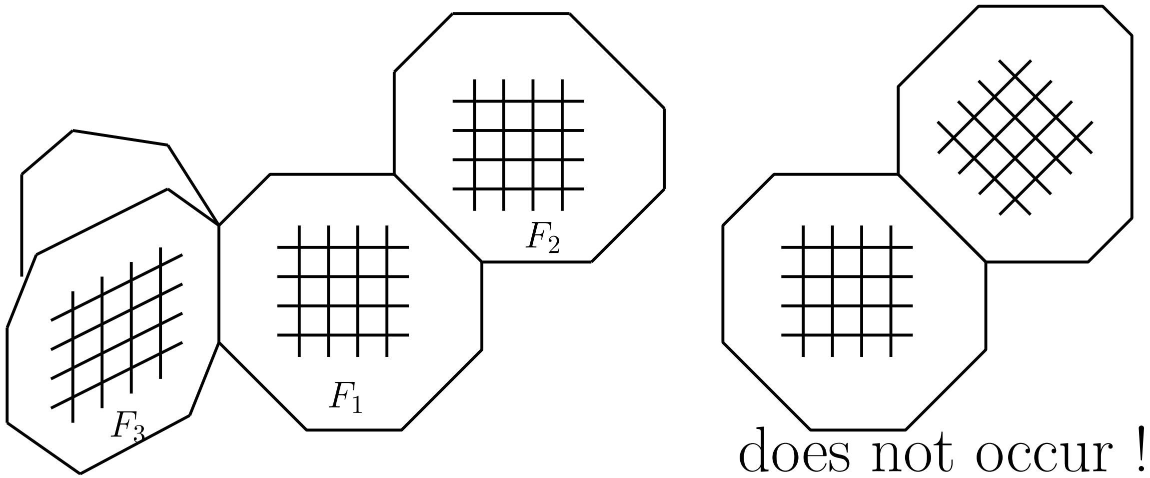
$\rho(s) = \rho(s') =$ $\quad \rho(s) = \rho(t') =$

$\rho(t) = \rho(t') =$ $\quad \rho(t) = \rho(s') =$

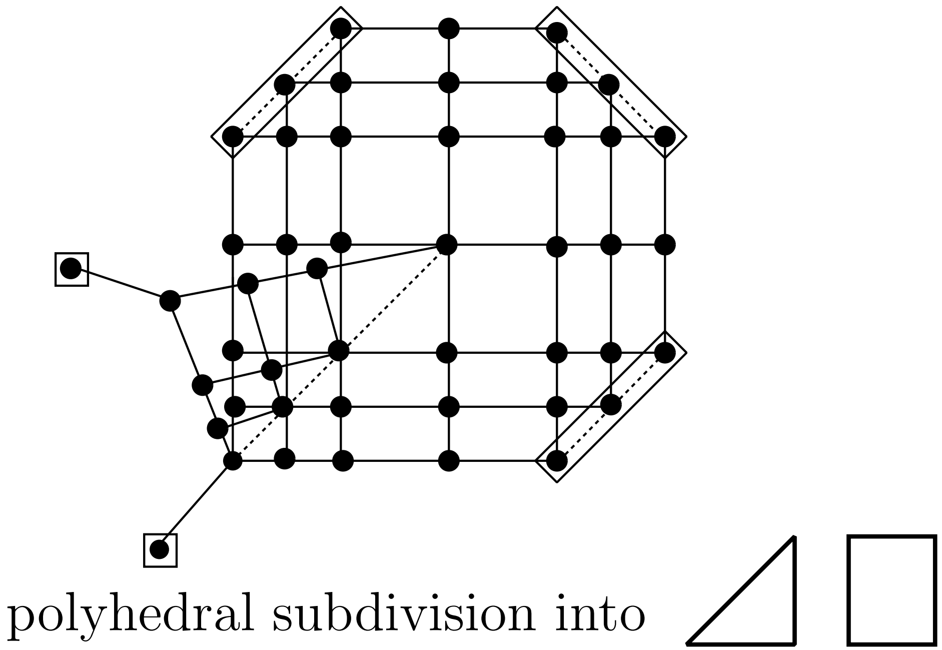
Lemma [H.07] 2-face of T_μ is isomorphic to



Lemma [H.07] 2-faces of T_μ are gluing *nicely*.



An l_1-grid on T_μ.



Lemma [H. 07]

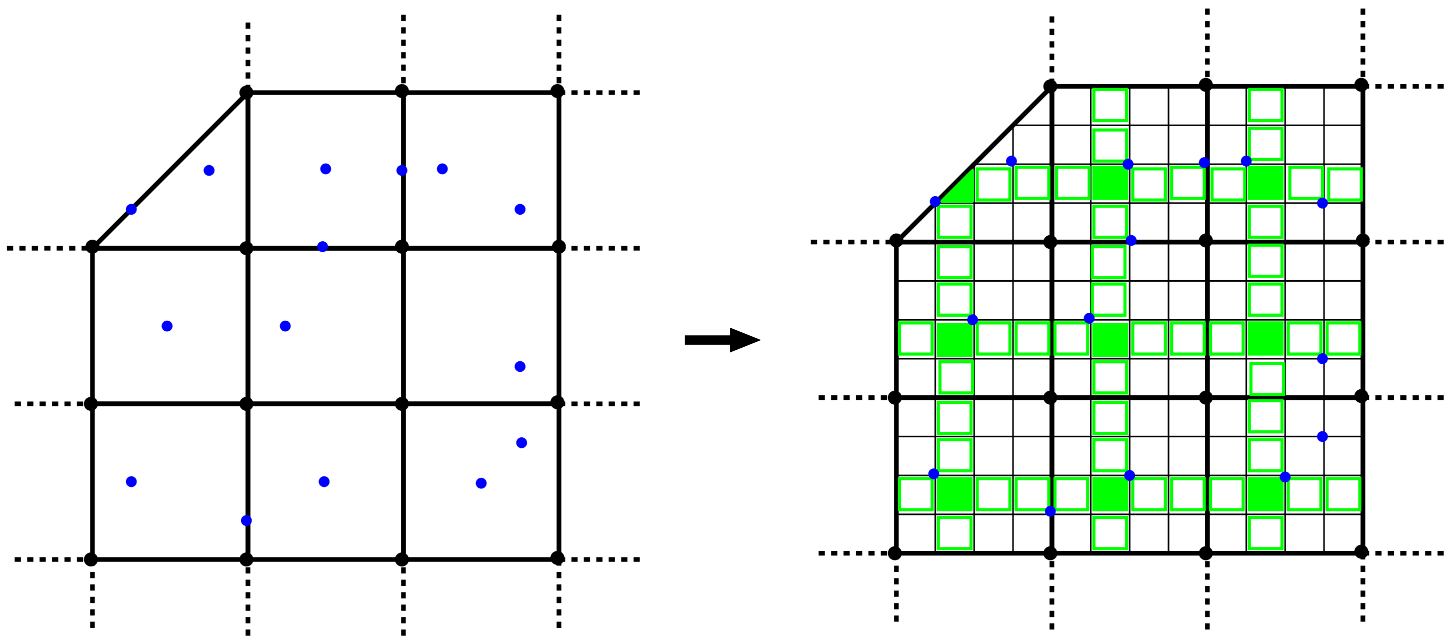
The graph of an l_1-grid is an isometric subspace of (T_μ, l_∞).

μ: a rational 2-dim distance on S.
Γ: the graph of an orientable l_1-grid on T_μ.
Γ_s: the subgraph of Γ induced by $T_{\mu,s}$ ($s \in S$).

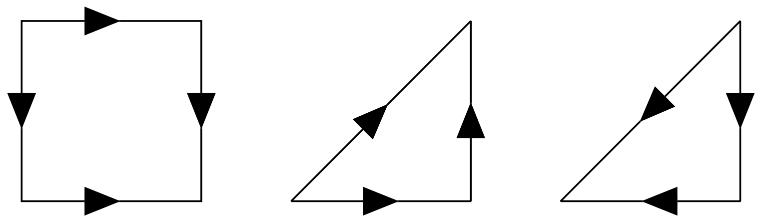
Theorem (H. 07)
T-dual is equivalent to

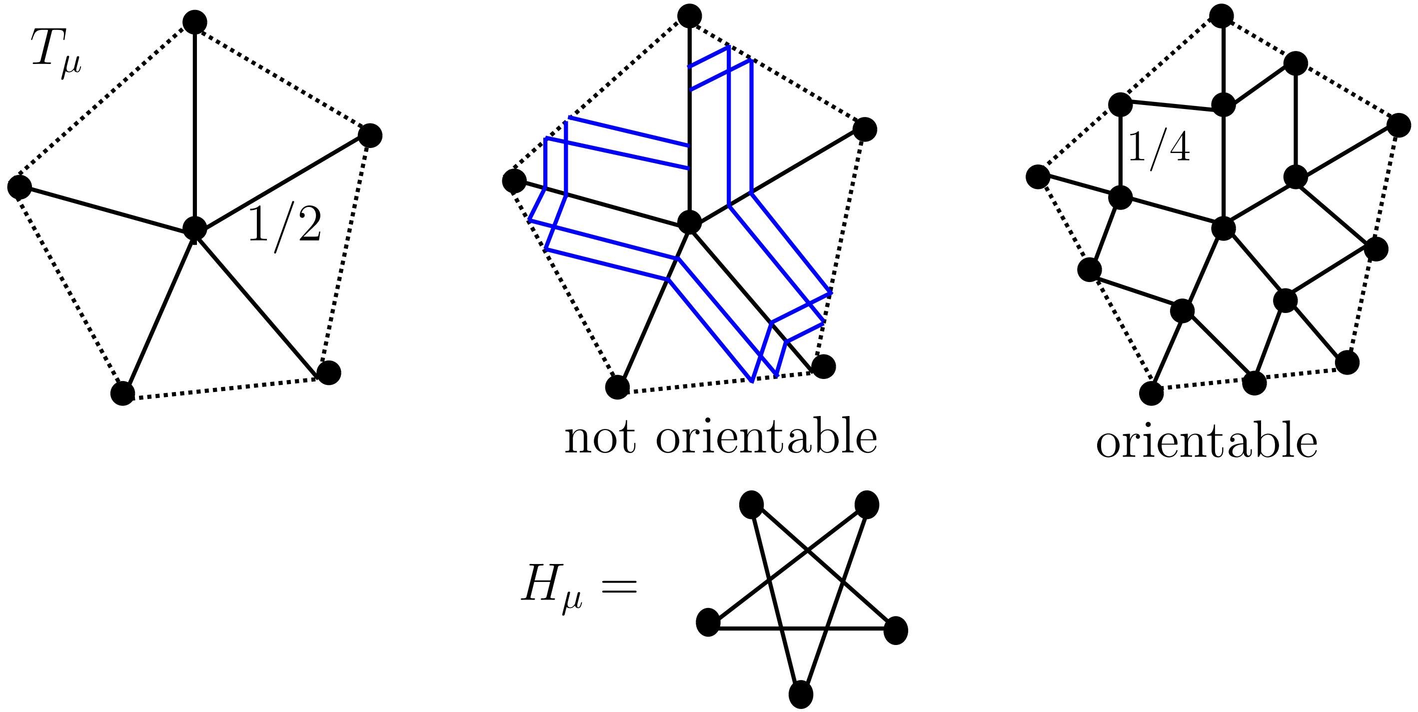
$$\begin{array}{rl} \text{Minimize} & \displaystyle\sum_{xy \in E} c(xy)\text{dist}_\Gamma(\rho(x), \rho(y)) \\ \text{Subject to} & \rho : V \to V\Gamma, \\ & \rho(s) \in V\Gamma_s \ (s \in S) \end{array}$$

- { the vertices of $\mathcal{P}_{\mu,V}$} $\subseteq \{d^\rho \mid \rho : \text{above}\}$,
  where $d^\rho(x, y) := \text{dist}_\Gamma(\rho(x), \rho(y))$.

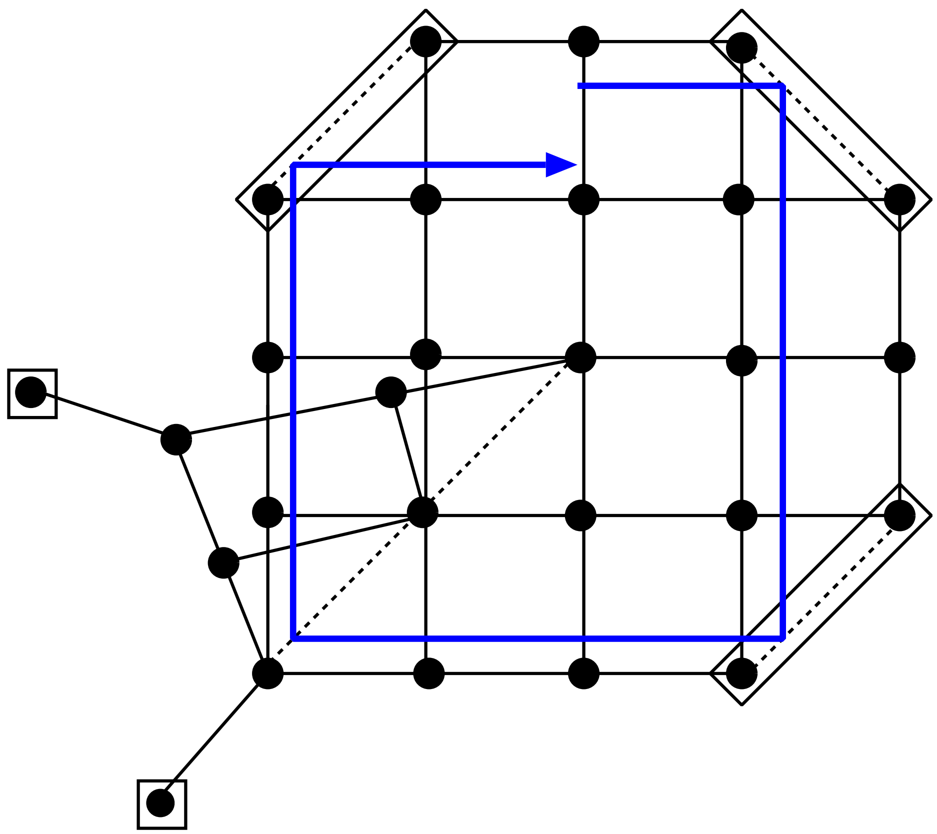


Orientablity is important.




T_μ
1/2
not orientable
1/4
orientable
$H_\mu =$

Nonorientablity

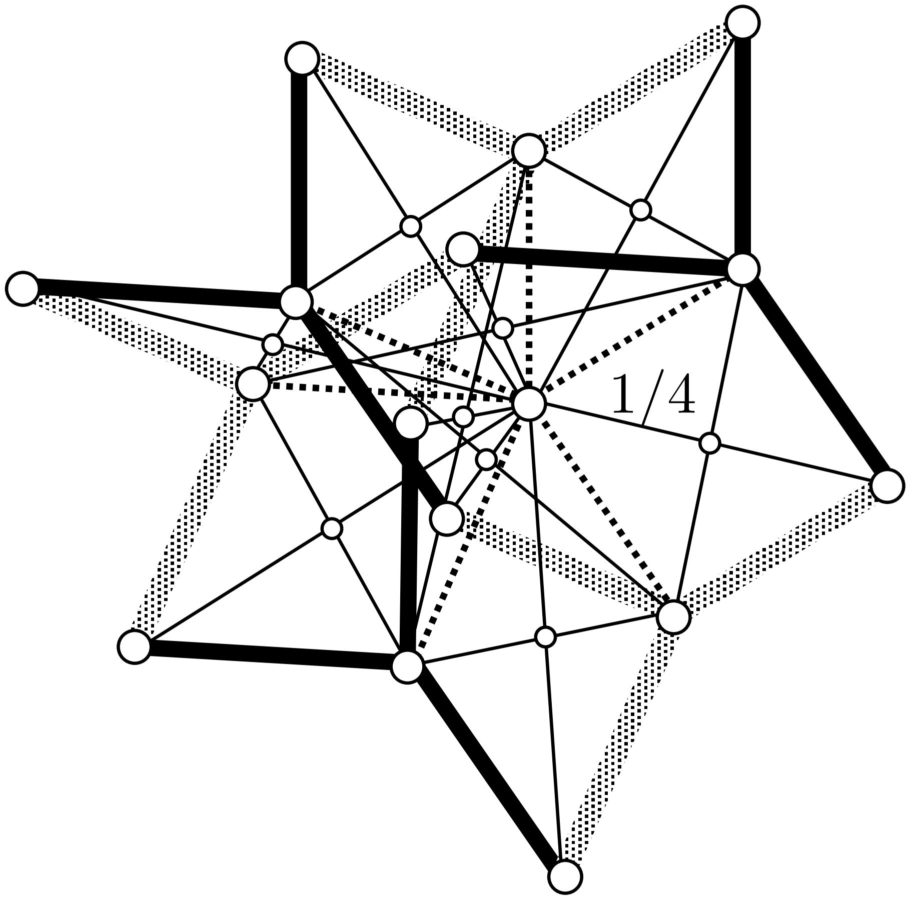


Proposition [H.07]
If μ is 2-dim 0-1 distance, then
$T_\mu \simeq$ one-point join of
the clique-vertex incidence graph of
the intersection graph of the maximal stable sets of H_μ.

- Karzanov-Lomonosov condition (1978) $\Leftrightarrow$ $\exists$ 1/2-l_1-grids.

- $T_\mu \simeq$ one-point join of
  the intersection graph of maximal stable sets of H_μ

- bipartiteness $\Leftrightarrow$ orientability



$$H_\mu = K_3 + K_3$$

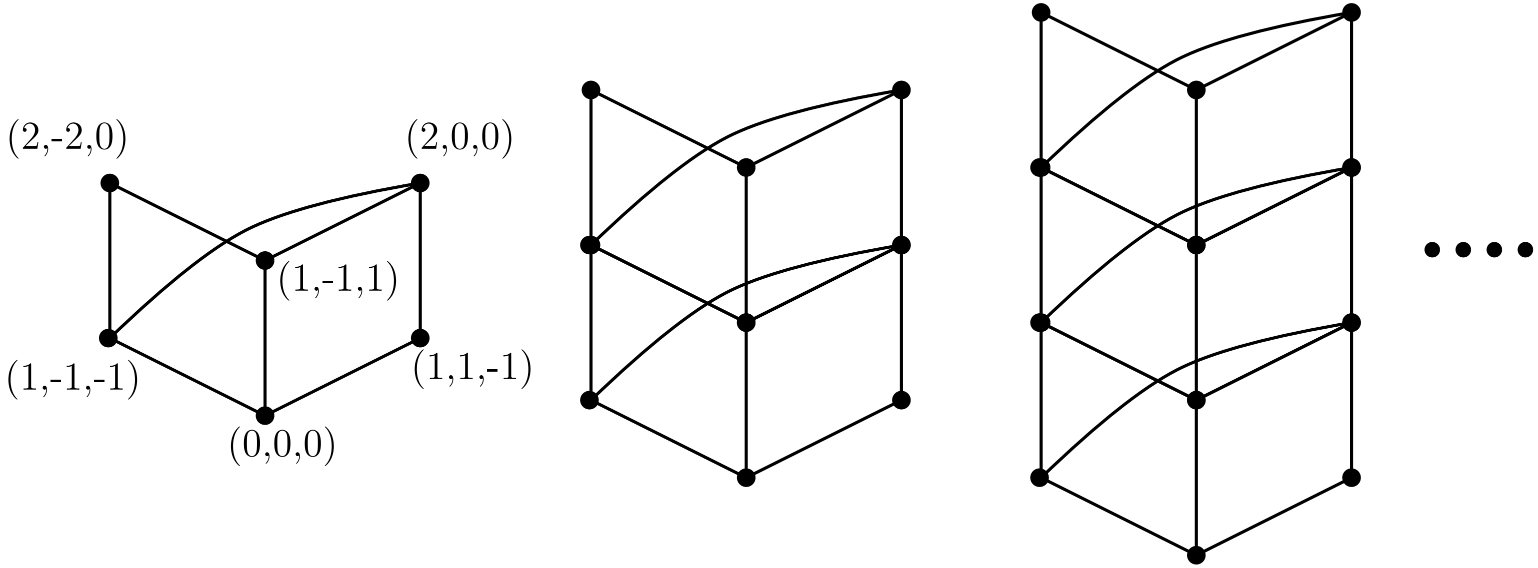
Theorem (H.07)

If μ is integral, then there is an orientable $1/4$-l_1-grid, and consequently $\mathcal{P}_{\mu,V}$ is $1/4$-integral.

- The existence of an $1/4$-l_1-grid is easy.
- P_μ is half-integral and $(x_1, x_2) \mapsto \left(\frac{x_1+x_2}{2}, \frac{x_1-x_2}{2}\right)$.
- The most difficult part is to prove that this $1/4$-l_1-grid is *orientable*.

Why is $\dim T_\mu \geq 3$ bad ?

• In $(\mathbf{R}^3, l_\infty)$, there exists an infinite family of finite sets $P_i (i = 1, 2, \ldots)$ such that d_{P_i, l_∞} $(i = 1, 2, \ldots)$ lie on all distinct extreme rays of the metric cone.



Karzanov's original approach (1998)

0-extension problem (metric labeling problem):

Given $G = (V, E)$, $c \in \mathbf{R}_+^E$, and Γ with $V\Gamma \subseteq V$

$$\begin{array}{rl} \text{Minimize} & \displaystyle\sum_{xy \in E} c(xy)\,\mathrm{dist}_\Gamma(\rho(x), \rho(y)) \\ \text{Subject to} & \rho : V \to V\Gamma, \\ & \rho|_{V\Gamma} = \mathrm{id}_{V\Gamma} \end{array}$$

$\Rightarrow$ NP-hard (3-terminal cut problem if $\Gamma = K_3$)

A relaxation problem:

$$\begin{array}{rl} \text{Minimize} & \displaystyle\sum_{xy \in E} c(xy)\,d(x, y) \\ \text{Subject to} & d\text{: metric on } V \\ & d|_{V\Gamma} = \mathrm{dist}_\Gamma \end{array}$$

(This is LP-dual of μ-max problem for $\mu = \mathrm{dist}_\Gamma$!)

Theorem (Karzanov 98)
Two problems are equivalent if and only if Γ is bipartite without isometric k-cycle for $k \geq 6$, and orientable.

$\Rightarrow$ a combinatorial characterization of 2-dim tight span (of metrics).

Karzanov's approach: graph theoretical, T_μ implicit.

Our approach: polyhedral geometry of T_μ.

Summary:

The tight span is very powerfull, and gives a unified understanding to multiflow problems.

Future works (for part I):

- Toward the generalized Karzanov conjecture (in preparation).
- Directed multiflows (in preparation, joint with Shungo Koichi).

$$P_\mu = \{(p,q) \in \mathbf{R}_+^S \times \mathbf{R}_+^S \mid p(s) + q(t) \geq \mu(s,t) \ (s,t \in S)\}$$
$$T_\mu = \text{the set of minimal elements of } P_\mu$$

- Discrete convex analysis for multiflows.
  - Network flow + convex analysis + discreteness (Iri 69, Rockafellar 84) $\Rightarrow$ Discrete convex analysis (Murota 98) afternoon today !
  - Multiflow + convex analysis + T-dual + discrete metrics $\Rightarrow$??

Part II: Metric packing for $K_3 + K_3$.

Multiflow feasiblity problem

$G = (V, E)$: an undirected graph with nonnegative capasity $c \in \mathbf{R}_+^E$
$H = (S, R)$: a demand graph $S \subseteq V$

Given a demand $q : R \to \mathbf{R}_+$, find a multiflow $f : \mathcal{P} \to \mathbf{R}_+$ such that

$$\sum\{f(P) \mid P \in \mathcal{P} : P \text{ is } st\text{-path}\} = q(st) \quad (st \in R).$$

Japanese Theorem (Onaga-Kakusho 71, Iri 71)

There exists a feasible multiflow if and only if

$$\langle c, d \rangle_E \geq \langle q, d \rangle_R \quad (\forall d\text{: metric on } V).$$

Cut condition:

$$\langle c, \delta_A \rangle_E \geq \langle q, \delta_A \rangle_R \quad (S \subseteq V)$$

When is the cut condition sufficient ?

Theorem (Papernov 76)
The cut condition is sufficient if and only if $H = K_4, C_5$ or the union of two star.

Theorem (Hu 63, Rothchild-Winston 66, Lomonosov 76, 85, Seymour 80)
If H is above and $G + H$ is Eulerian, then the cut condition implies an integer multiflow.

Polarity

Lemma (Seymour 79, Karzanov 84)
The cut condition is sufficient if and only if for any $l \in \mathbf{R}_+^E$ there are a familiy of cuts $\{\delta_{A_i}\}_i$ and its nonnegative weight $\{\lambda_i\}_i$ such that

$$\begin{aligned}\sum_i \lambda_i \delta_{A_i}(x,y) &\leq \mathrm{dist}_{G,l}(x,y) \quad (xy \in E),\\ \sum_i \lambda_i \delta_{A_i}(s,t) &= \mathrm{dist}_{G,l}(s,t) \quad (st \in R)\end{aligned}$$

Such a $(\delta_{A_i}, \lambda_i)$ is called an H-packing

Theorem (Seymour 80 for $H = K_2 + K_2$, Karzanov 85)
If H is above and G is bipartite, then there exists an integral H-packing by cut metrics.

Beyond the cut condition

Γ: undirected graph

Definition A metric d on V is called a Γ-metric if there is $\phi : V \to V\Gamma$ such that

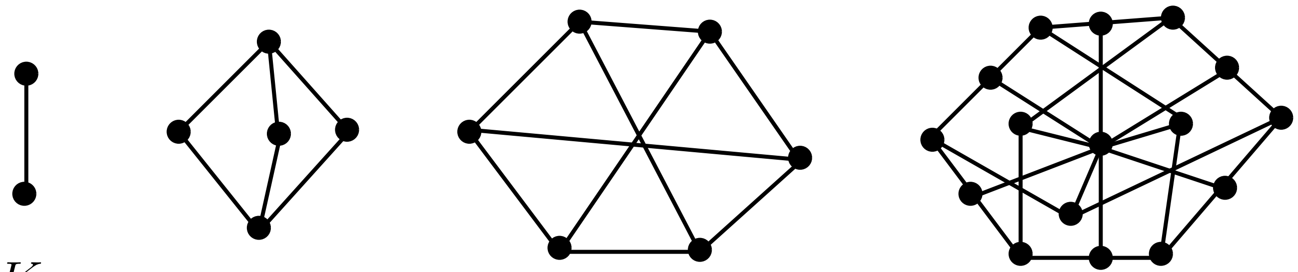
$$d(x,y) = \mathrm{dist}_{\Gamma}(\phi(x), \phi(y)) \quad (x, y \in V).$$

Remark: cut metric $\simeq K_2$-metric.

Lemma: For a set $\mathcal{G}$ of graphs, $\mathcal{G}$-metric condition is sufficient if and only if for $l \in \mathbf{R}_+^E$ there are familiy of $\mathcal{G}$-metrics $\{d_i\}_i$ and its nonnegative weight $\{\lambda_i\}_i$ such that

$$\begin{aligned} \sum_i \lambda_i d_i(x,y) &\leq \mathrm{dist}_{G,l}(x,y) \quad (xy \in E) \\ \sum_i \lambda_i d_i(s,t) &= \mathrm{dist}_{G,l}(s,t) \quad (st \in R) \end{aligned}$$

demand graph H	K_4, C_5, star + star	K_5, K_3+star	$K_3 + K_3$	other classes: H has 3-matching
multiflow for $G + H$:Eulerian	integer flow	integer flow (Karzanov 87)	$\exists k$, $1/k$-flow conjectured (Karzanov 90)	no fixed integer k, $1/k$-flow (Lomonosov 85)
feasibility condition	K_2 cut condition	K_2, $K_{2,3}$ (Karzanov 87)	$K_2, K_{2,3}, \Gamma_{3,3}$ (Karzanov 89)	infinite family of graphs (Karzanov 90)
H-packing for G: bipartite	integer packing	integer packing (Karzanov 90)	half-integer packing conjectured (Karzanov 90)	



K_2 $K_{2,3}$ $K_{3,3}$ $\Gamma_{3,3}$

Main result

Theorem [H. 07]

If $H = K_3 + K_3$ and G is bipartite, then there is an integral H-packing by cut, $K_{2,3}$, $K_{3,3}$, and $\Gamma_{3,3}$-metrics

Chepoi's approach (1997) with a modification by (H. 07)

μ: a bipartite metric on S ($\overset{\text{def}}{\Longleftrightarrow}$ $\mu(C)$ is even for cycle C)
L: a lattice on $\mathbf{Z}^S$ defined by

$$L = \{p \in \mathbf{Z}^S \mid p(s) + p(t) = 0 \mod 2 \quad (s, t \in S)\}$$

A_μ: an affine lattice defined by

$$A_\mu = \mu_s + L,$$

where μ_s is a s-th row vector of μ (well-defined).

Lemma (Chepoi 97, H. 07)

For a finite subset $Q \subseteq P_\mu \cap A_\mu$, there is a map $\phi : Q \to T_\mu \cap A_\mu$ such that

(1) $\phi(p) \leq p$ for $p \in Q$ (, and thus $\phi(p) = p$ if $p \in T_\mu$)

(2) $\|\phi(p) - \phi(q)\|_\infty \leq \|p - q\|_\infty$ for $p, q \in Q$.

This is a discrete version of Dress' lemma.

$H = (S, R)$: a commodity graph
$G = (V, E)$: a bipartite graph and $S \subseteq V$

$\Rightarrow$ dist_G is a bipartite metric on V

Define a (bipartite) metric μ on S by

$$\mu := \mathrm{dist}_G|_S.$$

Define a point $p^x \in \mathbf{R}^S$ for $x \in V$ by

$$p^x(s) = \mathrm{dist}_G(s, x) \quad (s \in S).$$

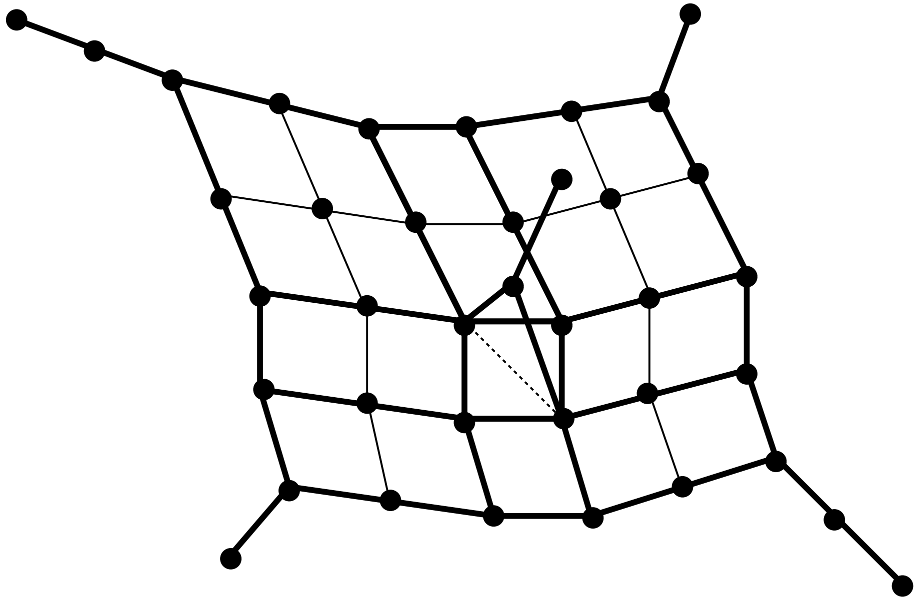
Lemma:

- $p^x \in P_\mu \cap A_\mu$ for $x \in V$, and $p^s \in T_\mu \cap A_\mu$ for $s \in S$.
- $\|p^x - p^y\| \leq \mathrm{dist}_G(x, y)$.
- $\|p^s - p^t\| = \mathrm{dist}_G(s, t) = \mu(s, t)$ for $s, t \in S$

integral H-packing $\Rightarrow$ decomposing $(T_\mu \cap A_\mu, l_\infty)$

Chepoi proved Karzanov's $K_2, K_{2,3}$-packing theorems by using the classification result of tight spans of five point metrics (Dress 84).

Remark. $\dim T_\mu \leq \#S/2$



Unfortunately, this approach cannot be applied to six-vertex commodity graph $K_3 + K_3$.

Definition A metric μ is called an H-*minimal* if there is no metric $\mu' \neq \mu$ with $\mu' \leq \mu$ such that

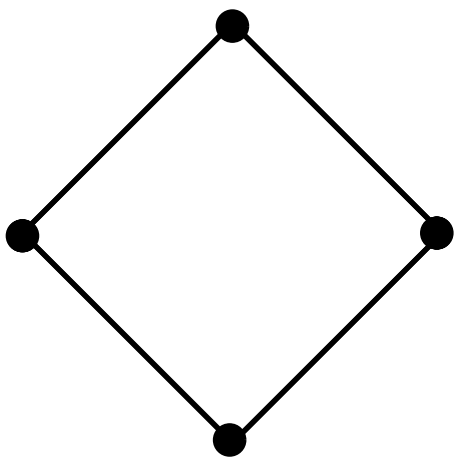
$$\mu'(s,t) = \mu(s,t) \quad (s,t \in R).$$

In the process above, we can replace μ by H-minimal bipartite metric μ' with $\mu(s,t) = \mu'(s,t)$ for $st \in R$.

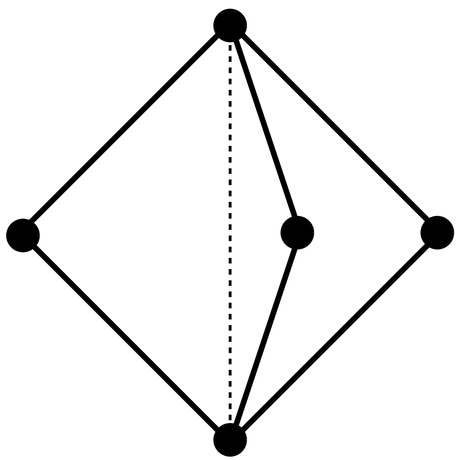
Lemma [H. 07] If H has no 3-matching, then any H-minimal metric μ is $\dim T_\mu \leq 2$.

Consider the graph Γ of $T_\mu \cap A_\mu$ connecting p, q by edge if $\|p - q\|_\infty = 1$

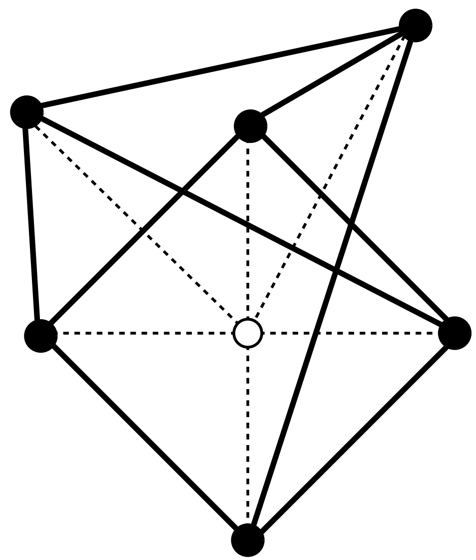
Proposition [H. 07] If H has no 3-matching and μ is H-minimal, the connected components of the closure of $T_\mu \setminus \Gamma$ are



square



$K_{2,3}$-folder



$K_{3,3}$-folder

Future works (for part II):

- A unified understanding to planer multiflows and some variations:
  - planar multiflows with demand edges on k holes ($k = 1$: Okamura-Seymour 81, $k = 2$: Okamura 83, $k = 3, 4$: Karzanov 94,95)
  - graph having no K_5-minor (Seymour 81), signed graph having no odd K_5-minor (Geelen-Guenin 01)